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Exactly Solvable Problems in Quantum Mechanics

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Preface

This work is devoted to various methods of obtaining exact solutions for quantum mechanical problems. It was originally intended to have two main parts. The first one was to contain a review of methods used in this field of quantum mechanics since its very beginning. The second part was supposed to focus on a relatively new method which leads to quasi–exactly solvable models.

During the preparation of the work it proved to be very convenient, both for the author and – I hope – for the readers, to gather the mathematical definitions and the most fundamental facts concerning the group theory in a separate, preliminary chapter.

Although there are a few to some extent original comments on the cited works in the Chapter 2, the central part of the work is the Chapter 3, devoted to the quasi–exactly solvable models in quantum mechanics. Its contents covers both the works of other authors on this subject and my own contribution to the problem.

The latter has been possible thanks to dr hab. Andrzej Radosz, who invited me to take part in his research work, and – especially – to prof. Henryk Konwent, who introduced me to the problems of exact and quasi exact solvability and who has always been willing to answer my questions. To both of them I owe my gratitude.

Introduction

Since the very beginning of quantum mechanics, it has been clear that the number of quantum mechanical problems that can be solved exactly is very limited. This fact gave rise to the development of many approximate methods, such as variational methods, perturbation theory or WKB method. On the other hand, the search began for exactly solvable problems. The first solutions were based either upon cleverly chosen ansätze or upon the idea of dynamical symmetry carried over from the classical physics. Later on, new methods appeared, some of which were based on some kind of hidden symmetry of the problem (or of the class of problems) while others made use e.g. of special functions. Although the latter did not use symmetry explicitely, in most cases it results after more thorough examination that there is an indirect relation to an algebraic approach. This is the case e.g. when the special functions approach is concerned. One may find out that there are other symmetry properties behind the systems of special functions used to generate the exactly solvable Hamiltonians. The same happens with the ansatz methods. The ansätze used usually prove to be the basis functions of a certain Lie algebra.

The close relation that appears to exist between one of the oldest systematic approaches – the factorization method and one of the most recent ones – the supersymmetry is another convincing example.

Having all this in mind, one is strongly tempted to conclude that any existing case of exact solvability can be explained and derived in terms of hidden symmetry. However, even if it were true, it would probably not mean unifying all the ways to approach the problem. In fact, although many methods are based on symmetry, or – to be more specific – on the properties of Lie algebras, and although the algebras used are in large majority of cases $sl(2)$ or related algebras, there are several independent ways of using these algebras to obtain the exact solutions. There is large difference between the classical method using constants of motion and all the other methods. Depending on the approach, the Hamiltonian may be obtained as an element of the algebra, as the Casimir operator or as an arbitrary quadratic form of the algebra elements.

As for the last of these possibilities, one encounters here a new phenomenon: quasi–exact solvability, which is completely different from all the examples of exactly solvable problems known before. Its characteristic feature is that only a certain number of the lowest lying levels can be found exactly and written in a closed form. The quasi–exact solvability arises as an effect of relation between the potential system and a spin system with the total spin S, resulting in $2S+1$ exactly known eigenstates.

As most of the methods discussed in this work are based on the theory of Lie algebras, it may be profitable to recapitulate the chief points of the theory at the beginning of the work. This is also a convenient form of providing the notation reference as well as the formulae that are most often referred to. For this reason, the mathematical facts, mostly quoted after textbooks, have been collected in the Chapter 1.

The Chapter 2 contains a review of some methods that can be used to obtain exactly solvable quantum mechanical problems. The review is in principle limited to the 1-dimensional problems. Several of them may, however, be interpreted as radial parts of 3-dimensional problems. Some of the results quoted in this chapter are commented on from the point of view of the relations between the methods. It must be stressed that there are many methods which are not discussed here. The lilmited volume of this work forced me to choose only some of the very great number of known results in this field.

The Chapter 3 contains the description and several examples of application of the relatively new technique which leads to quasi–exactly solvable problems. In the first sections of this chapter, the method is derived in a systematical way. Then, several simpler cases are discussed in more detail, resulting in a few new quasi–exactly solvable potentials. One of the results of this chapter, which – to my knowledge – has not been reported before, is the existence of exactly solvable potentials as limiting cases of the quasi–exactly solvable ones.

The Summary at the end of the work recapitulates the main results and the relations between them and indicates several possible further subjects to be examined.

Chapter 1

Mathematical Preliminaries

In order to provide a unified notation throughout the work as well as for further reference, the most important definitions and results of the group theory are collected in this Chapter. Some of them are basic handbook facts which we quote mostly for completeness. A few others are more sophisticated and are going to be of essential importance in the further chapters of the work. This chapter is based on the handbooks [1], [2], [3].

1.1 Groups

Following are several fundamental definitions concerning groups in the algebraic sense.

A group is a set G with a binary operation defined on it (called *multiplication*), satisfying the following conditions

- there exists an **identity element** e in G (a *unit*) such that $eg = ge = g$ for any $g \in G$;
- for every $g \in G$ there exists an **inverse element** $g^{-1} \in G$, satisfying $gg^{-1} = g^{-1}g = e$;
- the **associative law** is satisfied, $(gh)k = g(hk),\ g, h, k \in G$.

A subgroup of a group G is a subset $K \subset G$ which is closed with respect to the group multiplication, i.e. $g, h \in K \Rightarrow gh \in K$.

A homomorphism of groups is a mapping $G \to H$ transforming products into products.

An isomorphism is a homomorphism which is '1–1' and 'onto'. Two groups between which there exists an isomorphism are called **isomorphic**.

A representation of a group G is a homomorphism of the group into the group of invertible operators on a certain (most often complex) Hilbert space V (called **representation space**). The dimension of this space is called the dimension of the representation. If the representation is to be finite–dimensional, it is sufficient to consider homomorphisms $G \to GL(n)$, where $GL(n)$ is the group of non-singular matrices of the dimension n. Usually, the image of the group in this homomorphism is called a representation as well.

An irreducible representation is a representation whose representation space contains no proper subspace invariant under the operators of the representation.

1.2 Linear Lie Groups

Linear Lie groups constitute a special case of abstract Lie groups. All the commonly used continuous matrix groups are linear Lie groups. Some basic definitions are gathered below.

A local linear Lie group. Let $F = \mathcal{R}$ – the field of real numbers, or $F = \mathcal{C}$ – the field of complex numbers. Let $0 \in W \subset F^p$. A linear Lie group of dimension p is a set of non–singular $m \times m$ matrices $A(\vec{g})$, parametrized by a vector $\vec{g} \in W$, having the following properties:

- $A(0) = \mathrm{I}$ (the unit matrix);
- the function $\vec{g} \mapsto A(\vec{g})$ is analytical and '1–1' on W;
- p matrices $\frac{\partial A(\vec{g})}{\partial g_i}$ form a linearly independent set for any $\vec{g} \in W$. Thus, they span a p-dimensional subspace in the m^2-dimensional space of all matrices $m \times m$.
- there exists a neighbourhood $W' \in W$ containing the vector 0, such that $\vec{g}, \vec{h} \in W' \Rightarrow \exists k \in W,\ A(\vec{g})A(\vec{h}) = A(\vec{k})$.

If such a group as a whole possesses the algebraic structure of an abstract group, it is a **(global) linear Lie group**. Such a group is essentially an abstract (algebraic) group the elements of which form an analytical manifold such that the group invertion and the group multiplication are analytical functions with respect to the manifold structure. Note that in the case of a global Lie group the requirements of the previous definition are satisfied in some neighbourhood of the unit element as a consequence of the definitions of a manifold and of a map on a manifold.

A compact linear Lie group is a linear Lie group which is closed in the topological sense and the matrices belonging to it are commonly bounded with respect to the standard matrix norm, i.e.

$$\exists M > 0,\ ||A(\vec{g})|| < M,\ \vec{g} \in W.$$

It is possible to define an invariant measure on a compact Lie group and to rewrite for them many results of the theory of finite groups.

A local representation of a Lie group is a local analytical homomorphism of the group into a group of operators on a Hilbert space. The local homomorphism is defined on a certain neighbourhood of the unit element of the group.

A global representation is an analytical homomorphism of the group manifold as a whole.

1.3 Matrix Lie Algebras

Matrix Lie algebras are special cases of general Lie algebras. Below, a few definitions are presented, followed by several less fundamental facts which are going to be useful in the further development of the work.

Matrix Lie algebras. A linear space $\mathcal{G}$ of matrices is a matrix Lie algebra if a commutator of any pair of matrices from $\mathcal{G}$ belongs to $\mathcal{G}$, i.e. apart from the linear structure, there is another binary operation defined – the matrix commutator. Any basis of the linear space $\mathcal{G}$ is called the set of **generators** of the algebra. Only finite Lie algebras, i.e. algebras with finite dimension of the space $\mathcal{G}$, will be used in this work.

The Lie algebra $L(G)$ of a local linear Lie group G is the algebra that is spanned by the matrices

$$\mathcal{A}_i = \frac{\partial}{\partial g_i} A(\vec{g})|_{\vec{g}=0}$$

over the field of scalars F. It is straightforward to show that it may be equivalently defined as the set of all matrices of the form

$$\mathcal{A} = \frac{\mathrm{d}}{\mathrm{d}t} A(\vec{g}(t))|_{t=0},$$

where $\vec{g}(t)$ is an analytical curve in the parameter space F^p such that $\vec{g}(0) = 0$.
One can also show that the exponential mapping

$$L(G) \ni \mathcal{A} \to A = \exp(\mathcal{A}) \in G \tag{1.1}$$

transforms a sufficiently small neighbourhood of $\vec{g} = 0$ in the Lie algebra onto some neighbourhood of the unit matrix in the group. Moreover, this mapping is '1–1'. For global connected Lie groups this mapping may be extended to a mapping of the algebra onto the whole group.

The complexification $\mathcal{G}_c$ of a real matrix Lie algebra $\mathcal{G}$ (i.e. a Lie algebra over $\mathcal{R}$) is the algebra of all the complex linear combinations of the generators of $\mathcal{G}$.
The complexification of a real Lie algebra is obviously unique.

A real form of a complex Lie algebra is any its subset being itself a real Lie algebra. If $\mathcal{K}$ is a real form of $\mathcal{G}$, and $\mathcal{K}_C$ is the complexification of $\mathcal{K}$, then $\mathcal{K}_C = \mathcal{G}$.
A complex Lie algebra may have multiple real forms.
It is straightforward to show that the natural extension of an irreducible representation μ of a real Lie algebra $\mathcal{K}$ (we assume that the representation space is complex)

$$\mu_c(S) \stackrel{\text{def}}{=} \mu(S), \quad \mu_c(iS) \stackrel{\text{def}}{=} i\mu(S), \quad S \in \mathcal{G}$$

is an irreducible representation of its complexification $\mathcal{K}_c$. Conversely, a restriction of an irreducible representation of a complex Lie algebra to the elements of its real form is obviously a representation of this real form. One proves easily that such a representation is irreducible.

A representation of a Lie algebra is a homomorphism of the algebra into an algebra of operators, which preserves the linear structure and the commutators.
There exists a relation between representations (or, in general any homomorphisms) of linear Lie group and of their Lie algebras.

- An analytical homomorphism of groups $\mu : G \to G'$ induces a homomorphism of Lie algebras $\mu^* : L(G) \to L(G')$ ($L(G)$ is the Lie algebra of the linear Lie group G). This homomorphism is defined by
$$\mu^*(\mathcal{A}) = \frac{\mathrm{d}}{\mathrm{d}t}\mu\left(A(\vec{g}(t))\right)|_{t=0}\,,$$
where $\vec{g}(t)$ is an analytic curve such that
$$\mathcal{A} = \frac{\mathrm{d}}{\mathrm{d}t}A(\vec{g}(t)).$$
- If G, G' are local linear Lie groups and $\rho : L(G) \to L(G')$ is a homomorphism of the corresponding Lie algebras, then there exists a unique analytical local homomorphism $\mu : G \to G'$ such that $\mu^* = \rho$. This homomorphism is defined by
$$\mu\left(\exp(\alpha_i J_i)\right) = \exp\left(\alpha_i \rho(J_i)\right).$$

A Casimir operator is an operator which commutes with all the elements of the algebra. For compact Lie groups, the Casimir operator is proportional to the unit operator on every invariant subspace of the representation space (the Schur lemma), i.e. all the vectors belonging to one invariant subspace of the representation space belong to the same eigenvalue of the Casimir operator.

1.4 Lie Derivatives and Generalized Lie Derivatives

Many exact solutions of quantum mechanical problems are obtained by using algebras of operators that have the form of Lie derivatives and generalized Lie derivatives. It seems worthwile to quote the fragments of the theory that are of interest from the point of view of this work.

1.4.1 Lie derivatives

Let $F = \mathcal{R}$ – the field of real numbers, or $F = \mathcal{C}$ – the field of complex numbers. Consider a local p-parameter Lie transformation group G acting on an open connected subset $U \subset F^m$, i.e. a group of transformations

$$g : U \ni x \mapsto gx \in F^m$$

such that gx is an analytical function of the $m+p$ parameters of x and g and the following conditions are satisfied

- $ex = x$, $x \in U$;
- if $gx \in U$ then $h(gx) = (hg)x$, $g, h \in G$.

Let $\mathcal{F}$ be the space of analytic functions on U. One can define operators $T(g) : \mathcal{F} \to \mathcal{F}$ by

$$[T(g)f](x) = f(g^{-1}x). \tag{1.2}$$

It is obvious that $T(g)$ is a representation of the group G on $\mathcal{F}$.

The Lie derivative of a function $f \in \mathcal{F}$ is defined in the following way

$$L_i f(x) = \frac{\mathrm{d}}{\mathrm{d}x} \left[T(\exp(tS_i))f\right](x)|_{t=0}, \tag{1.3}$$

where S_i is a generator of the transformation group G.

In the explicite form, this reads

$$L_i = \sum_j P_{ij}(x) \frac{\partial}{\partial x_j}, \tag{1.4}$$

where $P_{ij}(x)$ are certain, determined functions of x.

The Lie derivatives span an algebra which is a homomorphic image of the Lie algebra $L(G)$ of the group G. The algebra of Lie derivatives is the Lie algebra induced by the representation $T(g)$. A more general form of a Lie algebra may be introduced.

1.4.2 Generalized Lie derivatives

A local multiplier representation of the group G on the function space $\mathcal{F}$ is defined by the formula

$$[Q(g)f](x) = \nu(x, g) f(g^{-1}x),$$

where $\nu(x, g)$ is a scalar–valued function analytic in g, x, satisfying the two relations

- $\nu(x, e) = 1$;
- $\nu(x, g_2)\nu(g_2^{-1}x, g_1) = \nu(x, g_2 g_1)$.

The generalized Lie derivative of a function $f \in \mathcal{F}$ corresponding to the generator S_i of the group G is

$$D_i f(x) = \frac{\mathrm{d}}{\mathrm{d}x} \left[Q(\exp(tS_i))f\right](x)|_{t=0}.$$

Performing a direct calculation we conclude that

$$D_i f(x) = \sum_j P_{ij}(x) \frac{\partial f}{\partial x_j} + R_i(x) f(x).$$

Generalized Lie derivatives form an algebra which is a homomorphic image (i.e. a representation) of the Lie algebra $L(G)$.

It is evident from the above formula and the formula (1.4) that an algebra of usual Lie derivatives is a special case of an algebra of generalized Lie derivatives.

1.5 Important Linear Lie Groups

- The largest group is the group of non-singular complex matrices of a certain dimension n – **the complex general linear group**

  $$GL(n) \equiv GL(n, \mathcal{C}) = \{A : A - n \times n \text{ matrix}, \det A \neq 0\}.$$

  The Lie algebra of this group consists of all the $n \times n$ matrices.
  Similarly, one can define the **real general linear group**.

- The **special linear group** consits of the matrices of unit determinant

$$SL(n, \mathcal{C}) = \{A : A \in GL(n),\ \det A = 1\}.$$

  Its Lie algebra is constituted by all traceless matrices $n \times n$.

- The special orthogonal group is the group of all orthogonal matrices of determinant $+1$

$$SO(n, F) = \{A : A \in GL(n, F),\ A\,A^T = I,\ \det A = 1\},$$

  where $F = \mathcal{C}$ or $F = \mathcal{R}$. The Lie algebra of this group consists of all skew-symmetric matrices of dimension n, $\mathcal{A} = -\mathcal{A}^T$.

- The special unitary group is the group of all complex $n \times n$ unitary matrices of determinant $+1$

$$SU(n) = \{A : A \in GL(n, F),\ A\,A^\dagger = I,\ \det A = 1\}.$$

  The Lie algebra of this group consists of all $n \times n$ skew-hermitian matrices, $\mathcal{A} = -\mathcal{A}^\dagger$.

1.6 Heisenberg–Weyl and Harmonic Oscillator Algebras

These two algebras are of different nature, as they are not related to linear Lie groups. Within the range of this work, they are used to solve the harmonic oscillator problem (Section 2.4) as well as in connection with the coherent states applications in the Section 2.7. However, their use is much wider, especially in the field theories.

The Heisenberg-Weyl algebra is generated by the bosonic creation and annihilation operators and the identity operator with the known commutation relations

$$\left[a, a^\dagger\right] = I, \quad [a, I] = \left[a^\dagger, I\right] = 0.$$

This algebra may be extended by adding the operator

$$H = a^\dagger a + \frac{1}{2}.$$

The resulting set of generators is closed with respect to the commutation relations

$$\left[H, a^\dagger\right] = a^\dagger, \quad [H, a] = -a, \quad [H, I] = 0.$$

This algebra is called the harmonic oscillator agebra [2], since the operator H in one of the realizations (the most commonly used, in fact) is the Hamiltonian for the harmonic oscillator problem. This is discussed in detail in the Section 2.4.

1.7 Representations of $su(2)$, $su(1,1)$ and $sl(2)$

The groups $SO(3)$ and $SU(2)$ are widely used in physics. The first of them describes rotations of 3-dimensional space; the second one is related to spin operators in quantum mechanics.

The relatively simple structure of these groups and of the corresponding Lie algebras allows of good understanding of their properties. This is probably one of the reasons why these groups are eagerly used in the search for exactly solvable problems.

Related with these groups are the groups $SO(2,1)$ and $SU(1,1)$. The $SO(2,1)$ group is the group of transformations of the $2+1$ dimensional spacetime. The $SU(1,1)$ is the group of transformations of the 2-dimensional complex space, conserving the quadratic form $x_1^2 - x_2^2$.

All of these groups are formally related to the $SL(2)$. This relation can be formally stated in terms of their algebras, as it will be done later on.

In the next two chapters of this work many approaches to exact solvability will be presented, almost all of which are based on particular realizations of the Lie algebras of these groups. It seems therefore worthwhile to recapitulate here some facts concerning their properties.

1.7.1 The groups $SU(2)$ and $SO(3)$

The Lie algebras $su(2)$ and $so(3)$ are both spanned over $\mathcal{R}$ by three generators. In the standard basis the commutation relations (i.e. the algebraic structure) of these generators for both these algebras are[1]

$$[J_i, J_j] = \epsilon_{ijk} J_k, \tag{1.5}$$

where J_i are generators of rotations with respect to the three mutually orthogonal axes. This can be expressed as the well known fact that the Lie algebras of these groups are isomorphic, while the groups themselves are locally isomorphic. To be more specific, the group $SO(3)$ can be parametrized in the standard way by the Euler angles:

$$0 \leq \phi < 2\pi, \quad 0 \leq \vartheta < \pi, \quad 0 \leq \psi < 2\pi,$$

whereas the Euler angles for the $SU(2)$ group can vary in the range

$$0 \leq \phi < 2\pi, \quad 0 \leq \vartheta < \pi, \quad -2\pi \leq \psi < 2\pi.$$

Thus, roughly speaking, the group $SU(2)$ is twice greater. Moreover, these groups differ in their topological properties. Both of them are connected but only $SU(2)$ is simply connected.

As it was stated in the Section 1.3, there is a 1–1 relationship betwen the representations of linear groups and the representations of their Lie algebras. Therefore, it is sufficient to classify either the group representations or the Lie algebra representations. For the $SU(2)$ group, the standard procedure involves the latter. Moreover, it is Lie algebras and their representations that play the essential role in the algebraic methods of solving quantum mechanical problems. Therefore, in future sections, we will quote the procedure for finding their representations.

As a preliminary remark which will prove to be essential in the following parts of the work, let us note that the discussed groups being compact, the global representations of their Lie algebras have one important feature. Namely, let us consider a representation of the Lie algebra $so(3)$ or $su(2)$, $\mu(J_i) = S_i$, corresponding to a global representation of the respective group. Assume that m is an eigenvalue of S_i. Then, $e^{\alpha m}$ is an eigenvalue of $\exp(\alpha S_i)$. This, however, must be periodical in α with the period 2π or 4π for $SO(3)$ and $SU(2)$, respectively, since we require that the representation should be globally analytical (one can think of such a group element as of a rotation). Thus, if f is the eigenvector belonging to m, we have

$$f = \exp(\alpha_0 S_i) f = e^{\alpha_0 m} f$$

and, consequently,

$$e^{\alpha_0 m} = 1,$$

where $\alpha_0 = 2\pi$ for $SO(3)$ and $\alpha_0 = 4\pi$ for $SU(2)$. Therefore, we see that im must be integer for $SO(3)$ and $2im$ must be integer for $SU(2)$. This property becomes important when the algebra generators are used to construct quantum mechanical Hamiltonians. It is clear that in such case the $su(2)$ algebra will be appropriate for constructing the discreete part of the spectrum (cf. Section 2.5). It should be stressed, that these conclusions are valid for global representations only.

Generally, one looks for unitary representations of groups[2]. It is clear from the relation (1.1) that the generators of the corresponding representation of the Lie algebra must be skew-hermitian operators.

1.7.2 The group $SO(2,1)$

Let us mention briefly some of the properties of the $SO(2,1)$ group. They will be made use of in the Section 2.5. The group $SO(2)$ is locally isomorphic with the $SU(1,1)$ group, the relation being

[1] Everywhere in this Section J_i denotes a generator of a given algebra, whereas S_i is the operator of the representation of the algebra, corresponding to J_i, i.e. $S_i = \mu(J_i)$.

[2] For a compact group, every representation is equivalent to a unitary one

analogous to the one between $SU(2)$ and $SO(3)$. The canonical basis of these groups satisfies the relations

$$[J_1, J_2] = -J_3, \quad [J_2, J_3] = J_1, \quad [J_3, J_1] = J_2. \tag{1.6}$$

Here, J_3 is the generator of the geometrical rotation and J_1, J_2 are Lorentz transformation. The discussion of the previous section may be now repeated for $S_3 = \mu(J_3)$ (called the compact generator) only, showing that its eigenvalues in any global representation may be integers or half-integers for $SO(2,1)$ and $SU(1,1)$, respectively. The two other generators admit any real eigenvalues – their spectrum is continuous. Therefore, they can be used for finding exact solutions within the continuous parts of spectra.

1.7.3 Representations of $su(2)$ and $su(1,1)$ using the $sl(2)$ algebra

It is straightforward to check, that if the operators S_1, S_2, S_3 satisfy the relations (1.5), the operators $-iS_1$, $-iS_2$, S_3 satisfy (1.6). This shows that the algebras $su(2)$ and $su(1,1)$ have a common complexification. This complexification is the $sl(2)$ algebra. The irreducible representations of the algebras $su(2)$ and $su(1,1)$ and of their complexification can be easily derived from one another (cf. Section 1.3). Therefore, it is sufficient to classify the representations of $sl(2)$ in order to find the representations of its real forms.

In the following, representations of the algebras $su(2)$ and $su(1,1)$ corresponding to global unitary representations of the corresponding groups will be derived. To this end we will use some representations of the $sl(2)$ algebra, which can be reduced to representations of $su(2)$ and $su(1,1)$ having the required properties. Every such representation will be derived in terms of the basis in which the generator S_3 is diagonal. The Casimir operator C together with S_3 form a complete set of commuting operators. The representations are characterized by the eigenvalue of the operator C and the set of eigenvalues of S_3 (as it will become clear, for the $su(2)$ algebra the former defines the representation completely; this, however, is not the case for $su(1,1)$).

The Casimir operators for the two groups are

$$C_{su(2)} = -S_1^2 - S_2^2 - S_3^2 \tag{1.7}$$

and

$$C_{su(1,1)} = S_1^2 + S_2^2 - S_3^2. \tag{1.8}$$

Many authors use the operator $C_{su(1,1)}$ with the opposite sign. The choice proposed here is convenient for unified approach to both these groups.

We shall use the basis of $sl(2)$ – the complexification of these algebras, defined separatly for each case

su(2):

$$\begin{aligned} S_0 &= -iS_3 \\ S_+ &= iS_1 + S_2 \\ S_- &= iS_1 - S_2 \end{aligned}$$

Inverting these defining relations we get

$$\begin{aligned} S_3 &= iS_0 \\ S_2 &= \frac{1}{2}(S_+ - S_-) \\ S_1 &= \frac{1}{2i}(S_+ + S_-) \end{aligned}$$

It is clear that the operators S_i, $i = 1, 2, 3$ are skew-hermitian when $S_+^\dagger = S_-$ and $S_0^\dagger = S_0$

su(1,1):

$$\begin{aligned} S_0 &= -iS_3 \\ S_+ &= S_1 - iS_2 \\ S_- &= S_1 + iS_2 \end{aligned}$$

Inverting these relations we get

$$\begin{aligned} S_3 &= iS_0 \\ S_2 &= \frac{1}{2i}(S_- - S_+) \\ S_1 &= \frac{1}{2}(S_- + S_+) \end{aligned}$$

Now the operators S_i, $i = 1, 2, 3$ are skew-hermitian when $S_+^\dagger = -S_-$ and $S_0^\dagger = S_0$

The operators S_+, S_-, S_3 defined in this way for both groups constitute a basis of $sl(2)$ satisfying the same commutation relations

$$[S_0, S_+] = S_+, \quad [S_0, S_-] = -S_-, \quad [S_+, S_-] = 2S_0. \tag{1.9}$$

The Casimir operator in terms of this basis is

$$Q = S_+S_- + S_0^2 - S_0 = S_-S_+ + S_0^2 + S_0. \tag{1.10}$$

In this way we can discuss the problem of representation in a uniform way.

Let $f_m^{(q)}$ be simultaneous eigenvectors of Q and S_0

$$\begin{aligned} Qf_m^{(q)} &= qf_m^{(q)} \\ S_0 f_m^{(q)} &= mf_m^{(q)}. \end{aligned}$$

As we are looking for global representations, $2m$ must be integer.

Applying the first of the relations (1.9) to $f_m^{(q)}$ one easily finds that either $S_+ f_m^{(q)} = 0$ or $S_0 S_+ f_m^{(q)} = (m+1)S_+ f_m^{(q)}$, i.e. $S_+ f_m^{(q)} \sim f_{m+1}^{(q)}$. Similarly, using the second identity (1.9), one finds that either $S_- f_m^{(q)} = 0$ or $S_- f_m^{(q)} \sim f_{m-1}^{(q)}$.

Thus, the matrix element $(S_+)_{km}$ may be non-zero only if $k = m+1$; the matrix element $(S_-)_{km}$ may be non-zero only if $k = m-1$.

Applying the Casimir operator Q (1.10) to any $f_m^{(q)}$ one obtains

$$S_+S_- f_m^{(q)} = [q - m(m-1)] f_m^{(q)}$$

S_+S_- is a diagonal operator and its elements satisfy

$$(S_+S_-)_{mm} = (S_+)_{mk}(S_-)_{km} = q - m(m-1). \tag{1.11}$$

Requiring that the generators S_i of the algebras $su(2)$ and $su(1,1)$ should be skew-hermitian, we can write (1.11) in the form

$$\pm \sum_k (S_+)_{mk}(S_+)^*_{mk} = q - m(m-1),$$

where the upper sign refers to $su(2)$ and the lower one to $su(1,1)$. However, $(S_+)_{mk} = 0$ for $k \neq m-1$. Therefore we may write

$$\pm|(S_+)_{m,m-1}|^2 = q - m(m-1)$$

or

$$\pm|(S_+)_{m+1,m}|^2 = q - m(m+1). \tag{1.12}$$

In the same way one obtains the equality

$$\pm|(S_-)_{m-1,m}|^2 = q - m(m-1). \tag{1.13}$$

The last two identities (1.12),(1.13) can now be used to discuss the representations of $su(2)$ and $su(1,1)$. Each of these groups will be now analyzed separatly.

1.7.4 Global unitary representations of $su(2)$

Because of the form (1.7) of the Casimir operator $C_{su(2)}$ its eigenvalues must be real positive numbers (S_i are skew-hermitian, so their spectrum is purely imaginary). From (1.12), we have

$$q - m(m-1) \geq 0$$

for any m allowed by the representation.

This means that the "ladder" of eigenvectors $f_m^{(q)}$ cannot extend to infinity. Thus, there must be a vector $f_u^{(q)}$ such that $S_+ f_u^{(q)}$=0. This is possible only if $q = u(u+1)$ (which can be proved by applying the Casimir operator (1.10) to $f_u^{(q)}$). Then, from (1.13) $S_- f_{-u}^{(q)} = 0$. Thus, the global unitary representations of $SU(2)$ always have a finite dimension $(2u+1)$, $2u$ being integer. This remains in accordance with the general theorem for compact Lie groups.

1.7.5 Global unitary representations of $su(1,1)$

The eigenvalue q of the Casimir operator Q may be an arbitrary real number here. For the generators of the representation to be skew-hermitian the following condition must be satisfied

$$q - m(m+1) \leq 0. \tag{1.14}$$

This is possible in three cases

1. $q \leq -1/4$

   The condition (1.14) is automatically fulfilled for any m such that $2m$ is integer.

2. $q \in (-1/4, 0)$

   The condition (1.14) is satisfied for integer m.

3. $q \geq 0$

   In this case the "ladder" $f_m^{(q)}$ must be cut at a proper m for the operators to have the required properties. Thus,

   $$q = k(k+1)$$

   for certain half-integer k, and for a given k there are two representations. One of them is spanned by $f_{k+1}^{(q)}, f_{k+2}^{(q)}, \ldots$. The other is spanned by $f_{-k-1}^{(q)}, f_{-k-2}^{(q)}, \ldots$.

In this way the Bargmann's classification [4] of the unitary representations of $SU(1,1)$ in the basis in which the compact generator S_3 is diagonal has been reproduced. It is also possible, although more complicated, to derive the representations in the basis in which one of the non-compact generators is diagonal [5].

1.7.6 Local and non-unitary representations

If it is not essential for a given purpose that the group representations be global and unitary, it is possible to derive a number of other representations. For any q it is possible to span the representation space on the vectors $f_{m+k}^{(q)}$, $m \in \mathcal{C}$, k – integer. For certain (real) q such a representation may generate a unitary (but not global) representation of $SU(1,1)$.

If a vector f_κ satisfies the equations $S_0 f_\kappa = \kappa f_\kappa$ and $S_- f^\kappa = 0$ then the vectors $f_{\kappa+m}$, $m \in N$, span a subspace that is invariant with respect to the algebra. In this way the representation (obviously irreducible) is defined, for which $q = \kappa(\kappa-1)$. This representation is denoted by $\uparrow_\kappa$.

In a similar way, if $S_- f_{-\kappa} = 0$, the representation $\downarrow^{-\kappa}$ may be defined, for which $q = \kappa(\kappa-1)$, as in the previous case.

Both these representations generate local unitary group representations if κ is real and $\kappa \geq 0$.

1.8 Coherent States

Coherent states (CS) are used in quantum mechanics and quantum field theory. They may be applied to solving quantum mechanical problems with time-dependent hamiltonians (see Section 2.7). They also allow establishing a relationship between eigenvalue equation with a certain spin (matrix) hamiltonian and the corresponding coordinate–representation hamiltonian. In this way new solvable quantum problems may be found (this is discussed in Chapter 3). In this Section several fundamental ideas will be introduced.

1.8.1 Introduction

The coherent states were first introduced for the harmonic oscillator by E. Schrödinger [6]. They describe non-spreading wave packets. The language of coherent states is very useful for describing harmonic oscillations that have relatively sharply determined phase which means that the uncertainity of the number of excitations is very large (the Heisenberg relation $\Delta\varphi\Delta n \geq 1$ holds). In the case of the harmonic oscillator, the coherent states are related with the Heisenberg – Weyl algebra describing the dynamical symmetry of this system.

The coherent states for an arbitrary system are defined as follows [7]. Consider a group G and its irreducible representation $T(g)$, $g \in G$ acting on a Hilbert space H. Chose one, fixed vector $|n_0\rangle \in H$. The set

$$\mathcal{G}(n_0) = \{|g\rangle \equiv T(g)\,|n_0\rangle : g \in G\} \tag{1.15}$$

is the set of (generalized) coherent states.

The set of CS is invariant under $T(g)$ which is irreducible, according to our assuption. Therefore, the linear envelope of $\mathcal{G}(n_0)$ must be equal to H. Hence, the CS form a complete set of states. They are, however, not orthogonal.

1.8.2 Coherent spin states

The coherent spin states (small CSS) were proposed by Radcliffe in the paper [8]. They are contained within the general scheme described above as the CS for the $SU(2)$ group.

Let us consider an irreducible representation $T^j(g)$ of the group $SU(2)$ on the space spanned by the spin states $|j,\mu\rangle$, $\mu = -j,\ldots,j$. Let S_1, S_2, S_3 be the generators of the corresponding Lie algebra, and define the spherical operators $S_\pm = S_1 \pm S_2$, $S_0 = S_3$.

Every operator of $T^j(g)$ can be expressed in terms of the Euler angles

$$T^j(g) = e^{-\phi S_3} e^{-\theta S_2} e^{-\psi S_3}. \tag{1.16}$$

If we chose the $|j,-j\rangle$ state as the $|n_0\rangle$ state in (1.15), the set of CSS will be

$$|\theta,\phi\rangle = e^{i\alpha(n)} e^{-\phi S_3} e^{-\theta S_2}\,|n_0\rangle \equiv D(\theta,\phi)\,|n_0\rangle\,, \tag{1.17}$$

where $\alpha(n)$ is an arbitrary phase. The difference between the formulae (1.16) and (1.17) is due to the fact, that the operator $e^{-\psi S_3}$ standing on the right and acting on $|j,-j\rangle$ changes only the phase of this vector, without changing the state[3]. The label n may be identified with the point on the unit sphere, given by the angles θ,ϕ.

It can be shown that the operator $D(n)$ can be written in the form

$$D(\zeta) = e^{\zeta S_+} e^{\beta S_0} e^{\gamma S_-}, \tag{1.18}$$

where

$$\zeta = -e^{-i\phi}\tan\frac{\theta}{2}, \quad \beta = \ln(1+|\zeta|^2), \quad \gamma = -\bar{\zeta}. \tag{1.19}$$

This corresponds, in a way, to the normal ordering of creation and aniquillation operators.

[3] In fact, the CS are characterized by a point of the quotient group H/K, where K is the stationary subgroup of $|n_0\rangle$, ie. the set of operations leaving the physical state unchanged – changing only the phase of the vector.

According to the definition (1.17), this gives easilly

$$|\zeta\rangle = (1+|\zeta|^2)^{-j} e^{\zeta S_+} |n_0\rangle \,.$$

Expanding the exponential and using the equation

$$|j,\mu\rangle = \sqrt{\frac{(j-\mu)!}{(2j)!(j+\mu)!}} (S_+)^{j+\mu} |j,-j\rangle$$

we get the decomposition of the coherent state $|\zeta\rangle$ in terms of the standard basis

$$|\zeta\rangle = (1+|\zeta|^2)^{-j} \sum_\mu \sqrt{\frac{(2j)!}{(j+\mu)!(j-\mu)!}} \zeta^{j+\mu} |j,\mu\rangle \,. \tag{1.20}$$

1.8.3 Symbols; differential realization of spin operators

The fact that the spin operators in the CS representation have the form of differential operators was used by some authors to build the correspondence between spin hamiltonians and Schrödinger–type Hamiltonians (see [9],[10]). To my knowledge, however, no formal construction of this correspondence, which would give it a strict meaning has been carried out. In the following, a possible way of passing formally from matrix to differential spin operators is described.

Since the CS form a complete set of states, it is possible to decompose any state in coherent states

$$|\psi\rangle = \int d\mu(\zeta) \psi(\bar{\zeta}) |\zeta\rangle \,,$$

where

$$\psi(\bar{\zeta}) = \langle\zeta|\psi\rangle \tag{1.21}$$

is called the symbol of the state $|\psi\rangle$ in the CS representation [7]. The measure $d\mu$ is a certain measure on the group whose explicit form is not important here.

A symbol is obviously a function of ζ. We will assume it is smooth enough (continuous second derivative is sufficient). The equation (1.21) defines a linear mapping of the Hilbert space of states $\mathcal{H}$ onto a subspace $\tilde{\mathcal{H}} \subset C_2(G)$ of the space of functions on the group G, spanned by the symbols of the states $|\psi\rangle \in H$. The properties of the scalar product guarantee linearity of this mapping, so it is a homomorphism. Moreover, this mapping is an isomorphism if no state has the symbol 0.

Consider the space $\mathcal{L}(\mathcal{H})$ of linear operators $S : \mathcal{H} \to \mathcal{H}$. We establish a homomorphism ϕ of this space into the space $\mathcal{L}(\tilde{\mathcal{H}})$ definig the action of the operators $\tilde{S} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ on the symbs in the following way (we denote $\phi(S) = \tilde{S}$)

$$[\tilde{S}\psi](\bar{\zeta}) = \langle\zeta\,|S|\,\psi\rangle \tag{1.22}$$

The latter shows that the value of the operator $\tilde{S}$ acting on a symbol is again a symbol. Some essential properties of the operators and mappings considered here are listed below.

1. The operators $\tilde{S}$ are linear. This is easily proved using (1.22) and the properties of scalar product.

2. The mapping ϕ is linear, which is also evident from the definition (1.22) and the properties of the scalar product.

3. In addition to its linearity, the mapping ϕ preserves the multiplicative structure of $\mathcal{L}(\mathcal{H})$, i.e. the products are mapped into products.

4. This means, in particular, that any commutator $[S_1, S_2]$ is mapped into the commutator $[\phi(S_1), \phi(S_2)]$.

If we consider a subspace $\mathcal{F}$ of $\mathcal{L}(\mathcal{H})$, as the image in the mapping ϕ we will get a subspace $\tilde{\mathcal{F}}$ of $\mathcal{L}(\mathcal{H})$. As a next step, let us assume that $\mathcal{F}$ is a Lie algebra of linear operators. Taking in mind the last point of the above listed properties, we conclude that the mapping ϕ is a homomorphism of Lie algebras. In other words, starting with a realization of a Lie algebra on the space of states $\mathcal{F}$, we generate a realization of this algebra on the subspace $\tilde{\mathcal{F}} \subset \tilde{H} \subset C_2$ containing smooth functions.

1.8.4 Generators of the $SU(2)$ algebra in the CS representation

The results of the previous Section show that to any operator acting on the Hilbert space $\mathcal{H}$ corresponds an operator acting on the space of symbols. Moreover, this correspondence is indeed a homomorphism preserving both the additive and multiplicative structure of algebras of operators.

Let us restrict ourselves to the coherent spin states. In this Section we are going to show that any realization of the $su(2)$ algebra in form of generalized Lie derivatives can be derived from the CSS representation.

First, let us determine the action of the $su(2)$ algebra generators in the CSS representation. The definition (1.22) together with the known properties of $su(2)$ yields directly the formulae for the action of the generators on symbols of basis vectors. The linearity of the generators allows extending the result onto the whole space.

For S_3 we have

$$\tilde{S}_0\Psi_{j,\mu}(\zeta) = \langle\zeta\,|S_0|\,j,\mu\rangle = \mu\,\langle\zeta|j,\mu\rangle = \mu\Psi_{j,\mu}(\bar{\zeta}),$$

where $\Psi_{j,\mu}(\zeta)$ is the state $|j,\mu\rangle$ in the CS representation.

Using the decomposition (1.20) and the definition (1.21) we can write the explicit formula for the symbol $\Psi_{j,\mu}$. In terms of the variables θ,ϕ (1.19) we obtain

$$\Psi_{j,\mu}(\theta,\phi) = \sqrt{\frac{(2j)!}{(j+\mu)!(j-\mu)!}}(-\sin\frac{\theta}{2})^{j+\mu}(\cos\frac{\theta}{2})^{j-\mu}e^{-i(j+\mu)\phi}. \tag{1.23}$$

Now, let us limit ourselves to a one–parameter curve on the sphere (θ,ϕ), parametrized by the variable x

$$\theta = \theta(x), \quad \phi = \phi(x).$$

This is a generalization of the idea of Zaslavsky and Ulyanov [10]. Using (1.23) we get after some algebra

$$\tilde{S}_0\Psi_{j,\mu}(x) = \left[\frac{1}{i\phi' - \frac{\theta'}{\sin\theta}}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{i\phi' - \theta'\cot\theta}{i\phi' - \frac{\theta'}{\sin\theta}}\right]\Psi_{j,\mu}(x).$$

It should be noted that the operator on the right depends neither on j nor on μ. This means that it has the same form for any linear combination of the basis vectors, i.e. for any vector of the space of symbols.

Choosing appropriately the functions $\phi(x)$, $\theta(x)$ it is possible to obtain the operator $\tilde{S}_0$ of the form $f(x)\frac{\mathrm{d}}{\mathrm{d}x} + g(x)$ for arbitrary functions f, g. In the same way one can obtain the operators $\tilde{S}_+$, $\tilde{S}_-$ (they may also be derived in an independent way, cf. the first sections of the Chapter 3). The commutation relations of the $sl(2)$ algebra hold because of the general properties of the mapping $S \mapsto \tilde{S}$. In this way, algebras of generalized Lie derivatives may be generated using the coherent states formalism.

The above procedure may be interpreted from the geometrical point of view. The coherent states correspond to the points of the sphere (θ,ϕ). From the definition (1.15) it is clear that a generator of the Lie algebra of the group G corresponds to an infinitesimal change from one coherent state to another or, equivalently, from one point on the sphere to another. If the starting state lies on the selected curve, the infinitesimal transformation has two componets: along the curve and transversal to the curve. The first of them corresponds to the $\mathrm{d}/\mathrm{d}x$ term, the second one to the other term in $\tilde{S}_0$. It is to be expected that similar thing happens always when one considers the representations of generators of a group in the representation of coherent states of the same group.

Chapter 2

Review of Results on Exactly Solvable Problems

In this chapter the most widely used methods of obtaining exact solutions for non–relativistic quantum mechanical problems are described. Some of them make no explicite use of group–theoretical and Lie–algebraic methods. Others have symmetry in a way "in the background". Finally, some methods are essentially based on the symmetry properties of the exactly solvable (ES) problem.

A characteristic feature is that most of the problems may be solved by many different methods. On one hand, this forces us to draw the conclusion that exact solvability is an inherent feature of a given problem and it does not depend merely of the cleverness of the method used. On the other hand, one is tempted to look for analogies between the methods, or even for a proof of their equivalence. A short discussion of the latter problem will be provided in this chapter.

2.1 Solutions via Ansatz

Consider a Schrödinger equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V(x)\right]\psi(x)=E\psi(x). \tag{2.1}$$

For some potentials $V(x)$ it is possible to guess the general form of the eigenfunction $\psi(x)$. Such a general solution depends on several parameters. As a result one obtains a set of recursive equations for the parameters, which normally leads to a kind of algebraic problem.

Such a method was used in [11] to solve the equation (2.1) with the generalized Morse potential

$$V(r)=A\left(1-e^{-\mu(r-r_\alpha)}\right)^2+B\left(1-e^{-\mu(r-r_\beta)}\right)^3+C\left(1-e^{-\mu(r-r_\gamma)}\right)^3.$$

In this case the condition of solvability of the obtained system was that the determinant of a certain matrix should be 0. This led to a restriction of the allowed values of the potential parameters and gave the corresponding eigenvalues of energy.

Let us quote here another, unpublished result, following the work [12].

The equation we are going to solve is

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{1}{4}B^2\sinh^2 x-B(S+\frac{1}{2})\cosh x\right]\psi=E\psi. \tag{2.2}$$

This equation describes a particle in the double–Morse potential and will be dicussed from another point of view in the Section 3.4.4.

One can look for solutions of this equation of the form

$$\psi(x)=\exp\left(-\frac{B}{2}\cosh x\right)\sum_{\infty}^{-\infty}a_m\exp(mx).$$

This ansatz leads to a difference equation for the coefficients a_m

$$(E+m^2)a_m + \frac{B}{2}\left[(S+1-m)a_{m-1} + (S+1+m)a_{m+1}\right] = 0.$$

There is no general theory for solving such three–term difference equations. However, in this case it is evident that in order for this difference equation to terminate at some finite $m_0 \equiv S$ we may assume $a_{-S-1} = a_{S+1} = 0$. Then $m = -S, -S+1, \ldots S$ and the difference equation may be then written in a matrix form (note that the energy E lies on the diagonal)

$$M\vec{a} = E\vec{a},$$

where M is a $(2S+1)\times(2S+1)$ matrix. For example, for $S = 1/2$ we have

$$M = \begin{bmatrix} \frac{1}{4} & \frac{B}{2} \\ \frac{B}{4} & \frac{1}{4} \end{bmatrix}.$$

Such a matrix equation can be solved analytically for lower values of S. For larger S the problem of diagonalization of the matrix M appears, since the roots of the characteristic polynomial cannot be found.

The relatation of this method to other methods should be stressed. Usually, the exact solvability has more profound reasons (e.g. some kind of symmetry). The ansatz used to solve a given problem can be, for example, a general linear combination of basis functions for the Lie algebra describing the problem (compare the last example with the development of the Section 3.4.4). Revealing the corresponding symmetry, one places the originally guessed ansatz within a wider scheme.

2.2 The Factorization Method

The factorization method, introduced by Schrödinger [6], is the standard method of solving the quantum mechanical problems. The systhematic review of this method is contained in the classical work by Infeld and Hull [13]. Important generalizatins are included in the works [14]–[16] Here, only a short résumé is given.

An equation of the general form

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y(x,m) + r(x,m)y(x,m) + \lambda y(x,m) = 0 \tag{2.3}$$

is said to be factorized if it is equivalent to the two first order equations

$$\begin{aligned} H^+_{m+1}H^-_{m+1}y(x,m) &= \left[\lambda - L(m+1)\right]y(\lambda,m) \\ H^-_m H^+_m y(x,m) &= \left[\lambda - L(m)\right]y(\lambda,m), \end{aligned}$$

where $L(m)$ is a certain function and

$$H^\pm_m = k(x,m) \pm \frac{\mathrm{d}}{\mathrm{d}x}.$$

The parameter m may take the values $m_0, m_0+1\ldots$.

Starting from any solution $y(x,m)$ of the equation (2.3), the operators H^-_{m+1} and H^+_m produce a solution corresponding to $m+1$ and $m-1$ respectively. Its worth stressing that the solutions belong to different problems (2.3) i.e. to different potentials $r(x,m)$.

There are theorems concerning normalizability of the functions $y(x,m)$ and others restricting the possible forms of the eigenvalue l. Finally, the form of $k(x,m)$ proves to be strongly restricted, resulting in a classification of all possible factorizations into 6 classes. Moreover, some of these classes are related with each other as well.

As a result of this procedure several problems admitting factorization, and therefore exactly solvable, are found. Among them there are "popular" ES quantum mechanical problems such as the generalized Pöschl–Teller potential

$$V(y) = \frac{A}{\cosh^2(y)} + \frac{B}{\sinh^2(y)}, \tag{2.4}$$

the Morse potential

$$V(y) = Ae^{-2y} + Be^{-y}, \tag{2.5}$$

the Rosen–Morse potential (called also Eckart Potential)

$$V(y) = A\frac{1}{\cosh^2(y)} + B\frac{\sinh(y)}{\cosh^2(y)}, \tag{2.6}$$

the oscillating rotator problem and others. The method is also useful for handling many types of special functions, e.g. the associated spherical harmonics, the hypergeometric and confluent hypergeometric functions or the Gegenbauer functions.

2.3 Approach via Special Functions

2.3.1 Description

Many sets of special functions have been defined and examined by mathematicians and physicists in relation to various problems of mathematical physics. These special functions are defined by differential equations which often have the form

$$\left[p_n(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q_n(x)\frac{\mathrm{d}}{\mathrm{d}x} + r_n(x)\right] f_n(x) = 0, \tag{2.7}$$

where $f_n(x)$ are special functions that can be written in explicit form, using e.g. their known recursive properties. If an appropriate transformation is performed on the equation (2.7), one may obtain another equation, satisfied by modified functions $\tilde{f}_n$.

The two usually considered transformations of the equation (2.7) are the change of variables

$$x = x(y) \tag{2.8}$$

and the "scaling" of functions

$$f(x) = \kappa(x)g(x), \tag{2.9a}$$

which may be formally viewed as a similarity transformations; the differential operator on the left side of (2.7) undergoes at the same time the transformation

$$\tilde{D} = \kappa(x)D\kappa^{-1}(x), \tag{2.9b}$$

where

$$D = p_n(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q_n(x)\frac{\mathrm{d}}{\mathrm{d}x} + r_n(x).$$

If the functions $x(y)$ and $\kappa(x)$ are appropriately chosen, one obtains the transformed differential operator of the form

$$\tilde{D} = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + V_n(y)$$

together with the set of transformed functions $\tilde{f}_n(y)$ satisfying this equation. If, in addition, a certain n-dependent constant may be extracted from the function $V_n(y)$ in such a way that the resting function does not depend on n any more

$$V_n(y) = V(y) - \epsilon_n,$$

we obtain a Schrödinger Hamiltonian whose eigenfunctions are $\tilde{f}_n(y)$ and whose eigenvalues are ϵ_n. The potential $V(y)$ may of course depend on multiple parameters, if they were present in the functions $p(x), q(x)$ and $r(x)$ in (2.7). However, it should not depend on n.

2.3.2 Results

Transformations of this kind were applied by Bhattacharjee and Sudarshan [17] to several equations. The hypergeometric equation yielded a class of generalized Pöschl–Teller potentials of the form (2.4). Transforming the confluent hypergeometric equation the authors were able to obtain the Schrödinger equations with the Morse potential (2.5) as well as with the potential

$$V(y) = \frac{l(l+1)}{y^2} + \frac{A}{y}, \tag{2.10}$$

which may be interpreted as the radial equation for the three-dimensional problem in the Coulomb field.

The third result of the same work was obtained by applying the appropriate transformation of the Bessel equation. The potential obtained in this case is singular and has the form

$$V(y) = \frac{1/4 - A^2}{y^2}.$$

In a later work [18], Ginocchio discusses a wide class of potentials which for a certain choice of parameters have the form of the Pöschl–Teller potential. The Schrödinger equation with these potentials is shown to reduce to the equation defining the Gegenbauer functions.

2.3.3 Relation to other methods

All the most interesting problems that may be solved using the special functions approach are also known to be exactly solvable by other methods. In fact, the method described in this Section is not very likely to allow obtaining new results. The reason is that the special functions themselves and the equations defining them usually may be derived by another method (this is why their solutions are known). The same method can be used to obtain in the parallel way the corresponding Schrödinger equations. For example, within the factorization method [13], both the Pöschl Teller problem and the hypergeometric functions belong to the class A factorization. Also the Morse problem and the confluent hypergeometric equation both belong to the same factorization – type B.

2.4 Harmonic Oscillator and Similar Problems

In the previous sections we presented several ways of generating and solving ES problems which made no explicit use of symmetry. Now, let us procede to symmetry–based methods. We are going to start with short introduction to the classical notion of dynamical symmetry as a continuation from classical physics.

2.4.1 The Idea of Dynamical Symmetry; Constants of Motion

It was understood very early that the physical reality underlying exact solvability of many systems are their dynamical symmetries. The notion of *dynamical symmetry* is used here in its original meaning – a transformation of canonical coordinates which leaves the system unchanged. Such a transformation is a symmetry transformation in the classical sense of this word. However, it cannot be assigned a purely geometrical meaning, since it involves both coordinate and momentum transformations. On the other hand it has all the properties of a "standard" symmetry, the most important of them being that its generator commutes with the Hamiltonian. Transformations of this kind are generated by operators corresponding to constants of motion. The important feature of such a symmetry is the analogy of the quantum mechanical problem with the classical one in canonical description.

The existence of an algebra of constants of motion may allow using group theoretical methods to diagonalize the quantum mechanical Hamiltonian. The best known example is the Pauli's algebraic solution for the hydrogen atom [19] (see. also [20]), extended later by Bander and Itzykson [21] (also to scattering states). In this problem the existence of the conserved Runge–Lentz vector is used.

A question arises in a natural way, concerning the possibility of transfering the known classical constants of motion to quantum mechanical problems. This problem was addressed by Jauch and

Hill [22]. They have shown that besides the hydrogen atom mentioned above, one may solve in a similar way the 2-dimensional Coulomb problem and the isotropic harmonic oscillator in an arbitrary dimension. However, for the anisotropic harmonic oscillator even in 2 dimensions there are classical constants of motion that have quantum mechanical counterparts and several others which cannot be carried over to quantum mechanics.

2.4.2 The harmonic oscillator

The clear and elegant meaning of dynamical symmetry as mentioned above was later extended to a little different properties of quantum mechanical systems. This notion began to be used in referrence to any problem which could be solved with the help of algebraic, symmetry–based methods. The classical example is the harmonic oscillator

$$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{k^2}{2}x^2 = \frac{1}{2}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + x\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + x\right) + \frac{1}{2} = a^\dagger a + \frac{1}{2}, \tag{2.11}$$

where

$$a = \left(\frac{\mathrm{d}}{\mathrm{d}x} + kx\right).$$

The operators $H, a, a^\dagger, I$ generate the harmonic oscillator algebra of the Section 1.6. It is easy to convince oneself (in the same way as for the spin algebra $su(2)$) that the operators $a, a^\dagger$ lower and raise, respectively, the eigenvalues of the hamiltonian H. The lowest level satisfies

$$a\psi_0 = 0$$

and its energy is

$$\epsilon_0 = 1/2.$$

Then, it is posible to obtain all the eigenstates of H as

$$\psi_n = (a^\dagger)^n \psi_0.$$

2.4.3 Problems related to $sl(2)$

The solution of the harmonic oscillator problem is possible due to the fact that the $a, a^\dagger$ operators raise and lower the eigenvalues of the Hamiltonian H. The same happens, however, in the $sl(2)$ algebra with the operators S_-, S_+ and S_0, respectively. One is therefore tempted to search for such realizations of this algebra that S_0 is a second order differential operator and S_-, S_+ are operators that raise and lower its eigenvalues. After appropriate change of variables (2.8) and/or transformation of functions (2.9) such an operator may yield a quantum mechanical hamiltonian. Its lowest state would then satisfy

$$S_-\psi_0 = 0$$

and the higher states could be obtained from it using the S_+ operator.

In fact, such realizations exist, leading to the exactly solvable Morse potential (2.5), radial Coulomb equation (2.10), and the radial equation for the 3-dimensional isotropic harmonic oscillator [23],[24]. A systematic study of such realization was performed by Brajamani and Singh [25]. Several additional potentials were found, mostly having the form of

$$V(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials in x.

2.5 Projections of Two–Dimensional Problems

2.5.1 Bound states

At a certain point in the search for exactly solvable 1-dimensional quantum mechanical systems it was realized that models having the property of exact solvability can appear while considering 2-dimensional differential realizations of Lie algebras. This gave rise to the whole theory, developped among others by Alhassid, Gürsey, Iachello and Wu [26]–[31]. The starting point for this method is a realization of the $su(2)$ algebra in terms of Lie derivatives or generalized Lie derivatives in two dimensions. This may be done in a natural way by one of the two methods [26]:

Schwinger realization of $su(2)$

If $a, a^\dagger$ and $b, b^\dagger$ are bosonic anihilation and creation operators, the following operators satisfy the commutation relations of the $sl(2)$ algebra (1.9)

$$\begin{aligned} J_0 &= \frac{1}{2}(a^\dagger a - b^\dagger b) \\ J_+ &= b^\dagger a \\ J_- &= a^\dagger b. \end{aligned}$$

Tis is called the Schwinger representation of the $sl(2)$ algebra. The Casimir operator is equal to one half of the number operator

$$C = \frac{1}{2}(a^\dagger a + b^\dagger b).$$

Choosing the bosonic operators in the form of differential operators

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right), \\ b &= \frac{1}{\sqrt{2}}\left(y + \frac{\partial}{\partial y}\right) \end{aligned}$$

and then going back to the generators J_1, J_2, J_3 of $su(2)$ we obtain a realization of $su(2)$ on the plane. The Casimir operator and the second generator in this realization in polar coordinates read

$$\begin{aligned} C &= \frac{1}{4}\left[r^2 - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} - 2\right], \\ J_2 &= -\frac{1}{2}\frac{\partial}{\partial \phi} \end{aligned}$$

This realization has an important feature: the generator J_y contains only a derivative with respect to one of the variables. Therefore it is easy to extract the subspace of eigenfunctions belonging to a given eigenvalue m_y of this generator. This subspace consists of all the functions of the form

$$\Psi_{j,m_y}(r,\phi) = R_{j,m_y}(r)e^{-2m_y\phi}.$$

Acting with the Casimir operator C on such a function and performing the change of variables

$$r^2 = (N+1)e^{-\rho}$$

we obtain the equation which is satisfied by $R_{j,m_y}(\rho)$

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \left(\frac{j/2+1}{2}\right)^2\left(e^{-2\rho} - 2e^{-\rho}\right)\right]R_{j,m_y}(\rho) = -m_y^2 R_{j,m_y}.$$

This is the Schrödinger equation for the Morse potential (cf. (2.5)).

The function R_{j,m_y} may be found using the theory of representations of $su(2)$ (see Section 1.7). In this case, however it is not straightforward and requires some nontrivial use of the Schwinger representation. The details can be found in [26]. Here, let us only mention one aspect of those results: the construction of the eigenfunctions requires that the numbers j, m_y should be integer or half-odd integer. This is a considerable restriction of the class of ES Morse potentials in this approach.

Realization on the sphere

The generators are the Lie derivatives corresponding to the standard representation of the $SO(3)$ group as a transformation group on unit sphere

$$\begin{aligned} J_0 &= -i\frac{\partial}{\partial\phi}, \\ J_\pm &= e^{\pm i\phi}\left[\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right] \\ C &= -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \end{aligned}$$

Let us diagonalize simultaneously the operators J_0, C. The resulting functions are, of course, spherical harmonics

$$Y_{j,m}(\theta,\phi) = P_{j,m}(\cos\theta)e^{im\phi},$$

where $P_{j,m}$ are the associated Legendre functions. The eigenvalue equation

$$CY_{j,m}(\theta,\phi) = j(j+1)Y_{j,m}(\theta,\phi)$$

upon the substitution

$$\cos\theta = \tanh\rho; \quad \theta \in (-\pi, 0) \tag{2.12}$$

yields the Schrödinger equation with the Pöschl–Teller potential

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{j(j+1)}{\cosh^2\rho}\right] u_{j,m}(\rho) = -m^2 u_{j,m}(\rho),$$

where $u_{j,m}(\rho) = P_{j,m}(\tanh\rho)$.

Note that due to the substitution (2.12) the whole range of the variable ρ corresponds to $\theta \in (-\pi, 0)$. The ϕ-dependent factor does not describe any physical reality since we are going to reduce the 2-dimensional problem to a 1-dimensional one. Therefore we may consider the above problem as one on a rectangle on the plane (θ, ϕ) rather than on a sphere. In other words, the one valuedness condition on the sphere may be omitted in this case which means that m may be any real number provided that the resulting function $u_{j,m}$ is normalizable. A set of such functions may be generated as a basis of the $\uparrow_{-j}$ representation of the $sl(2)$ algebra (Section 1.7) for any $j > 0$. In the construction one takes advantage of the fact that the explicit form of the generators allows analytical (and simple) solution of both the equations

$$J_0 Y_{j,-j} = -jY_{j,-j}$$

and

$$J_- Y_{j,-j} = 0.$$

The functions

$$Y_{j,k} \equiv (J_+)^{j+k} Y_{j,-j}$$

satisfy the condition mentioned above for $k < 0$. Thus the class of exact solutions is wider that it was apparent in [26].

2.5.2 Generalizations

New potentials from old ones

Using the two realizations of the $su(2)$ group introduced in the Section 2.5.1 and allowing for a wider class of transformation, it is possible to obtain other classes of ES problems [32]. The additional transformation which is used in addition to the change of variables is "scaling" of functions, i.e. the transformation of the type (2.9). The quantum mechanical problems that can be obtained from the Schwinger realization are: the equation of harmonic oscillator with an extra $1/x^2$ barrier, which can be

interpreted also as the radial equation for the 3-dimensional problem and the 1-dimensional Coulomb problem. The realization on the sphere may lead to the modified Pöschl–Teller potential

$$V(x) = \frac{m^2 - 1/4}{\sinh^2(x)}$$

which, upon the imaginary translation of variable $x = x' + i\pi$, is transformed into the Pöschl-Teller potential (2.4). Hence, this result is not essentially new.

Another exactly solvable system obtained in this way is the square well which appears to be a limiting case of the previous one.

General form of the Hamiltonian

It is possible to construct a general Hamiltonian which is diagonal in the basis of the representation space using the whole set of generators of the algebra that can be simultaneously diagonalized (which can be also referred to as using the Casimir operators of the chain of algebras $su(1) \subset su(2)$). The most general Hamiltonian has the form

$$H = J_x^2 + \kappa J_y^2.$$

On inserting a function satisfying

$$C\Psi(\theta, \phi) = j(j+1)\Psi(\theta, \phi)$$

to the equation

$$H\Psi(\theta, \phi) = \epsilon\Psi(\theta, \phi),$$

it reduces, in conical coordinates, to the Schrödinger equation [27]

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + j(j+1)\kappa^2 \mathrm{sn}^2(\theta, \kappa)\right)\Theta(\theta) = \epsilon\Theta(\theta),$$

where $\mathrm{sn}(\theta, \kappa)$ is the Jacobi sine amplitude function with modulus κ. This equation describes a particle in a periodic potential of an interesting form. It is also known as the Lamé equation.

More general form of the realization

In order to search for other ES systems one may try to introduce a more general realization of the $su(2)$ algebra. This was done by Sukumar [33]. The method consisted in fitting the most general form of the generators $J_\pm$ to the fixed $J_0 = -i\partial/\partial\phi$ in such a way that the commutation relations hold. Of course, the earlier results of Alhassid are reproduced as special cases within this general approach. Unfortunately, no more "nice–looking" ES potentials were derived in more general cases.

A similar method allowed Englefield and Quesne [34] to obtain some more potentials, including Eckart, Pöschl–Teller and generalized Pöschl–Teller potentials.

Special functions

As the approach involving projections from two dimensions leads to potentials related to special functions (see Section 2.3), it is to be expected that the systems of special functions themselves can be derived within this method. Indeed, some of them may be obtained in this way. For example, the work by Wu and Alhassid [31] contains some results concerning the hypergeometric equation.

2.5.3 Scattering states

This work is not meant to comprise ES scattering problems. Nevertheless, the approach to these problems using the methods described above seems worth at least mentioning shortly for its formal elegance. This concerns first of all the euclidean connection [29] (cf. also [30]). Moreover, the method of Alhassid *et al.* [26],[29] is one of very few methods that allow dealing with scattering problems.

Analytical continuation

In order to obtain continuous eigenvalues of energy and at the same time to use well defined mathematical objects, it is convenient to use the $su(1,1)$ algebra instead of the previously used $su(2)$. As it was pointed out in the Section 1.7, the $su(1,1)$ algebra contains non-compact generators which have continuous spectrum while the generators of $su(2)$ allow only discrete spectrum. It is possible to define the realization analogous to both Schwinger and spherical realization of $su(2)$, obtaining algebras of operators which yield solutions to the same problems as in the Section 2.5.1, but corresponding to scattering states. Once the wave functions are derived explicitely using the Lie algebra theory, one may examine their asymptotic behaviour for $\rho \to \pm\infty$ and determine the elements of the transfer and scattering matrices.

It is also possible [28] to use an algebra containing both $su(2)$ and $su(1,1)$ as subalgebras. This algebra – the $Sp(4,\mathcal{R})$ algebra – not only provides the unified approach to bound and scattering states but it also contains operators corresponding to transitions between these two groups of states.

Euclidean connection

The scattering matrix can by found also in a purely algebraic way, i.e. without writing the wave functions in the coordinate realization [29]. To this end one considers a fictional two-dimensional problem. The asymptotic behaviour of the solutions is free particle–like

$$|j,m\rangle^{\infty} = A_m e^{-ik\rho} e^{im\phi} + B_m e^{ik\rho} e^{im\phi},$$

since this is the way the real 1-dimensional solutions behave and the 2-dimensional functions are obtained by multiplying them by $e^{im\phi}$. On the other hand, the free waves form the basis for a representation of the euclidean group in two dimensions $E(2)$. The asymptotic forms of the generators of $su(1,1)$ may be expressed in terms of the generators of a certain realization of the Lie algebra of the group $E(2)$. Acting with these asymptotic forms on both sides of the last equation one obtains recurrence relations for A_m, B_m which allows one to write a recurrence formula for the reflection coefficient. In this way the problem is solved purely analytically.

2.6 Supersymmetry

The supersymmetric approach to exact solvability is one of the most recent achievements in this field. The idea of supersymmetry itself first appeared in the field theory [35]. Later it was used to generate exactly solvable quantum mechanical problems in one dimension [36], [37] and for problems in higher dimensions [38]. Very recently, a wide review article on supersymmetric quantum mechanics was published [39]. In this section we will provide a very brief outlook of this method.

Given two operators

$$\begin{aligned} Q_+ &= \left(-\frac{\mathrm{d}}{\mathrm{d}x} + \chi(x)\right), \\ Q_- &= \left(\frac{\mathrm{d}}{\mathrm{d}x} + \chi(x)\right), \end{aligned}$$

we can construct two Hamiltonians

$$H_0 = Q_+ Q_- = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \chi^2(x) - \chi', \tag{2.13a}$$

$$H_1 = Q_- Q_+ = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \chi^2(x) + \chi', \tag{2.13b}$$

where $\chi' = \mathrm{d}\chi/\mathrm{d}x$. The two relations hold

$$H_1 Q_- = Q_- H_0, \quad Q_+ H_1 = H_0 Q_+.$$

Using these identities it is easy to show that if Ψ_0 is an eigenvector of H_0 than either $Q_-\Psi_0$ is an eigenvector of H_1 belonging to the same eigenvalue or $Q_-\Psi_0 = 0$. In the same way, if Ψ_1 is an

eigenvector of H_1 than either $Q_+\Psi_1$ is an eigenvector of H_0 belonging to the same eigenvalue or $Q_+\Psi_1 = 0$. Thus the Hamiltonians (2.13a) and (2.13b) have the same spectrum except the lowest energy level of one of them, for which

$$Q_-\Psi = 0 \text{ or } Q_+\Psi = 0.$$

Such two Hamiltonians are called supersymmetric partners.

Repeating the procedure of factorizing the Hamiltonians into the operators Q_+ and Q_- we can create a hierarchy of Hamiltonians, each next one having a spectrum differing from the spectrum of the previous one by exactly one – the lowest – energy level. If the potentials entering these hamiltonians differ only in values of their parameters and additive constants the eigenvalues of each of them can be found (see [36]).

Note the direct relation of the supersymmetric approach with the factorization method. Indeed, any of the hierarchy of the ES hamiltonians may be expressed by the operators Q_+,Q_-, which yields instantly its factorization in the sense of the Section 2.2.

2.7 Time Dependent Hamiltonians – Application of Coherent States

All the previously described methods concern only stationary Schrödinger equations. They allow finding exact solutions for problems without explicit dependence on time. However, there are many important problems described by time-dependent Hamiltonians, where separation of the time variable cannot be carried out and the standard methods fail. The need for another approach, suitable in this case, arises. The method that fits this need is the coherent state approach. It is useful in the cases where the interaction part of the Hamiltonian can be expressed as a linear combination of algebra generators.

For example [7], the harmonic oscillator under time dependent external force is described by the hamiltonian

$$H = H_0 + H_{int},$$

where H_0 is the harmonic oscillator Hamiltonian (2.11) and

$$H_{int}(t) = -f(t)x = -f(t)(a + a^\dagger)$$

is the interaction Hamiltonian. In the interaction picture we are to solve the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi(t)\right\rangle = H_{int}(t)\left|\psi(t)\right\rangle.$$

As the interaction Hamiltonian is an element of the Heisenberg–Weyl algebra, the evolution operator is an element of the corresponding group. Thus, if we use the coherent states of harmonic oscillator, a coherent state will always evolve into another coherent state (cf. the definition (1.15)). We keep in mind that coherent states correspond to (and are labeled by) points on the group manifold, up to the phase factor. Therefore, if the initial state is a coherent state, the evolution may be viewed as a classical evolution of a point on the group manifold. Another equation describes the time evolution of the phase. In this way the problem is reduced to a classical problem.

More problems (e.g. an electron in time-dependent homogeneous magnetic field) are solved within the same scheme in [3]. The most interesting thing, however, is that using the coherent spin states one is able to reduce time-dependent problems with spin Hamiltonians to classical equations of motion exactly in the same way as it was presented above [7]. In this way, we can speak of a classical picture of spin systems.

Chapter 3

Quadratic Forms of Generators

In this chapter a new method of generating solvable problems, based on symmetry, is introduced. The quantum mechanical Hamiltonians are obtained with the aid of certain realizations of the $sl(2)$ algebra. In contrast to the previous methods, the operators of the algebra used here have the form of generalized Lie derivatives (see Section 1.4). Allowing such a form of generators permits us to perform the whole development without neccessity of using two–dimensional models as "parent" models for the one–dimensional ones. As it will become clear, such approach results in quasi–exactly solvable (QES) models.

In the Section 1.8.4 we showed that realizations of $sl(2)$ algebra which are used in this Section appear when the spin operators are written in the coherent states representation. Thus, the coordinate systems obtained in this Chapter may be considered as spin systems in a special representation. The spectra of both systems are identical; the $2S+1$ lowest states of the coordinate system correspond to the states of the spin system.

Nevertheless, it is convenient to consider the possible realizations of the algebra without referring to their spin origin. This seems reasonable since most of the considered Hamiltonians have no clear physical meaning, not being hermitian as spin Hamiltonians (see e.g. (3.7)). One of the exceptions is the Hamiltonian (3.27). For an appropriate choice of parameters it describes a uniaxial paramagnet in transverse magnetic field.

The chapter is organized as follows. First, the general form of the algebra of operators is derived. A systematic, general approach is given which in principle allows classifying all the 1-dimensional QES models obtained within this scheme. Next, the description of the method is given, followed by the analysis of several special cases – both new and earlier known ones.

3.1 Generators of the sl(2) Algebra

We are going to analyze the realizations of the $sl(2)$ algebra in terms of generalized Lie derivatives. This algebra has been chosen because it is simple and therefore it is well examined from the mathematical point of view. In this Section, it will be shown that the commutation relations determine the general form of the generators of the $sl(2)$ algebra in this case up to two arbitrary functions and two constants, one of which is the "total spin" S.

It is convenient to use the basis of the $sl(2)$ algebra formed by the ladder operators satisfying the relations (1.9). Let us denote

$$S_3 = v_3(x)\frac{\mathrm{d}}{\mathrm{d}x} + u_3(x), \tag{3.1a}$$

$$S_+ = v_+(x)\frac{\mathrm{d}}{\mathrm{d}x} + u_+(x), \tag{3.1b}$$

$$S_- = v_-(x)\frac{\mathrm{d}}{\mathrm{d}x} + u_-(x). \tag{3.1c}$$

Suppose the functions v_3 and u_3 are given. The relation

$$[S_3, S_+] = S_+$$

after insertion of (3.1a,b) and comparison of the corresponding terms can be rewritten in terms of the functions u, v

$$v_3 v_+' - v_+ v_3' = v_+, \tag{3.2a}$$

$$v_3 u_+' - v_+ u_3' = u_+. \tag{3.2b}$$

The first equation (3.2a) is a homogeneous ordinary differential equation and may be easily solved for v_+. The result may be written in the following form, convenient for further use

$$v_+(x) = v_3(x) e^{F_1(x)}$$

where

$$F_1(x) = \int \frac{\mathrm{d}x}{v_3(x)}$$

is any integral of $1/v_3(x)$,

$$\frac{\mathrm{d}}{\mathrm{d}x} F_1(x) = \frac{1}{v_3(x)}.$$

The non–zero multiplicative factor that appears, in general, in front of the solution is included in $F_1(x)$. The equation (3.2b) after insertion of v_3 is a first order ordinary differential equation, as well, and it may be easily solved for u_+. The solution is

$$u_+(x) = (u_3(x) - S_1)\, e^{F_1(x)},$$

depending on one more constant S_1.

In the same way the condition

$$[S_3, S_-] = -S_-$$

determines the functions v_-, u_-[1]. The solutions are

$$\begin{aligned} v_-(x) &= -v_3(x) e^{-F_2(x)} \\ u_-(x) &= -(u_3(x) + S_2)\, e^{-F_2(x)}, \end{aligned}$$

where F_2 is an integral of $1/v_3$ and S_2 is an arbitrary constant.

From the third of the relations (1.9) we easily get the conditions

$$\begin{aligned} S_1 &= S_2, \\ F_1(x) &= F_2(x). \end{aligned}$$

Hence, the functions defining the generators (3.1) are

$$\begin{aligned} & v_3(x), \\ & u_3(x), \\ v_+(x) &= v_3(x) e^{F(x)}, \\ u_+(x) &= (u_3(x) - S)\, e^{F(x)}, \\ v_-(x) &= -v_3(x) e^{-F(x)} \\ u_-(x) &= -(u_3(x) + S)\, e^{-F(x)}. \end{aligned} \tag{3.3}$$

The function $F(x)$ may be an arbitrary indefinite integral of $1/v_3(x)$. The sense of the constant S becomes clear when the Casimir operator for this realization is calculated.

$$C = S_- S_+ + S_3^2 + S_3 \equiv S(S+1) \cdot \mathrm{Id}. \tag{3.4}$$

We encounter an interesting feature of this realization of the $sl(2)$ algebra. The Casimir operator is equal to a number which means that there may be at most one invariant finite-dimensional subspace of the Hilbert space.

[1] The differential equations for these functions are the same as (3.2) if the change $v_3 \to -v_3$, $u_3 \to -u_3$ is made.

3.2 General Description of the Method

Consider a general operator constructed as a quadratic form of generators of an algebra of generalized Lie derivatives isomorphic to the sl(2) algebra.

$$H = \sum_{ij} a_{ij} S_i S_j + \sum_i b_i S_i. \tag{3.5}$$

Such an operator contains, in general, first and second derivative terms. It is a standard one-particle non-relativistic Hamiltonian if the following conditions are satisfied

- The coefficient at $\mathrm{d}^2/\mathrm{d}x^2$ should be constant;
- The first-derivative term should vanish;
- The potential obtained should have stationary states.

Assume that it is possible to construct such a Hamiltonian using a certain realization of the algebra. Then the $sl(2)$ algebra describes the dynamical symmetry of the system, i.e. any subspace of the Hilbert space $\mathcal{H}$ invariant under the algebra is obviously invariant under the action of the Hamiltonian, as well. The existence of a finite-dimensional invariant subspace allows to replace the original problem

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right]\psi = \epsilon\psi$$

by a matrix problem (i.e. to restrict the problem to a finite-dimensional subspace).

From the analysis performed in the Section 1.7, it is clear that such a subspace can exist only if $2S$ is a non-negative integer. A standard (cf. [20]) basis in which S_3 is diagonal is obtained by using the recursive definition

$$\begin{aligned} f_{-S} &- \text{ the eigenfunction of } S_3 \text{ belonging to the eigenvalue } -S, \\ f_{m+1} &= \left[(S-m)(S+m+1)\right]^{-1/2} S_+ f_m, \quad m = -S, -S+1, \ldots, S-1. \end{aligned}$$

If the dimension of this subspace is not too high, the problem can be solved analytically – the eigenvalues of energy and the corresponding eigenfunctions can be found.

For $S > 2$, $2S$ integer, the problem is reduced to a finite matrix problem in the same way as described above. However, the existence of analytical solutions cannot be guaranteed, as a general matrix of a higher dimension cannot be analytically diagonalized. On the other hand, as it will become clear in the next section, the matrix Hamiltonian we obtain is always a finite band–diagonal (in many cases even tri-diagonal) matrix. From the point of view of numerical computations, the solutions of such a problem arbitrarily close to the exact ones may be achieved without changing the dimension of the matrix, since the invariant character of the subspace is an analytic result. This means that no information is lost when passing from the infinite dimensional problem to the finite matrix one. This is not the case for a general eigenvalue problem in an infinite space.

3.3 General Second Order Hamiltonian

In order to perform a possibly systematical analysis of one-particle Hamiltonians that are possible to obtain as quadratic forms of the $su(2)$ generators, it is convenient to begin with some reduction of the full expression (3.5).

Using the commutation relations (1.9), we eliminate one term out of each pair $S_3S_+ - S_+S_3$ *etc.*, making the neccessary corrections to the coefficients of the linear terms (this was done also in [40], although it was not stated explicitely). Next, we note that due to the identity (3.4), using the Casimir operator in the form (1.10), we can eliminate the term S_+S_- introducing corrections to the coefficients at S_3^2 and S_3 and changing the Hamiltonian by a constant.

The above considerations lead to the following general form of the Hamiltonian

$$\begin{aligned} H &= a_3 S_3^2 + a_+ S_+^2 + a_- S_-^2 + a_{+3} S_+ S_3 + a_{-3} S_- S_3 \\ &\quad + b_3 S_3 + b_+ S_+ + b_- S_-. \end{aligned} \tag{3.6}$$

Any Hamiltonian that can be expressed in the form (3.5) can also be expressed in the above form, if only the generators are defined in the appropriate way.

In the forthcoming, we search for possible choices of parameters in the expression (3.6) which give good quantum mechanical one-dimensional Hamiltonians of the normal form

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x).$$

Normally, the potential $V(x)$ would belong to a class of potentials depending on several parameters. Out of such a class, only some potentials, corresponding to a special choice of the parameters are quasi exactly solvable. This may be referred to as a Riemann surface of QES models in the parameter space of the whole class of models [41]. It may happen (and it usually does) that within a given class of models, the same QES potentials may be obtained by different choice of parameters which means that the dimension of the Riemann surface is lower than the apparent number of parameters of the QES model. It is then possible to reduce the number of parameters fixing some of them at any convenient value. We will call such redundant parameters *irrelevant*.

For the expression obtained from (3.6) for a certain choice of the parameters and of the functions $v(x)$, $u(x)$ to be a Hamiltonian of the usuall form, the three essential conditions listed in the Section 3.2 must be satisfied.

The first two conditions are the base for further calculation. However, it does not seem possible to take the third one *a priori* into account.

3.4 Potentials Obtained as Quadratic Forms

This section contains several QES systems related to the $sl(2)$ algebra, generated by quadratic forms of the Lie algebra generators. For completeness, the results which are believed to be new will be completed with those obtained up to now.

3.4.1 The x^6 potential

Derivation.

In this subsection the potential first discussed by Singh [42] and by Flessas [43] and later – from a group–theoretical point of view – by Deenen in [40] is obtained within the general scheme of Section 3.2. The discussion here is more general than in [40], the cases of $S > 0$ being discussed as well. Choosing $a_3 = a_- = 0$, $a_+ = -a \neq 0$ we obtain a Hamiltonian of the type

$$H = -aS_+S_3 + \tilde{b}_3 S_3 + \tilde{b}_+ S_+ + \tilde{b}_- S_-. \tag{3.7}$$

The first condition now reads

$$-av_3v_+ = -D$$

which can be rewritten in the form

$$v_3^2 \exp\left[\int \frac{\mathrm{d}x}{v_3}\right] = \frac{D}{a}; \quad D \neq 0 \tag{3.8}$$

Taking the derivative of both sides of this equation we see that for any integral in the exponent, v_3 must satisfy the equation

$$v_3'(x) = -\frac{1}{2}$$

(prime denotes differentiation with respect to x). Hence,

$$v_3 = -\frac{1}{2}x + \alpha$$

and

$$\int \frac{\mathrm{d}x}{v_3} = -2\ln(-\frac{1}{2}x + \alpha) + \ln C.$$

For the equation (3.8) to be satisfied, we must put $C = D/a$, whereas α is an arbtrary constant. We can translate the coordinate x and in this way suppress the term α. For the exponent we therefore have the formula

$$w(x) = \exp\left[\int \frac{\mathrm{d}}{v_3(x)}\right] = \frac{D}{a}\left(-\frac{1}{2}x\right)^{-2}. \tag{3.9}$$

Now it is possible to write the formulae for v_+, v_-, according to (3.3)

$$v_+ = \frac{D}{a}\left(-\frac{1}{2}x\right)^{-1}; \quad v_- = \frac{a}{D}\left(\frac{1}{2}x\right)^{3}. \tag{3.10}$$

The generators (3.1) of the Lie algebra have the form

$$S_3 = -\frac{1}{2}x\frac{\mathrm{d}}{\mathrm{d}x} + u_3(x),$$

$$S_+ = -\frac{2D}{a}\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x} + (u_3(x) - S)\frac{4}{x^2},$$

$$S_- = \frac{a}{8D}x^3\frac{\mathrm{d}}{\mathrm{d}x} - (u_3(x) + S)\frac{x^2}{4}.$$

The Hamiltonian (3.7) after insertion of the generators in the form given by (3.1) reads

$$H = -D\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k(x)\frac{\mathrm{d}}{\mathrm{d}x} + \tilde{V}(x)$$

where

$$k(x) = \frac{4Du}{x} - \frac{(2S+1)D}{x} - \frac{\tilde{b}_3 x}{2} - \frac{2\tilde{b}_+ D}{ax} + \frac{\tilde{b}_- a}{8D}x^3,$$

$$\begin{aligned}\tilde{V}(x) \quad = \quad & \frac{2Du'(x)}{x} - \frac{4Du(x)(u(x)-S)}{x^2} + \tilde{b}_3 u(x) + \\ & \frac{4D\tilde{b}_+(u(x)-S)}{ax^2} - \frac{\tilde{b}_- a(u(x)+S)x^2}{4D}.\end{aligned}$$

The second requirement for the Hamiltonian, i.e. $k(x) = 0$ gives

$$u(x) = \frac{2S+1}{4} + \frac{\tilde{b}_+}{2a} + \frac{\tilde{b}_3}{8D}x^2 - \frac{\tilde{b}_- a}{32D^2}x^4.$$

The non–derivative term obtained after insertion of $u(x)$ and $u'(x)$ has the form

$$\begin{aligned}\tilde{V}(x) \quad = \quad & \frac{a^2\tilde{b}_-^2}{256D^2}x^6 - \frac{a\tilde{b}_3\tilde{b}_-}{32D^2}x^4 + \frac{\tilde{b}_3^2 - 2\tilde{b}_-\tilde{b}_+ - 6\tilde{b}_- S - 4a\tilde{b}_-}{16D}x^2 + \\ & \frac{D(a(2S-1) - 2\tilde{b}_+)(a(2S+1) - 2\tilde{b}_+)}{4a^2x^2} + \frac{\tilde{b}_3\tilde{b}_+ + a\tilde{b}_3 S + a\tilde{b}_3}{2a}.\end{aligned}$$

Note that it is possible without any loss of generality to rescale the parameters in the folowing way ($a \neq 0$, $D \neq 0$)

$$\tilde{b}_3 = -Db_3; \quad \tilde{b}_+ = -ab_+; \quad \tilde{b}_- = \frac{D^2 b_-}{a}.$$

The QES equation we obtain is

$$\left[-D\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \tilde{V}(x)\right]\Psi(x) = \varepsilon\Psi(x),$$

which, after dividing by D, has the form of the Schrödinger equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V(x)\right]\Psi(x)=\epsilon\Psi(x), \tag{3.11}$$

with

$$\begin{aligned} V(x) &= \frac{b_-^2}{256}x^6+\frac{b_3b_-}{32}x^4+\frac{b_3^2+2b_-b_+-6b_-S-4b_-}{16}x^2+ \\ &\left(b_++S+\frac{1}{2}\right)\left(b_++S-\frac{1}{2}\right)\frac{1}{x^2}+\frac{b_3b_+-b_3S-b_3}{2} \end{aligned} \tag{3.12}$$

and $\epsilon=\varepsilon/D$, where ε is the eigenvalue of the spin Hamiltonian.

This means that the coefficient a and the constant D are irrelevant, i.e. fixing them at any value we get the same family of potentials.

The substitution $b_+=-S\pm 1/2$ in (3.12) leads to a potential of the polynomial form

$$V(x)=\alpha x^6+\beta x^4+\gamma x^2+\delta, \tag{3.13}$$

where

$$\alpha=\frac{b_-^2}{256};\quad \beta=\frac{b_3b_-}{32};\quad \gamma=\frac{b_3^2-b_-(8S+4\mp 1)}{16};\quad \delta=b_3(-2S-1\pm\frac{1}{2}).$$

This potential for the appropriate choice of parameter may be a one–well, double–well or triple–well potential. Note that it is impossible to obtain a quartic potential $x^4+\gamma x^2$ from it since, according to the above formulae, $\beta=0$ whenever one tries to put $\alpha=0$.

Solutions; normalization

The algebraic solution of the problem provides us with an invariant subspace of the Hamiltonian with a determined basis and with solutions in form of vectors – solutions of the corresponding matrix problem which can be expressed in terms of the basis functions. In this way, one obtains the solution as a linear combination of basis functions.

The basis is constituted for a given S by the eigenfunctions of S_3

$$S_3^S f_{S,m}=mf_{S,m};\quad m=-S,-S+1,\ldots,S. \tag{3.14}$$

A direct calculation yields

$$f_{S,m}=C_m x^{b_++S-2m+1}\exp\left[-\frac{b_-}{64}x^4+\frac{b_3}{8}x^2\right] \tag{3.15}$$

where C_m is a normalization constant.

Note that the exponent depends neither upon S nor upon m. The condition of normalizability requires that $b_->0$, whereas the requirement that $f_{S,m}(0)$ should be finite means that $b_+\geq S-1$.

3.4.2 Radial equation for a central potential

Let us remark that this result gives one more possibility which was unnoticed by Deenen. Consider a 3–dimensional Schrödinger equation with a central potential $U(r)$. The equation factorizes in the usual way due to the $SO(3)$ symmetry. We have

$$\Delta=\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}r-\frac{\boldsymbol{L}^2}{r^2}$$

and in the subspace belonging to the eigenvalue $l(l+1)$ of $\boldsymbol{L}^2$ the Schrödinger equation takes the form

$$\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}r\Phi_{l,m}(r,\theta,\phi)+\frac{l(l+1)}{r^2}\Phi_{l,m}(r,\theta,\phi)+V(r)\Phi_{l,m}(r,\theta,\phi)=\epsilon\Phi_{l,m}(r,\theta,\phi).$$

The operators do not depend on θ, ϕ. After the substitution

$$r\Phi(r,\theta,\phi) = \Psi(r)$$

we obtain

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2} + V(r)\right]\Psi(r) = \epsilon\Psi(r). \tag{3.16}$$

It is clear that the one-dimensional QES problem with the potential (3.12) is equivalent to the radial part of a three-dimensional problem with a central potential of the type (3.13). The orbital quantum number is

$$l = b_+ + S + \frac{1}{2}.$$

Hence, the results of Deenen may also be used to solve the radial Schrödinger equation for a spherically symmetrical problem with a potential of the type (3.13) for certain values of the parameters.

3.4.3 Generalized Morse potential; QES systems of the second type

Recently Znojil [11] proposed a generalization of the Morse potential of the form

$$V(r) = A\left(1 - e^{-\mu(r-r_\alpha)}\right)^2 + B\left(1 - e^{-\mu(r-r_\beta)}\right)^3 + C\left(1 - e^{-\mu(r-r_\gamma)}\right)^4 \tag{3.17}$$

which may be useful e.g. for description of molecular spectra. The Schrödinger equation with this potential

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V(r)\right]\psi(r) = E\psi(r)$$

proves to be quasi-exactly solvable. The quasi-exact solvability of this problem can be described within the scheme of this chapter. However, as we shall show, in order to draw the connection between Lie algebras and the Schrödinger problem in this case, one needs to introduce the notion of a QES problem of the II type according to the classification introduced by Turbiner [41]. It will become clear that the Znojil's problem is related to the QES potential (3.12), inheriting its exact solvability.

The following development is based on the work [44].

Let us introduce new notation and write the Schrödinger equation with the potential (3.12) in the form

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{L(L+1)}{x^2} + g_6x^6 + g_4x^4 + g_2x^2\right]\Phi(x) = \epsilon\Phi(x) \tag{3.18}$$

with

$$g_6 = \frac{b_-^2}{256}, \tag{3.19a}$$

$$g_4 = -\frac{b_3 b_-}{32}, \tag{3.19b}$$

$$g_2 = \frac{b_3^2 - 2b_-b_+ - 6b_-S - 4b_-}{16}, \tag{3.19c}$$

$$L = b_+ - S - \frac{1}{2}. \tag{3.19d}$$

After the transformation

$$\Phi(x) = x^{1/2}\psi(x) \tag{3.20}$$

followed by the change of variables

$$x = e^{-r}, \tag{3.21}$$

we obtain the equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \epsilon e^{-2r} + g_2e^{-4r} + g_4e^{-6r} + g_6e^{-8r}\right]\psi(r) = -(L+\frac{1}{2})^2\psi(r). \tag{3.22}$$

We would like to remark that this potential appears in the work [41]. However, neither discussion nor derivation is provided there.

The solutions of the equation (3.18) can be found only for some special choices of the coefficients. To be specific, the QES potentials can be viewed as hyper-surfaces parametrized by the three spin Hamiltonian parameters b_3, b_+, b_- in the four-dimensional coefficient space corresponding to a general problem of the type (3.18). The algebraic approach provides us with a series of $2S+1$ solutions for a determined value of S. These solutions form a multiplet of solutions for a given problem, i.e. they are different energy eigenvalues and eigenfunctions found for the same single set of potential coefficients.

Transforming the known exact solutions of (3.18) according to (3.20),(3.21) one obtains, for a determined S, a set of $2S+1$ exact solutions, corresponding to a definite set of coefficients g_2, g_4, g_6, L and differing in the value of ϵ corresponding to each of them. Looking at the equation (3.22) one realizes that this means that the known solutions belong now to $2S+1$ different generalized Morse problems. This is so because the original energy ϵ plays the role of a parameter in (3.22) whereas we interprete one of the former parameters as the energy of the transformed system. This is a characteristic feature of quasi exactly solvable problems of the II kind.

Some comments may be made on the normalization of the solutions. Transforming any linear combination of the functions (3.15) (i.e. any exact solution of the problem (3.18)) using (3.20),(3.21) we obtain the following general form of a solution of (3.22)

$$\begin{aligned}\psi(r) &= \sum_m C_m e^{-(b_+ + S - 2m + 1/2)r} \exp\left[-\frac{b_-}{64}e^{-4r} + \frac{b_3}{8}e^{-2r}\right], \\ m &= -S, -S+1, \ldots, S.\end{aligned}$$

This function is analytical on $(-\infty, \infty)$. The sufficient condition of its normalizability is

$$b_+ > S - \frac{1}{2}.$$

Note that b_- may now be arbitrary. However, $b_- = 0$ leads to the trivial case of Morse potential, related to the harmonic oscillator.

It may also be noted that the form of the mapping (3.21) causes that the n-th eigenfunction of the original problem (3.18) is not neccessarily mapped into the n-th eigenfunction of the system (3.22). The reason is that the positive half-axis is mapped onto the whole real axis and some nodes of the wave function may disappear.

In order to find the solutions of the equation (3.22), one solves the equation (3.18) and then uses the mapping (3.21),(3.20) which transforms the eigenfunctions of the ϕ^6 problem to the ones of the generalized Morse problem. Single solutions for different potentials may be found in this way. Finding multiplets of solutions, however, requires more effort.

We want to search for multiplets of exact solutions of the problem (3.22), i.e for doublets or triplets of solutions of (3.18) corresponding to the same values of g_6, g_4, g_2 and ϵ but to different values of L. Comparing the roles played by the coefficients in (3.18) and (3.22) we see that such solutions, are then mapped into multiplets of states of one generalized Morse problem (3.22). It is convenient to formulate this problem in terms of the spin Hamiltonian parameters.

First, note that the potential (3.12) is shifted by a constant with respect to (3.18). Thus, if λ is the eigenvalue of the matrix Hamiltonian (3.7) and therefore of the Hamiltonian with the potential (3.12), the corresponding eigenvalue of the equation (3.18) is

$$\epsilon = \lambda - \frac{b_3 b_+ + b_3 S + b_3}{2}. \tag{3.23}$$

Assume we can solve the equation (3.18) for a certain set of parameters g_i obtaining the same energy ϵ with two different values of L. This, of course, is possible only if the two solutions correspond to different values of S, say $S^{(1)}$ and $S^{(2)}$. Denote the corresponding coefficients in (3.7) by $b^{(1)}_{\pm,3}$ and $b^{(2)}_{\pm,3}$, respectively. As the parameters g_i, $i = 2, 4, 6$ and ϵ must be the same in both cases, we have from (3.19a,b,c) and (3.23)

$$b^{(1)}_- = b^{(2)}_-, \tag{3.24a}$$

$$b_3^{(1)} = b_3^{(2)}, \tag{3.24b}$$

$$b_+^{(1)} + 3S^{(1)} = b_+^{(2)} + 3S^{(2)}, \tag{3.24c}$$

$$\lambda^{(1)} + b_3 S^{(1)} = \lambda^{(2)} + b_3 S^{(2)}, \tag{3.24d}$$

(we omit the unneccessary upper index in b_3 and b_-).

For example, for $S^{(1)} = 0$, $S^{(2)} = 1/2$ we have at once $\lambda^{(1)} = 0$ and, after diagonalizing the corresponding 2×2 matrix Hamiltonian obtained by insertion of the corresponding matrix operatiors into (3.7),

$$\lambda^{(2)} = \pm\sqrt{\frac{b_3^2}{4} + b_-(b_+^{(2)} + \frac{1}{2})}.$$

The last one of the above conditions reads

$$b_-(b_+^{(2)} + \frac{1}{2}) = 0$$

which has no interesting and pysically correct solutions.

For $S^{(1)} = 0$, $S^{(2)} = 1$ we have

$$\lambda^{(2)} + b_3 = \lambda^{(1)} = 0, \tag{3.25}$$

and $\lambda^{(2)}$ is the solution of the characteristic equation for the corresponding 3–dimensional matrix Hamiltonian

$$\lambda(b_3 + \lambda)(b_3 - \lambda) - 2b_-(b_3 - \lambda) + 4b_+b_-\lambda = 0, \tag{3.26}$$

where $\lambda \equiv \lambda^{(2)}$. The only physically correct condition obtained after insertion of (3.25) into (3.26) is

$$b_3 = 0.$$

This defines a restriction of the 3-dimensional manifold of quasi exactly solvable potentials (3.22) to a 2-dimensional manifold of potentials for which two exact solutions can be found. In fact, this is only one of the multiple sheets of this manifold. More of them can be found by considering other values of $S^{(1)}$, $S^{(2)}$.

1-dimensional hyper-surfaces of potentials for which triplets of exact solutions exist can be obtained in the same way. Generalizing the equations (3.24c,d) we may write

$$\begin{aligned} b_+^{(n)} &= b_+ - 3S^{(n)} \\ \lambda^n &= \lambda - b_3 S^{(n)}, \qquad n = 1, 2, 3, \end{aligned}$$

where λ and b_+ are certain numbers. The parameters b_3 and b_- are the same in all three cases. In this way, for any definite set $S^{(1)}, S^{(2)}, S^{(3)}$, we deal with four variables λ, b_3, b_+, b_-, which are bound by three characteristic equations

$$\det\left[H_S^{(n)} - \lambda^{(n)}\right] = 0, \quad n = 1, 2, 3,$$

together with the additional condition required for the normalizability of the obtained results. The calculation is complicated, although it seems a bit clarified due to the introduction of the spin Hamiltonian parameters and to the transparent formulation of the problem achieved with their use.

Some analytical, as well as numerical results may be found in [11]

3.4.4 The double–Morse potential

On choosing $a_- = a = 0$, $a_3 = -a^2 \neq 0$ in (3.6) the Hamiltonian takes the form

$$H = -a^2 S_3^2 + \tilde{b}_3 S_3 + \tilde{b}_+ S_+ + \tilde{b}_- S_-. \tag{3.27}$$

In this case a simple condition for a constant coefficient at $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ is found

$$-a^2 v_3^2(x) = -D^2,$$

hence

$$v_3 = \frac{D}{a}.$$

It is straightforward to obtain from the above the following expressions using (3.3)

$$v_+(x) = \frac{D}{a} e^{ax/D+\alpha}; \quad v_-(x) = -\frac{D}{a} e^{-ax/D-\alpha}. \tag{3.28}$$

We translate the origin to suppress the term α in the exponent. Then, the generators, according to (3.1), are

$$S_3 = \frac{D}{a}\frac{\mathrm{d}}{\mathrm{d}x} + u_3(x) \tag{3.29a}$$

$$S_+ = \frac{D}{a} e^{ax/D}\frac{\mathrm{d}}{\mathrm{d}x} + [u_3(x) - S]\, e^{ax/D} \tag{3.29b}$$

$$S_- = -\frac{D}{a} e^{-ax/D}\frac{\mathrm{d}}{\mathrm{d}x} - [u_3(x) + S]\, e^{-ax/D}. \tag{3.29c}$$

Introducing the new variable

$$\zeta = \frac{ax}{D} \tag{3.30}$$

we rewrite (3.29) in the following form

$$S_3 = \frac{\mathrm{d}}{\mathrm{d}\zeta} + u_3(\zeta) \tag{3.31a}$$

$$S_+ = e^{\zeta}\frac{\mathrm{d}}{\mathrm{d}\zeta} + [u_3(\zeta) - S]\, e^{\zeta} \tag{3.31b}$$

$$S_- = -e^{-\zeta}\frac{\mathrm{d}}{\mathrm{d}\zeta} - [u_3(\zeta) + S]\, e^{-\zeta}.$$

Inserting the operators (3.31) into the Hamiltonian (3.27) we obtain the coefficient at the first derivative

$$k(\zeta) = -2a^2 u_3(\zeta) + \tilde{b}_+ e^{\zeta} - \tilde{b}_- e^{-\zeta} + \tilde{b}_3$$

and the condition $k(\zeta) = 0$ gives

$$u_3(\zeta) = \frac{1}{2a^2}\left(\tilde{b}_+ e^{\zeta} - \tilde{b}_- e^{-\zeta} + \tilde{b}_3\right). \tag{3.32}$$

The non–derivative term is

$$\begin{aligned}\tilde{V}(\zeta) &= -a^2 u_3'(\zeta) - a^2 u_3^2(\zeta) + \tilde{b}_+ u_3(\zeta) e^{\zeta} - \\ &\quad \tilde{b}_+ S e^{\zeta} - \tilde{b}_- u_3(\zeta) e^{-\zeta} - \tilde{b}_- S e^{-\zeta} + \tilde{b}_3 u(\zeta).\end{aligned}$$

After insertion of (3.32) and some simple algebra this reads

$$\tilde{V}(\zeta) = \frac{1}{4a^2}\left(\tilde{b}_+ e^{\zeta} - \tilde{b}_- e^{-\zeta} + \tilde{b}_3\right)^2 - \left(S + \frac{1}{2}\right)\left(\tilde{b}_+ e^{\zeta} - \tilde{b}_- e^{-\zeta}\right). \tag{3.33}$$

The identities

$$e^{\zeta} = \sinh\zeta + \cosh\zeta; \quad e^{-\zeta} = \sinh\zeta - \cosh\zeta$$

allow us to rewrite (3.33) in the form

$$\begin{aligned}\tilde{V}(\zeta) &= \frac{1}{4a^2}\left[(\tilde{b}_+ + \tilde{b}_-)\sinh\zeta + (\tilde{b}_+ - \tilde{b}_-)\cosh\zeta + \tilde{b}_3\right]^2 \\ &\quad - \left(S + \frac{1}{2}\right)\left[(\tilde{b}_+ - \tilde{b}_-)\sinh\zeta + (\tilde{b}_+ + \tilde{b}_-)\cosh\zeta\right].\end{aligned}$$

The Schrödinger–type equation we have received has the form

$$\left[-a^2\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}+\tilde{V}(\zeta)\right]\Psi(\zeta)=\varepsilon\Psi(\zeta),$$

Performing the transformation (3.30) back to the original variable x and dividing this equation by the factor D^2 appearing at the second–derivative term one gets

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V(x)\right]\Psi(x)=\epsilon\Psi(x), \tag{3.34}$$

with $\epsilon=\varepsilon/D^2$ and

$$\begin{aligned} V(x) &= \frac{a^2}{D^2}\left[(b_+ + b_-)\sinh(\frac{a}{D}x)+(b_+ - b_-)\cosh(\frac{a}{D}x)+b_3\right]^2 \\ &- \frac{a^2}{D^2}\left(S+\frac{1}{2}\right)\left[(b_+ - b_-)\sinh(\frac{a}{D}x)+(b_+ + b_-)\cosh(\frac{a}{D}x)\right], \end{aligned} \tag{3.35}$$

where the parameters b_3, b_+, b_- are obtained from the old ones by the rescaling

$$\tilde{b}_{\pm,3}=a^2 b_{\pm,3}.$$

This shows again that all the solvable potentials related to this kind of spin Hamiltonians, up to a linear change of variable, are obtained by fixing a and D at any value and appropriately chosing the three parameters $b_{\pm,3}$.

Choosing $b_+=b_-=B$, $b_3=0$ one gets a symmetrical potential of the form

$$V(x)=\alpha^2B^2\sinh^2(\alpha x)-\alpha^2\left(S+\frac{1}{2}\right)B\cosh(\alpha x).$$

The potential (3.35) was proposed in different forms in [9] and [45]. In the former one, solutions were given depending on certain coefficients defined by a matrix equation equivalent to the eigenvalue problem of the spin Hamiltonian (3.27). The relation between the double–Morse problem and the underlying symmetry was analyzed in [46].

It is possible to make the substitution $x=i\zeta$ and to obtain from (3.35) a periodical potential. The solutions found by the algebraic method correspond then to the center and to the edge of the Brillouin zone, for S integer and half–odd integer, respectively.

The exactly solvable limiting case – the Morse potential

Let us put $b_-=B\neq 0$, $b_3=A$, $b_+=0$. For simplicity, let $D=a=1$ (this does not matter, as we have shown). The potential (3.35) is

$$V(x)=B^2e^{2x}-B\left[2A+S+\frac{1}{2}\right]e^x. \tag{3.36}$$

This is equivalent to the well known Morse potential[2]. Its most attractive feature is that in the algebraic picture the corresponding matrix Hamiltonian is a lower–diagonal matrix for any S. This allows finding the eigenvalues at once, and the eigenvectors after very little algebra, no matter how large S is. Moreover, as the coefficients A and B are arbitrary, the potentials (3.36) cover the whole family of Morse potentials (up to a constant shift of energy), which was not the case for the double–Morse potential. This means that the Morse potential is exactly solvable within the frame of the metod discussed here. For example, if we are interested in 101 lowest eigenvalues of energy for a certain Morse potential (provided it has so many levels), we put $S=50$, adjust A, B to get the proper values of the coefficients and just read the energies off the diagonal of the matrix Hamiltonian.

The exact solvability of the Morse problem has been known and discussed for long and nearly a dozen of different approaches have been applied, cf. e.g. [28],[47],[32],[25],[34],[23],[33],[48]. Now we can state there is one more way of treating this problem. We must admit, however, that this approach apparently does not provide the possibility of defining in a simple manner the shift operators relating directly the eigenfunctions at a given energy for Morse potentials of different strengths, as it is possible in other approaches (cf. [24] and the references therein).

[2]If we put $b_-=0$, $b_+\neq 0$ we get a mirror reflection of this potential

3.5 The Pöschl–Teller Potential

Choosing the class of spin Hamiltonians of the form

$$H = a\left[S_3^2 + a_+S_+S_3 + b_+S_+ + b_-S_-\right],$$

and repeating the procedure described in the previous cases one obtains the following formulae for the functions v_3, u_3

$$\begin{aligned} v_3 &= \frac{D}{\sqrt{a}}\tan\left(-\frac{\sqrt{a}}{2D}x + C\right) \\ u_3 &= \left(\frac{1}{4} + \frac{S}{2} + \frac{b_3}{2} - \frac{b_+}{2a_+}\right)\frac{1}{\cos^2\left(-\frac{\sqrt{a}}{2D}x + C\right)}. \end{aligned}$$

This leads to the Schrödinger equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right]\psi(x) = \frac{a}{D^2}\epsilon_0\psi(x),$$

where ϵ_1 is the eigenvalue of the spin problem with $a = 1$. The potential $V(x)$ has the form

$$V(x) = \frac{a}{D^2}(A - B)\frac{1}{\sin^2\left(-\frac{\sqrt{a}}{D}x + C\right)} + \frac{a}{D^2}B\frac{\cos\left(-\frac{\sqrt{a}}{D}x + C\right)}{\sin^2\left(-\frac{\sqrt{a}}{D}x + C\right)} - \frac{b_3^2}{8}, \tag{3.37}$$

where

$$B = -\frac{1}{4}\left(b_3 + 1 - 2b\right)\left(b_3 + 1 + 2S\right),$$

$$A = \frac{1}{8}\left(2S + 2b + 1\right)\left(2S + 2b - 1\right),$$

$$b = \frac{b_+}{a_+}.$$

Note, that none of the parameters a, D is essentially relevant. Their combination a/D^2 is in fact only a scaling factor.

The obtained potential is a periodic singular potential for real a/D^2. It may be transformed into an aperiodic potential by choosing $a = \alpha^2 i$, $D = 1$. The resulting potential is

$$V(x) = (A - B)\alpha\frac{1}{\sinh^2(\alpha x + \gamma)} + B\alpha\frac{\cosh(\alpha x + \gamma)}{\sin^2(\alpha x + \gamma)}.$$

It is also singular. It is impossible to use an imaginary shift to obtain the Eckart potential (2.6) from it because of the imaginary factor appearing at the second term which, in general case, cannot be got rid of by any choice of parameters. Nevertheless, one may obtain the whole class of Pöschl–Teller potentials, putting

$$b_3 = 2b - 1 \;\text{ or }\; b_3 = -2S - 1$$

and

$$\gamma = i\frac{\pi}{2}.$$

3.6 Other Spin Hamiltonians

Apart from the models discussed above, a few others have been examined but have not led to any interesting results.

The spin Hamiltonian

$$S_3^2 + aS_+^2 + b_+S_+ + b_-S_- + b_3S_3$$

results in a very complicated potential consisting of many independent terms involving hyperbolic functions. No simpler special cases have been found.

The Hamiltonian of the form

$$S_+^2 + b_+ S_+ + b_- S_- + b_3 S_3$$

yields a x^4 type potential. However, the basis functions of the $sl(2)$ representation space are not normalizable and the results obtained within the algebraic scheme have no physical meaning.

3.7 Conclusion

The methods involving the quadratic forms of the $sl(2)$ Lie algebra generators allows obtaining several interesting QES quantum mechanical models. It yields, among others, potentials that were not known before and are probably not exactly solvable (double Morse potential). The cases analyzed in this Chapter allow to draw the general conclusion that only some spin Hamiltonians lead to interesting problems and to their physically correct solutions. Moreover, simple spin models are more likely to generate an interesting QES coordinate system.

Not all the possible cases have been examined. Some of those that have not are related to the examined ones by the correspondence between S_+ and S_- with the substitution $v_3 \to -v_3$. This means that systems obtained from a given spin Hamiltonian are the same as those obtained from the Hamiltonian transformed by the replacement $S_+ \to S_-$. Other combinations of coefficients are still to be examined. However, the calculation becomes more complicated and less transparent as the number of non-linear terms in the spin Hamiltonian grows.

An interesting point of this Chapter that should be stressed is the existence of exactly solvable problems within the method of quadratic forms. Such ES problems may be viewed as limiting cases of QES problems.

Summary

There are numerous methods of solving the stationary Schrödinger equation. Some of them have been discussed in this work. Several general remarks arise from this discussion.

The first interesting thing is that most of the methods are based on symmetry or closely related to it. The symmetry–based methods are very numerous and differ essentially from one another. Nevertheless there are some features common for all these methods. One of them is that the algebra used in most cases is $su(2)$. Admittedly, this may be caused by the place it occupies in physics and by the amount of work that has been therefore devoted to it. On the other hand, the structure of this group is favourable of using it to generate exactly solvable Hamiltonians. The raising and lowering operators often allow obtaining the solutions in a very natural way. The interesting task for the closest future will be to check if there are other groups of known structure that could be used to generalize some of the approaches of the Chapters 2 and 3. The group that coud be successfully used is $SU(3)$ which, in addition, would contain the models related to $SU(2)$ as special cases.

One is tempted to look for general statements concerning the application of symmetry in quantum mechanics. Since the time when the notion of "dynamical symmetry" lost its original meaning related to transformations of the phase space the physical meaning of Lie algebras from the point of view of exact solutions has been rather vague. It would be interesting to search for a physical explanation of the formal methods used in this work. This seems, however, to be a complicated problem, perhaps involving the most fundamental questions of physics.

And, last but not least, it may be expected that the exact or quasi–exact solvability is reflected in the alternative formulation of quantum mechanics – the path integral one. Several simple one-dimensional problems have already been solved, and so has the hydrogen atom. However, no direct analogy between the methods used in the Schrödinger formulation and in the path integral approach has been revealed. With this respect, the hydrogen atom with its physically clear $O(4)$ symmetry could be the first one to be examined.

As any work in physics, this one opened more questions than it answered. Nevertheless, I hope it contributes to some extent to the understanding of the problems related to exactly solvable problems in quantum mechanics.

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