MATH 262/CME 372: Applied Fourier Analysis and Elements of Modern Signal Processing Winter 2016

Lecture 9 — February 2, 2016

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1 Outline

Agenda: X-ray tomography

1. Radon transform
2. Backprojection
3. Radon inversion formula

Last Time: We concluded our brief discussion of discrete Fourier transforms by mentioning the multidimensional and non-uniform transforms that arise in a number of important applications. One such application is medical imaging, and so we introduced the topic of X-ray tomography. After suggesting a model that simplifies the relevant physics, we arrived at the problem of determining an object from a collection of line integrals, each giving the projection of the object in a particular spatial direction. We asked first, is this possible? And second, if so, is there an inversion formula?

2 Radon transform

To answer the questions posed last time, we shall develop the machinery needed to handle the projection information given. In particular, for $f \in L^2(\mathbb{R}^2)$, define the **Radon transform** of f as the function $\mathcal{R}f : \mathbb{R} \times [0, \pi) \mapsto \mathbb{C}$ given by

$$\mathcal{R}f(t, \theta) = \int_{L_{t,\theta}} f(\boldsymbol{x}) dS(\boldsymbol{x}) = \iint f(\boldsymbol{x}) \delta(\langle \boldsymbol{x}, \boldsymbol{n}_\theta \rangle - t) \, d\boldsymbol{x}, \tag{1}$$

where $\boldsymbol{n}_\theta = (\cos\theta, \sin\theta)$ is the unit vector with angle θ. $\mathcal{R}f(t, \theta)$ is the integral of f along the line $L_{t,\theta}$ whose direction is perpendicular to $\boldsymbol{n}_\theta$ and whose distance from the origin is t:

$$L_{t,\theta} = \{\boldsymbol{x} : \langle \boldsymbol{x}, \boldsymbol{n}_\theta \rangle = t\}.$$

Indeed, using the notation $\boldsymbol{n}_\theta^\perp = (-\sin\theta, \cos\theta)$, we can alternatively write

$$\mathcal{R}f(t, \theta) = \int f(t\boldsymbol{n}_\theta + s\boldsymbol{n}_\theta^\perp) \, ds.$$

Note that we take θ only in $[0, \pi)$ since the integral in (1) gives $\mathcal{R}f(t, \theta + k\pi) = \mathcal{R}f((-1)^k t, \theta)$, and so it suffices to consider the Radon transform for $\theta \in [0, \pi)$.

The Radon transform as given in (1) is clearly well-defined for smooth functions f that decay rapidly at infinity. This allows us to extend its definition by duality to the space of distributions, and in particular, to functions in $L^2(\mathbb{R}^2)$. Extensions to other spaces are possible using these arguments.

As an example, consider

$$f(\boldsymbol{x}) = \begin{cases} 1 & \|\boldsymbol{x}\| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then an easy calculation gives

$$\mathcal{R}f(t,\theta) = \begin{cases} 2\sqrt{1-t^2} & |t| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that in this case $\mathcal{R}f(t,\theta)$ has no dependence on θ, which follows from the fact that f is radially symmetric (or **isotropic**).

The Radon transform formalizes the collection of line integrals we began to study in Lecture 8: For θ fixed, we can interpret the function $t \mapsto \mathcal{R}f(t,\theta)$ as the intensity of rays that traverse f across $L_{t,\theta}$. Consequently, the problem of recovering the absorption coefficient is equivalent to that of inverting the Radon transform.

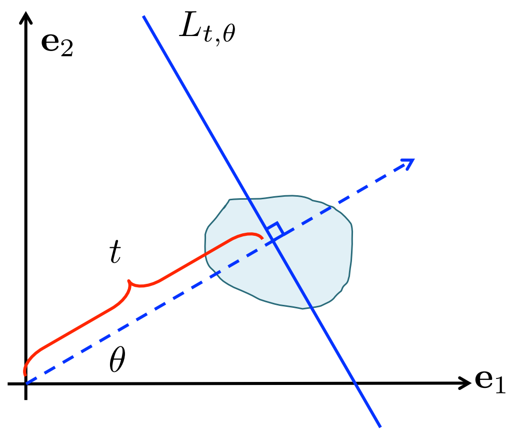


Figure 1: Parameterization of the lines along which the Radon transform is computed

3 The backprojection formula

We start our attempt to invert the Radon transform by discussing backprojection. Intuitively, the value $f(\boldsymbol{x})$ contributes to $\mathcal{R}f(t,\theta)$ whenever $\boldsymbol{x} \in L_{t,\theta}$. In the absence of any additional information, we will assume $f(\boldsymbol{x})$ contributes equally in all cases. Therefore, to compute the value of $f(\boldsymbol{x})$ we could try to average the values of $\mathcal{R}(t,\theta)$ along all lines that contain $\boldsymbol{x}$. By definition, $\boldsymbol{x} \in L_{t,\theta}$ if and only if $t = \langle \boldsymbol{x}, \boldsymbol{n}_\theta \rangle$, and so the lines containing $\boldsymbol{x}$ are parameterized by θ. We thus define the **backprojection** of f by

$$\tilde{f}(\boldsymbol{x}) = \frac{1}{\pi} \int_0^\pi \mathcal{R}f(\langle \boldsymbol{x}, \boldsymbol{n}_\theta \rangle, \theta)\, d\theta.$$

However, $\tilde{f}$ is not equal to f in general, meaning this is *not* an inversion formula. In the case of the indicator function of the unit disk (see Fig. 2a), we have

$$\tilde{f}(\boldsymbol{x}) = \frac{2}{\pi}\int_0^\pi \sqrt{1 - |\langle \boldsymbol{x}, \boldsymbol{n}_\theta \rangle|^2}\, d\theta,$$

In particular, while f is compactly supported, $\tilde{f}$ vanishes nowhere (see Fig. 2b). Nevertheless, the backprojection of the unit disk closely resembles the unit disk; $\tilde{f}$ appears to be a blurred version of f. This gives us hope that some additional operations might be performed to obtain a true inversion formula.

4 The inverse Radon transform

To deduce an inverse for the Radon transform, we will make use of its connection to the Fourier transform. This result is called the **projection-slice theorem**.

Theorem 1 (projection-slice)**.** *For any $\theta \in [0, \pi)$, the Fourier transform of the projection $\mathcal{R}f(\,\cdot\,, \theta)$ satisfies*

$$\int \mathcal{R}f(t,\theta)e^{-i\omega t}\, dt = \hat{f}(\omega\cos\theta, \omega\sin\theta).$$

In other words, measuring the Radon transform is equivalent to acquiring the Fourier transform of f along radial lines.

Proof. By direct calculation:

$$\int \mathcal{R}f(t,\theta)e^{-i\omega t}\, dt = \int\iint f(\boldsymbol{x})\delta(\langle \boldsymbol{x}, \boldsymbol{n}_\theta\rangle - t)e^{-i\omega t}\, d\boldsymbol{x}dt = \iint f(\boldsymbol{x})e^{-i\omega\langle \boldsymbol{x}, \boldsymbol{n}_\theta\rangle}\, d\boldsymbol{x} = \hat{f}(\omega\boldsymbol{n}_\theta).$$

$\square$

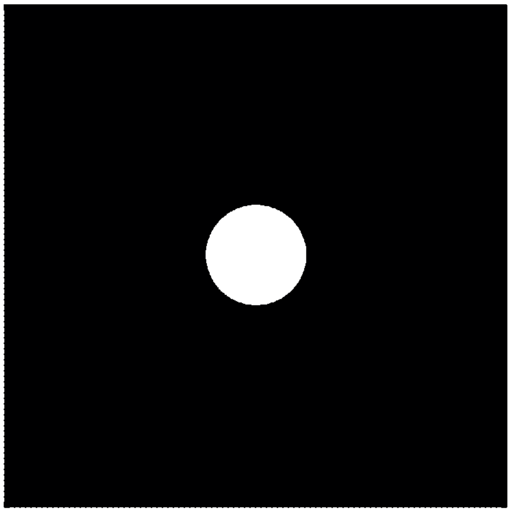
Consequently, we can to recover f from its Radon transform using the fact we know $\hat{f}$ along radial lines. This leads to an expression similar to the backprojection formula, suitably called **filtered backprojection** or the **Radon inversion formula**.

Theorem 2 (filtered backprojection)**.** *We have*

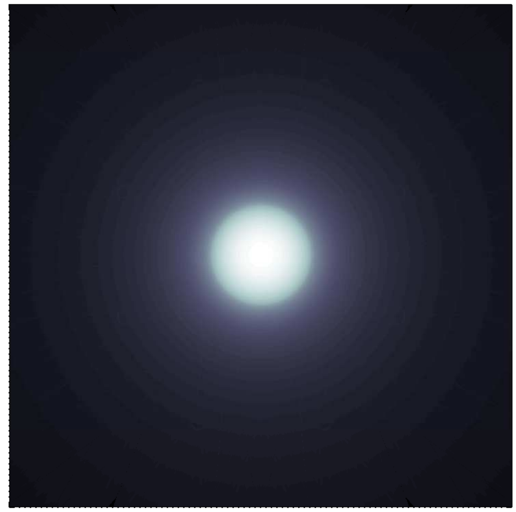
$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^2}\int_0^\pi (\mathcal{R}f(\,\cdot\,,\theta) * h)(\langle \boldsymbol{x}, \boldsymbol{n}_\theta\rangle)\, d\theta,$$

where h is such that $\hat{h}(\omega) = |\omega|$.

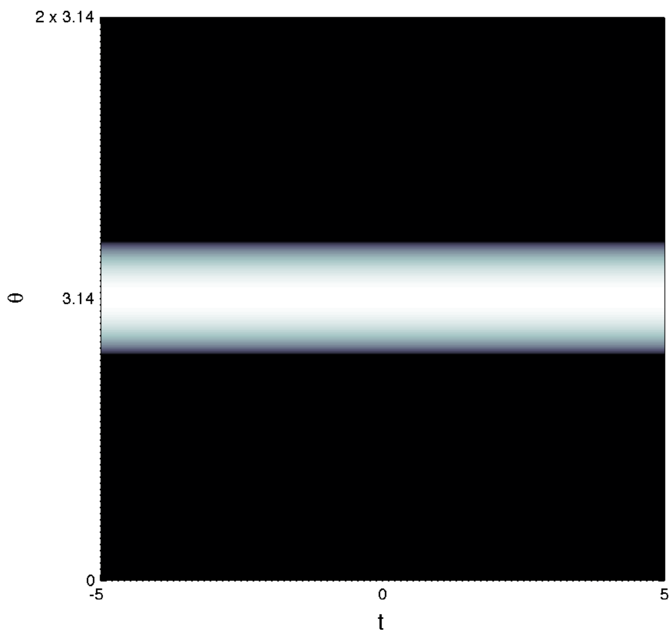
The designation "filtered" comes from the fact that this inversion formula has the same form as the backprojection formula, but instead of directly averaging the contributions as described in the previous section, we first filter the Radon transform along the radial variable with the convolution kernel h.



(a) $f(\boldsymbol{x}) = \mathbb{I}\{\|\boldsymbol{x}\| \leq 1\}$



(b) $\mathcal{R}^*\mathcal{R}f(\boldsymbol{x})$



(c) $\mathcal{R}f(t, \theta) = 2\mathbb{I}\{t \leq 1\}\sqrt{1-t^2}$

Figure 2: (a) Indicator function of the disk; (b) Backprojection reconstruction; (c) Radon transform of the indicator function of the disk.

Proof. Using the inverse Fourier transform, we have

$$
\begin{aligned}
f(\boldsymbol{x}) &= \frac{1}{(2\pi)^2} \iint \hat{f}(\boldsymbol{\omega}) e^{i\langle \boldsymbol{\omega}, \boldsymbol{x} \rangle} d\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{\infty} \hat{f}(r\boldsymbol{n}_\theta) e^{ir\langle \boldsymbol{n}_\theta, \boldsymbol{x} \rangle} r dr d\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^2}\int_0^{\pi}\int_0^{\infty}\hat{f}(r\boldsymbol{n}_\theta)e^{ir\langle \boldsymbol{n}_\theta,\,\boldsymbol{x}\rangle}r\,dr\,d\theta \\
&\quad + \frac{1}{(2\pi)^2}\int_{\pi}^{2\pi}\int_0^{\infty}\hat{f}(r\boldsymbol{n}_\theta)e^{ir\langle \boldsymbol{n}_\theta,\,\boldsymbol{x}\rangle}r\,dr\,d\theta,
\end{aligned}
$$

where we have changed to polar coordinates. The second integral can be rewritten as

$$
\begin{aligned}
\int_{\pi}^{2\pi}\int_0^{\infty}\hat{f}(r\boldsymbol{n}_\theta)e^{ir\langle \boldsymbol{n}_\theta,\,\boldsymbol{x}\rangle}r\,dr\,d\theta &= -\int_{\pi}^{2\pi}\int_0^{-\infty}\hat{f}(-r\boldsymbol{n}_\theta)e^{-ir\langle \boldsymbol{n}_\theta,\,\boldsymbol{x}\rangle}(-r)\,dr\,d\theta \\
&= \int_{\pi}^{2\pi}\int_{-\infty}^{0}\hat{f}(r\boldsymbol{n}_{\theta-\pi})e^{ir\langle \boldsymbol{n}_{\theta-\pi},\,\boldsymbol{x}\rangle}(-r)\,dr\,d\theta \\
&= \int_{0}^{\pi}\int_{-\infty}^{0}\hat{f}(r\boldsymbol{n}_{\theta})e^{ir\langle \boldsymbol{n}_{\theta},\,\boldsymbol{x}\rangle}(-r)\,dr\,d\theta,
\end{aligned}
$$

from which it follows that

$$
f(\boldsymbol{x}) = \frac{1}{(2\pi)^2}\int_0^{\pi}\int_{-\infty}^{\infty}\hat{f}(r\boldsymbol{n}_\theta)e^{ir\langle \boldsymbol{n}_\theta,\,\boldsymbol{x}\rangle}|r|\,dr\,d\theta = \frac{1}{2\pi}\int_0^{\pi}(\mathcal{R}f(\,\cdot\,,\theta)*h)(\langle \boldsymbol{n}_\theta,\,\boldsymbol{x}\rangle)\,d\theta.
$$

The last equality is justified by the convolution theorem with $\hat{h}(r) = |r|$, applied alongside the projection-slice theorem. □

In practice one cannot acquire information coming from rays in all of the continuum possible directions, and so one has to discretize in both t and θ. By the projection-slice theorem, this is equivalent to acquiring only a fraction of the radial lines in the Fourier domain, and therefore one has only partial information about the Fourier transform of f. A straightforward approach is to sum the corresponding filtered backprojections

$$
f_{\text{rec}}(\boldsymbol{x}) \approx \sum_k (\mathcal{R}f(\cdot,\theta_k)*h)(\langle \boldsymbol{x},\,\boldsymbol{n}_{\theta_k}\rangle),
$$

where the convolution is also computed discretely.