Harvard-MIT Mathematics Tournament

March 15, 2003

Guts Round — Solutions

1. Simplify $\sqrt[2003]{2\sqrt{11}-3\sqrt{5}} \cdot \sqrt[4006]{89+12\sqrt{55}}$.

   Solution: $\boxed{-1}$

   Note that $(2\sqrt{11}+3\sqrt{5})^2 = 89+12\sqrt{55}$. So, we have

$$\begin{aligned} \sqrt[2003]{2\sqrt{11}-3\sqrt{5}} \cdot \sqrt[4006]{89+12\sqrt{55}} &= \sqrt[2003]{2\sqrt{11}-3\sqrt{5}} \cdot \sqrt[2003]{2\sqrt{11}+3\sqrt{5}} \\ &= \sqrt[2003]{(2\sqrt{11})^2-(3\sqrt{5})^2} = \sqrt[2003]{-1} = -1. \end{aligned}$$

2. The graph of $x^4 = x^2y^2$ is a union of n different lines. What is the value of n?

   Solution: $\boxed{3}$

   The equation $x^4 - x^2y^2 = 0$ factors as $x^2(x+y)(x-y) = 0$, so its graph is the union of the three lines $x = 0$, $x+y = 0$, and $x-y = 0$.

3. If a and b are positive integers that can each be written as a sum of two squares, then ab is also a sum of two squares. Find the smallest positive integer c such that $c = ab$, where $a = x^3+y^3$ and $b = x^3+y^3$ each have solutions in integers (x, y), but $c = x^3+y^3$ does not.

   Solution: $\boxed{4}$

   We can't have $c = 1 = 1^3+0^3$ or $c = 2 = 1^3+1^3$, and if $c = 3$, then a or $b = \pm 3$ which is not a sum of two cubes (otherwise, flipping signs of x and y if necessary, we would get either a sum of two nonnegative cubes to equal 3, which clearly does not happen, or a difference of two nonnegative cubes to equal 3, but the smallest difference between two successive cubes ≥ 1 is $2^3 - 1^3 = 7$). However, $c = 4$ does meet the conditions, with $a = b = 2 = 1^3+1^3$ (an argument similar to the above shows that there are no x, y with $4 = x^3+y^3$), so 4 is the answer.

4. Let $z = 1-2i$. Find $\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$.

   Solution: $\boxed{(2i-1)/4}$

   Let $x = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$, so $z \cdot x = \left(1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \cdots\right)$. Then $z \cdot x - x = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots = \frac{1}{1-1/z} = \frac{z}{z-1}$. Solving for x in terms of z, we obtain $x = \frac{z}{(z-1)^2}$. Plugging in the original value of z produces $x = (2i-1)/4$.

5. Compute the surface area of a cube inscribed in a sphere of surface area π.

   Solution: $\boxed{2}$

   The sphere's radius r satisfies $4\pi r^2 = \pi \Rightarrow r = 1/2$, so the cube has body diagonal 1, hence side length $1/\sqrt{3}$. So, its surface area is $6(1/\sqrt{3})^2 = 2$.

6. Define the Fibonacci numbers by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For how many n, $0 \leq n \leq 100$, is F_n a multiple of 13?

Solution: $\boxed{15}$

The sequence of remainders modulo 13 begins $0, 1, 1, 2, 3, 5, 8, 0$, and then we have $F_{n+7} \equiv 8F_n$ modulo 13 by a straightforward induction. In particular, F_n is a multiple of 13 if and only if $7 \mid n$, so there are 15 such n.

7. a and b are integers such that $a + \sqrt{b} = \sqrt{15 + \sqrt{216}}$. Compute a/b.

   Solution: $\boxed{1/2}$

   Squaring both sides gives $a^2+b+2a\sqrt{b} = 15+\sqrt{216}$; separating rational from irrational parts, we get $a^2 + b = 15, 4a^2b = 216$, so a^2 and b equal 6 and 9. a is an integer, so $a^2 = 9, b = 6 \Rightarrow a/b = 3/6 = 1/2$. (We cannot have $a = -3$, since $a + \sqrt{b}$ is positive.)

8. How many solutions in nonnegative integers (a, b, c) are there to the equation

   $$2^a + 2^b = c! \quad ?$$

   Solution: $\boxed{5}$

   We can check that $2^a + 2^b$ is never divisible by 7, so we must have $c < 7$. The binary representation of $2^a + 2^b$ has at most two 1's. Writing $0!, 1!, 2!, \ldots, 6!$ in binary, we can check that the only possibilities are $c = 2, 3, 4$, giving solutions $(0, 0, 2), (1, 2, 3), (2, 1, 3), (3, 4, 4), (4, 3, 4)$.

9. For x a real number, let $f(x) = 0$ if $x < 1$ and $f(x) = 2x - 2$ if $x \geq 1$. How many solutions are there to the equation

   $$f(f(f(f(x)))) = x?$$

   Solution: $\boxed{2}$

   Certainly $0, 2$ are fixed points of f and therefore solutions. On the other hand, there can be no solutions for $x < 0$, since f is nonnegative-valued; for $0 < x < 2$, we have $0 \leq f(x) < x < 2$ (and $f(0) = 0$), so iteration only produces values below x, and for $x > 2$, $f(x) > x$, and iteration produces higher values. So there are no other solutions.

10. Suppose that A, B, C, D are four points in the plane, and let Q, R, S, T, U, V be the respective midpoints of AB, AC, AD, BC, BD, CD. If $QR = 2001, SU = 2002, TV = 2003$, find the distance between the midpoints of QU and RV.

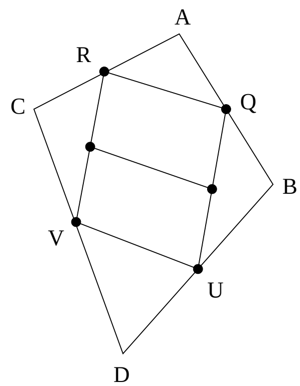
    Solution: $\boxed{2001}$

    This problem has far more information than necessary: QR and UV are both parallel to BC, and QU and RV are both parallel to AD. Hence, $QUVR$ is a parallelogram, and the desired distance is simply the same as the side length QR, namely 2001. (See figure, next page.)

11. Find the smallest positive integer n such that $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2$ is divisible by 100.

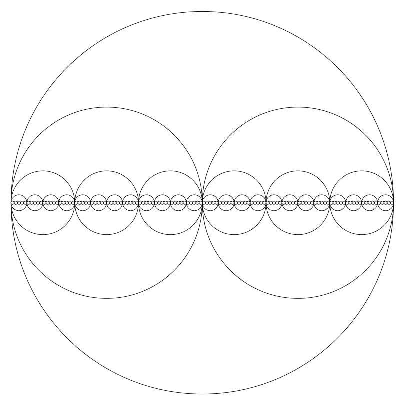
    Solution: $\boxed{24}$

    The sum of the first n squares equals $n(n+1)(2n+1)/6$, so we require $n(n+1)(2n+1)$ to be divisible by $600 = 24 \cdot 25$. The three factors are pairwise relatively prime, so



one of them must be divisible by 25. The smallest n for which this happens is $n = 12$ $(2n + 1 = 25)$, but then we do not have enough factors of 2. The next smallest is $n = 24$ $(n + 1 = 25)$, and this works, so 24 is the answer.

12. As shown in the figure, a circle of radius 1 has two equal circles whose diameters cover a chosen diameter of the larger circle. In each of these smaller circles we similarly draw three equal circles, then four in each of those, and so on. Compute the area of the region enclosed by a positive even number of circles.



Solution: $\boxed{\pi/e}$

At the nth step, we have $n!$ circles of radius $1/n!$ each, for a total area of $n! \cdot \pi/(n!)^2 = \pi/n!$. The desired area is obtained by adding the areas of the circles at step 2, then subtracting those at step 3, then adding those at step 4, then subtracting those at step 5, and so forth. Thus, the answer is

$$\frac{\pi}{2!} - \frac{\pi}{3!} + \frac{\pi}{4!} - \cdots = \pi\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots\right) = \pi e^{-1}.$$

13. If $xy = 5$ and $x^2 + y^2 = 21$, compute $x^4 + y^4$.

Solution: $\boxed{391}$

We have $441 = (x^2 + y^2)^2 = x^4 + y^4 + 2(xy)^2 = x^4 + y^4 + 50$, yielding $x^4 + y^4 = 391$.

14. A positive integer will be called "sparkly" if its smallest (positive) divisor, other than 1, equals the total number of divisors (including 1). How many of the numbers $2, 3, \ldots, 2003$ are sparkly?

Solution: $\boxed{3}$

Suppose n is sparkly; then its smallest divisor other than 1 is some prime p. Hence, n has p divisors. However, if the full prime factorization of n is $p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, the number of divisors is $(e_1+1)(e_2+1)\cdots(e_r+1)$. For this to equal p, only one factor can be greater than 1, so n has only one prime divisor — namely p — and we get $e_1 = p-1 \Rightarrow n = p^{p-1}$. Conversely, any number of the form p^{p-1} is sparkly. There are just three such numbers in the desired range (2^1, 3^2, 5^4), so the answer is 3.

15. The product of the digits of a 5-digit number is 180. How many such numbers exist?

Solution: $\boxed{360}$

Let the digits be a, b, c, d, e. Then $abcde = 180 = 2^2 \cdot 3^2 \cdot 5$. We observe that there are 6 ways to factor 180 into digits a, b, c, d, e (ignoring differences in ordering): $180 = 1 \cdot 1 \cdot 4 \cdot 5 \cdot 9 = 1 \cdot 1 \cdot 5 \cdot 6 \cdot 6 = 1 \cdot 2 \cdot 2 \cdot 5 \cdot 9 = 1 \cdot 2 \cdot 3 \cdot 5 \cdot 6 = 1 \cdot 3 \cdot 3 \cdot 4 \cdot 5 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$. There are (respectively) 60, 30, 60, 120, 60, and 30 permutations of these breakdowns, for a total of 360 numbers.

16. What fraction of the area of a regular hexagon of side length 1 is within distance $\frac{1}{2}$ of at least one of the vertices?

Solution: $\boxed{\pi\sqrt{3}/9}$

The hexagon has area $6(\sqrt{3}/4)(1)^2 = 3\sqrt{3}/2$. The region we want consists of six 120° arcs of circles of radius $1/2$, whichcan be reassembled into two circles of radius $1/2$. So its area is $\pi/2$, and the ratio of areas is $\pi\sqrt{3}/9$.

17. There are 10 cities in a state, and some pairs of cities are connected by roads. There are 40 roads altogether. A city is called a "hub" if it is directly connected to every other city. What is the largest possible number of hubs?

Solution: $\boxed{6}$

If there are h hubs, then $\binom{h}{2}$ roads connect the hubs to each other, and each hub is connected to the other $10-h$ cities; we thus get $\binom{h}{2} + h(10-h)$ distinct roads. So, $40 \geq \binom{h}{2} + h(10-h) = -h^2/2 + 19h/2$, or $80 \geq h(19-h)$. The largest $h \leq 10$ satisfying this condition is $h = 6$, and conversely, if we connect each of 6 cities to every other city and place the remaining $40 - [\binom{6}{2} + 6(10-6)] = 1$ road wherever we wish, we can achieve 6 hubs. So 6 is the answer.

18. Find the sum of the reciprocals of all the (positive) divisors of 144.

Solution: $\boxed{403/144}$

As d ranges over the divisors of 144, so does $144/d$, so the sum of $1/d$ is $1/144$ times the sum of the divisors of 144. Using the formula for the sum of the divisors of a number (or just counting them out by hand), we get that this sum is 403, so the answer is $403/144$.

19. Let r, s, t be the solutions to the equation $x^3 + ax^2 + bx + c = 0$. What is the value of $(rs)^2 + (st)^2 + (rt)^2$ in terms of a, b, and c?

Solution: $\boxed{b^2 - 2ac}$

We have $(x - r)(x - s)(x - t) = x^3 + ax^2 + bx + c$, so

$$a = -(r + s + t), \qquad b = rs + st + rt, \qquad c = -rst.$$

So we have

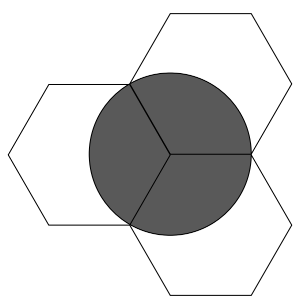
$$(rs)^2 + (st)^2 + (rt)^2 = (rs + st + rt)^2 - 2rst(r + s + t) = b^2 - 2ac.$$

20. What is the smallest number of regular hexagons of side length 1 needed to completely cover a disc of radius 1?

Solution: $\boxed{3}$

First, we show that two hexagons do not suffice. Specifically, we claim that a hexagon covers less than half of the disc's boundary. First, a hexagon of side length 1 may be inscribed in a circle, and this covers just 6 points. Translating the hexagon vertically upward (regardless of its orientation) will cause it to no longer touch any point on the lower half of the circle, so that it now covers less than half of the boundary. By rotational symmetry, the same argument applies to translation in any other direction, proving the claim. Then, two hexagons cannot possibly cover the disc.

The disc can be covered by three hexagons as follows. Let P be the center of the circle. Put three non-overlapping hexagons together at point P. This will cover the circle, since each hexagon will cover a $120°$ sector of the circle.



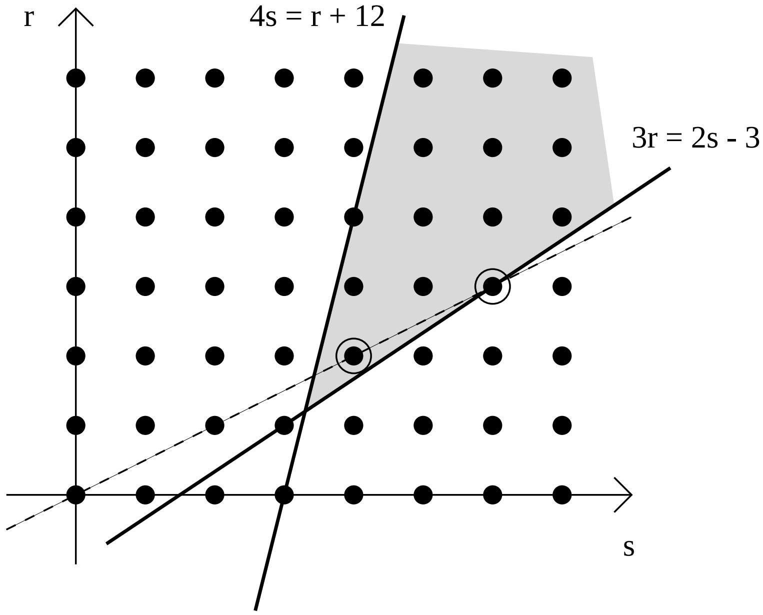
21. r and s are integers such that

$$3r \geq 2s - 3 \text{ and } 4s \geq r + 12.$$

What is the smallest possible value of r/s?

Solution: $\boxed{1/2}$

We simply plot the two inequalities in the sr-plane and find the lattice point satisfying both inequalities such that the slope from it to the origin is as low as possible. We find that this point is $(2, 4)$ (or $(3, 6)$), as circled in the figure, so the answer is $2/4 = 1/2$.



22. There are 100 houses in a row on a street. A painter comes and paints every house red. Then, another painter comes and paints every third house (starting with house number 3) blue. Another painter comes and paints every fifth house red (even if it is already red), then another painter paints every seventh house blue, and so forth, alternating between red and blue, until 50 painters have been by. After this is finished, how many houses will be red?

Solution: $\boxed{52}$

House n ends up red if and only if the largest odd divisor of n is of the form $4k+1$. We have 25 values of $n = 4k+1$; 13 values of $n = 2(4k+1)$ (given by $k = 0, 1, 2, \ldots, 12$); 7 values of $n = 4(4k+1)$ ($k = 0, 1, \ldots, 6$); 3 values of $n = 8(4k+1)$ ($k = 0, 1, 2$); 2 of the form $n = 16(4k+1)$ (for $k = 0, 1$); 1 of the form $n = 32(4k+1)$; and 1 of the form $n = 64(4k+1)$. Thus we have a total of $25+13+7+3+2+1+1 = 52$ red houses.

23. How many lattice points are enclosed by the triangle with vertices $(0, 99)$, $(5, 100)$, and $(2003, 500)$? Don't count boundary points.

Solution: $\boxed{0}$

Using the determinant formula, we get that the area of the triangle is

$$\begin{vmatrix} 5 & 1 \\ 2003 & 401 \end{vmatrix} \Big/ 2 = 1.$$

There are 4 lattice points on the boundary of the triangle (the three vertices and $(1004, 300)$), so it follows from Pick's Theorem that there are 0 in the interior.

24. Compute the radius of the inscribed circle of a triangle with sides 15, 16, and 17.

Solution: $\boxed{\sqrt{21}}$

Hero's formula gives that the area is $\sqrt{24 \cdot 9 \cdot 8 \cdot 7} = 24\sqrt{21}$. Then, using the result that the area of a triangle equals the inradius times half the perimeter, we see that the radius is $\sqrt{21}$.

25. Let ABC be an isosceles triangle with apex A. Let I be the incenter. If $AI = 3$ and the distance from I to BC is 2, then what is the length of BC?

Solution: $\boxed{4\sqrt{5}}$

Let X and Y be the points where the incircle touches AB and BC, respectively. Then AXI and AYB are similar right triangles. Since I is the incenter, we have $IX = IY = 2$. Using the Pythagorean theorem on triangle AXI, we find $AX = \sqrt{5}$. By similarity, $AY/AX = BY/IX$. Plugging in the numbers given, $5/\sqrt{5} = BY/2$, so $BY = 2\sqrt{5}$. Y is the midpoint of BC, so $BC = 4\sqrt{5}$.

26. Find all integers m such that $m^2 + 6m + 28$ is a perfect square.

Solution: $\boxed{6, -12}$

We must have $m^2+6m+28 = n^2$, where n is an integer. Rewrite this as $(m+3)^2+19 = n^2 \Rightarrow n^2 - (m+3)^2 = 19 \Rightarrow (n-m-3)(n+m+3) = 19$. Let $a = n-m-3$ and $b = n+m+3$, so we want $ab = 19$. This leaves only 4 cases:

- $a = 1$, $b = 19$. Solve the system $n - m - 3 = 1$ and $n + m + 3 = 19$ to get $n = 10$ and $m = 6$, giving one possible solution.
- $a = 19$, $b = 1$. Solve the system, as above, to get $n = 10$ and $m = -12$.
- $a = -1$, $b = -19$. We get $n = -10$ and $m = -12$.
- $a = -19$, $b = -1$. We get $n = -10$ and $m = 6$.

Thus the only m are 6 and -12.

27. The rational numbers x and y, when written in lowest terms, have denominators 60 and 70, respectively. What is the smallest possible denominator of $x + y$?

Solution: $\boxed{84}$

Write $x + y = a/60 + b/70 = (7a + 6b)/420$. Since a is relatively prime to 60 and b is relatively prime to 70, it follows that none of the primes $2, 3, 7$ can divide $7a+6b$, so we won't be able to cancel any of these factors in the denominator. Thus, after reducing to lowest terms, the denominator will still be at least $2^2 \cdot 3 \cdot 7 = 84$ (the product of the powers of $2, 3,$ and 7 dividing 420). On the other hand, 84 is achievable, by taking (e.g.) $1/60 + 3/70 = 25/420 = 5/84$. So 84 is the answer.

28. A point in three-space has distances $2, 6, 7, 8, 9$ from five of the vertices of a regular octahedron. What is its distance from the sixth vertex?

Solution: $\boxed{\sqrt{21}}$

By a simple variant of the British Flag Theorem, if $ABCD$ is a square and P any point in space, $AP^2+CP^2 = BP^2+DP^2$. Four of the five given vertices must form a square $ABCD$, and by experimentation we find their distances to the given point P must be $AP = 2, BP = 6, CP = 9, DP = 7$. Then A, C, and the other two vertices E, F also form a square $AECF$, so $85 = AP^2 + CP^2 = EP^2 + FP^2 = 8^2 + FP^2 \Rightarrow FP = \sqrt{21}$.

29. A *palindrome* is a positive integer that reads the same backwards as forwards, such as 82328. What is the smallest 5-digit palindrome that is a multiple of 99?

Solution: $\boxed{54945}$

Write the number as $\underline{XYZYX}$. This is the same as $10000X + 1000Y + 100Z + 10Y + X = 99(101X + 10Y + Z) + 20Y + 2X + Z$. We thus want $20Y + 2X + Z$ to be a multiple of 99, with X as small as possible. This expression cannot be larger than $20 \cdot 9 + 2 \cdot 9 + 9 = 207$, and it is greater than 0 (since $X \neq 0$), so for this to be a multiple of 99, it must equal 99 or 198. Consider these two cases.

To get 198, we must have $Y = 9$, which then leaves $2X + Z = 18$. The smallest possible X is 5, and then Z becomes 8 and we have the number 59895.

To get 99, we must have $Y = 4$. Then, $2X + Z = 19$, and, as above, we find the minimal X is 5 and then $Z = 9$. This gives us the number 54945. This is smaller than the other number, so it is the smallest number satisfying the conditions of the problem.

30. The sequence $a_1, a_2, a_3, \ldots$ of real numbers satisfies the recurrence

$$a_{n+1} = \frac{a_n^2 - a_{n-1} + 2a_n}{a_{n-1} + 1}.$$

Given that $a_1 = 1$ and $a_9 = 7$, find a_5.

Solution: $\boxed{3}$

Let $b_n = a_n + 1$. Then the recurrence becomes $b_{n+1} - 1 = (b_n^2 - b_{n-1})/b_{n-1} = b_n^2/b_{n-1} - 1$, so $b_{n+1} = b_n^2/b_{n-1}$. It follows that the sequence (b_n) is a geometric progression, from which $b_5^2 = b_1 b_9 = 2 \cdot 8 = 16 \Rightarrow b_5 = \pm 4$. However, since all b_n are real, they either alternate in sign or all have the same sign (depending on the sign of the progression's common ratio); either way, b_5 has the same sign as b_1, so $b_5 = 4 \Rightarrow a_5 = 3$.

31. A cylinder of base radius 1 is cut into two equal parts along a plane passing through the center of the cylinder and tangent to the two base circles. Suppose that each piece's surface area is m times its volume. Find the greatest lower bound for all possible values of m as the height of the cylinder varies.

Solution: $\boxed{3}$

Let h be the height of the cylinder. Then the volume of each piece is half the volume of the cylinder, so it is $\frac{1}{2}\pi h$. The base of the piece has area π, and the ellipse formed by the cut has area $\pi \cdot 1 \cdot \sqrt{1 + \frac{h^2}{4}}$ because its area is the product of the semiaxes times π. The rest of the area of the piece is half the lateral area of the cylinder, so it is πh. Thus, the value of m is

$$\begin{aligned} \frac{\pi + \pi\sqrt{1 + h^2/4} + \pi h}{\pi h/2} &= \frac{2 + 2h + \sqrt{4 + h^2}}{h} \\ &= \frac{2}{h} + 2 + \sqrt{\frac{4}{h^2} + 1}, \end{aligned}$$

a decreasing function of h whose limit as $h \to \infty$ is 3. Therefore the greatest lower bound of m is 3.

32. If x, y, and z are real numbers such that $2x^2 + y^2 + z^2 = 2x - 4y + 2xz - 5$, find the maximum possible value of $x - y + z$.

Solution: $\boxed{4}$

The equation rearranges as $(x-1)^2 + (y+2)^2 + (x-z)^2 = 0$, so we must have $x = 1$, $y = -2$, $z = 1$, giving us 4.

33. We are given triangle ABC, with $AB = 9$, $AC = 10$, and $BC = 12$, and a point D on BC. B and C are reflected in AD to B' and C', respectively. Suppose that lines BC' and $B'C$ never meet (i.e., are parallel and distinct). Find BD.

    Solution: $\boxed{6}$

    The lengths of AB and AC are irrelevant. Because the figure is symmetric about AD, lines BC' and $B'C$ meet if and only if they meet at a point on line AD. So, if they never meet, they must be parallel to AD. Because AD and BC' are parallel, triangles ABD and ADC' have the same area. Then ABD and ADC also have the same area. Hence, BD and CD must have the same length, so $BD = \frac{1}{2}BC = 6$.

34. $OKRA$ is a trapezoid with OK parallel to RA. If $OK = 12$ and RA is a positive integer, how many integer values can be taken on by the length of the segment in the trapezoid, parallel to OK, through the intersection of the diagonals?

    Solution: $\boxed{10}$

    Let $RA = x$. If the diagonals intersect at X, and the segment is PQ with P on KR, then $\triangle PKX \sim \triangle RKA$ and $\triangle OKX \sim \triangle RAX$ (by equal angles), giving $RA/PX = AK/XK = 1 + AX/XK = 1 + AR/OK = (x+12)/12$, so $PX = 12x/(12+x)$. Similarly $XQ = 12x/(12+x)$ also, so $PQ = 24x/(12+x) = 24 - \frac{288}{12+x}$. This has to be an integer. $288 = 2^5 3^2$, so it has $(5+1)(3+1) = 18$ divisors. $12 + x$ must be one of these. We also exclude the 8 divisors that don't exceed 12, so our final answer is 10.

35. A certain lottery has tickets labeled with the numbers $1, 2, 3, \ldots, 1000$. The lottery is run as follows: First, a ticket is drawn at random. If the number on the ticket is odd, the drawing ends; if it is even, another ticket is randomly drawn (without replacement). If this new ticket has an odd number, the drawing ends; if it is even, another ticket is randomly drawn (again without replacement), and so forth, until an odd number is drawn. Then, every person whose ticket number was drawn (at any point in the process) wins a prize.

    You have ticket number 1000. What is the probability that you get a prize?

    Solution: $\boxed{1/501}$

    Notice that the outcome is the same as if the lottery instead draws *all* the tickets, in random order, and awards a prize to the holder of the odd ticket drawn earliest and each even ticket drawn before it. Thus, the probability of your winning is the probability that, in a random ordering of the tickets, your ticket precedes all the odd tickets. This, in turn, equals the probability that your ticket is the first in the setconsisting of your own and all odd-numbered tickets, irrespective of the other even-numbered tickets. Since all 501! orderings of these tickets are equally likely, the desired probability is $1/501$.

36. A teacher must divide 221 apples evenly among 403 students. What is the minimal number of pieces into which she must cut the apples? (A whole uncut apple counts as one piece.)

    Solution: $\boxed{611}$

    Consider a bipartite graph, with 221 vertices representing the apples and 403 vertices representing the students; each student is connected to each apple that she gets a piece of. The number of pieces then equals the number of edges in the graph. Each student gets a total of $221/403 = 17/31$ apple, but each component of the graph represents a complete distribution of an integer number of apples to an integer number of students and therefore uses at least 17 apple vertices and 31 student vertices. Then we have at most $221/17 = 403/31 = 13$ components in the graph, so there are at least $221 + 403 - 13 = 611$ edges. On the other hand, if we simply distribute in the straightforward manner — proceeding through the students, cutting up a new apple whenever necessary but never returning to a previous apple or student — we can create a graph without cycles, and each component does involve 17 apples and 31 students. Thus, we get 13 trees, and 611 edges is attainable.

37. A *quagga* is an extinct chess piece whose move is like a knight's, but much longer: it can move 6 squares in any direction (up, down, left, or right) and then 5 squares in a perpendicular direction. Find the number of ways to place 51 quaggas on an 8×8 chessboard in such a way that no quagga attacks another. (Since quaggas are naturally belligerent creatures, a quagga is considered to attack quaggas on any squares it can move to, as well as any other quaggas on the same square.)

Solution: $\boxed{68}$

Represent the 64 squares of the board as vertices of a graph, and connect two vertices by an edge if a quagga can move from one to the other. The resulting graph consists of 4 paths of length 5 and 4 paths of length 3 (given by the four rotations of the two paths shown, next page), and 32 isolated vertices. Each path of length 5 can accommodate at most 3 nonattacking quaggas in a unique way (the first, middle, and last vertices), and each path of length 3 can accommodate at most 2 nonattacking quaggas in a unique way; thus, the maximum total number of nonattacking quaggas we can have is $4 \cdot 3 + 4 \cdot 2 + 32 = 52$. For 51 quaggas to fit, then, just one component of the graph must contain one less quagga than its maximum.

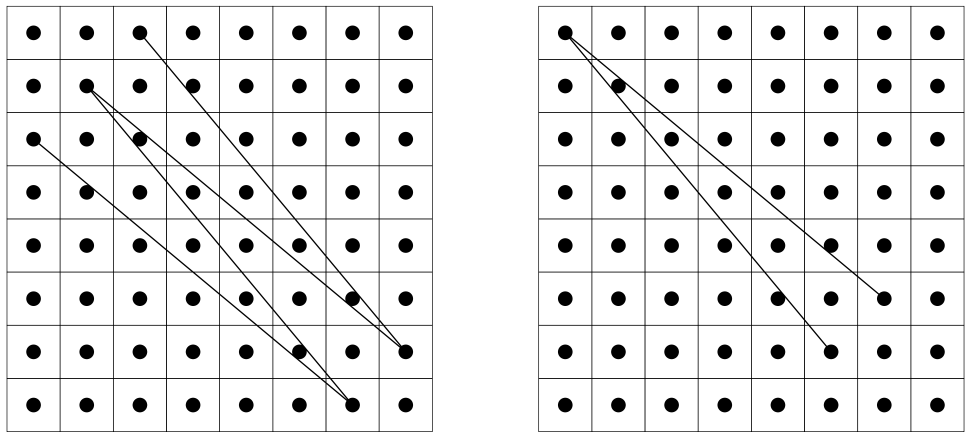
If this component is a path of length 5, there are $\binom{5}{2} - 4 = 6$ ways to place the two quaggas on nonadjacent vertices, and then all the other locations are forced; the 4 such paths then give us $4 \cdot 6 = 24$ possibilities this way. If it is a path of length 3, there are 3 ways to place one quagga, and the rest of the board is forced, so we have $4 \cdot 3 = 12$ possibilities here. Finally, if it is one of the 32 isolated vertices, we simply leave this square empty, and the rest of the board is forced, so we have 32 possibilities here. So the total is $24 + 12 + 32 = 68$ different arrangements.

38. Given are real numbers x, y. For any pair of real numbers a_0, a_1, define a sequence by $a_{n+2} = xa_{n+1} + ya_n$ for $n \geq 0$. Suppose that there exists a fixed nonnegative integer m such that, for every choice of a_0 and a_1, the numbers a_m, a_{m+1}, a_{m+3}, in this order, form an arithmetic progression. Find all possible values of y.

Solution: $\boxed{0, 1, (1 \pm \sqrt{5})/2}$

Note that $x = 1$ (or $x = 0$), $y = 0$ gives a constant sequence, so it will always have the desired property. Thus, $y = 0$ is one possibility. For the rest of the proof, assume $y \neq 0$.

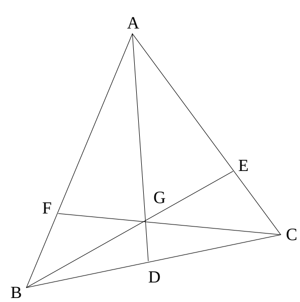
We will prove that a_m and a_{m+1} may take on any pair of values, for an appropriate choice of a_0 and a_1. Use induction on m. The case $m = 0$ is trivial. Suppose that a_m and a_{m+1} can take on any value. Let p and q be any real numbers. By setting



$a_m = \frac{q-xp}{y}$ (remembering that $y \neq 0$) and $a_{m+1} = p$, we get $a_{m+1} = p$ and $a_{m+2} = q$. Therefore, a_{m+1} and a_{m+2} can have any values if a_m and a_{m+1} can. That completes the induction.

Now we determine the nonzero y such that a_m, a_{m+1}, a_{m+3} form an arithmetic sequence; that is, such that $a_{m+3} - a_{m+1} = a_{m+1} - a_m$. But because $a_{m+3} = (x^2 + y)a_{m+1} + xya_m$ by the recursion formula, we can eliminate a_{m+3} from the equation, obtaining the equivalent condition $(x^2 + y - 2)a_{m+1} + (xy + 1)a_m = 0$. Because the pair a_m, a_{m+1} can take on any values, this condition means exactly that $x^2 + y - 2 = xy + 1 = 0$. Then $x = -1/y$, and $1/y^2 + y - 2 = 0$, or $y^3 - 2y^2 + 1 = 0$. One root of this cubic is $y = 1$, and the remaining quadratic factor $y^2 - y - 1$ has the roots $(1 \pm \sqrt{5})/2$. Since each such y gives an x for which the condition holds, we conclude that the answer to the problem is $y = 0, 1$, or $(1 \pm \sqrt{5})/2$.

39. In the figure, if $AE = 3$, $CE = 1$, $BD = CD = 2$, and $AB = 5$, find AG.



Solution: $\boxed{3\sqrt{66}/7}$

By Stewart's Theorem, $AD^2 \cdot BC + CD \cdot BD \cdot BC = AB^2 \cdot CD + AC^2 \cdot BD$, so $AD^2 = (5^2 \cdot 2 + 4^2 \cdot 2 - 2 \cdot 2 \cdot 4)/4 = (50 + 32 - 16)/4 = 33/2$. By Menelaus's Theorem applied to line BGE and triangle ACD, $DG/GA \cdot AE/EC \cdot CB/BD = 1$, so $DG/GA = 1/6 \Rightarrow AD/AG = 7/6$. Thus $AG = 6 \cdot AD/7 = 3\sqrt{66}/7$.

40. All the sequences consisting of five letters from the set $\{T, U, R, N, I, P\}$ (with repetitions allowed) are arranged in alphabetical order in a dictionary. Two sequences are called "anagrams" of each other if one can be obtained by rearranging the letters of the

other. How many pairs of anagrams are there that have exactly 100 other sequences between them in the dictionary?

Solution: $\boxed{0}$

Convert each letter to a digit in base 6: $I \mapsto 0, N \mapsto 1, P \mapsto 2, R \mapsto 3, T \mapsto 4, U \mapsto 5$. Then the dictionary simply consists of all base-6 integers from 00000_6 to 55555_6 in numerical order. If one number can be obtained from another by a rearrangement of digits, then the numbers are congruent modulo 5 (this holds because a number $\underline{abcde}_6$ $= 6^4 \cdot a + 6^3 \cdot b + 6^2 \cdot c + 6 \cdot d + e$ is congruent modulo 5 to $a + b + c + d + e$), but if there are 100 other numbers between them, then their difference is 101, which is not divisible by 5. So there are no such pairs.

41. A hotel consists of a 2×8 square grid of rooms, each occupied by one guest. All the guests are uncomfortable, so each guest would like to move to one of the adjoining rooms (horizontally or vertically). Of course, they should do this simultaneously, in such a way that each room will again have one guest. In how many different ways can they collectively move?

    Solution: $\boxed{1156}$

    Imagine that the rooms are colored black and white, checkerboard-style. Each guest in a black room moves to an adjacent white room (and vice versa). If, for each such guest, we place a domino over the original room and the new room, we obtain a covering of the $2 \times n$ grid by n dominoes, since each black square is used once and each white square is used once. Applying a similar procedure to each guest who begins in a white room and moves to a black room, we obtain a second domino tiling. Conversely, it is readily verified that any pair of such tilings uniquely determines a movement pattern. Also, it is easy to prove by induction that the number of domino tilings of a $2 \times n$ grid is the $(n + 1)$th Fibonacci number (this holds for the base cases $n = 1, 2$, and for a $2 \times n$ rectangle, the two rightmost squares either belong to one vertical domino, leaving a $2 \times (n - 1)$ rectangle to be covered arbitrarily, or to two horizontal dominoes which also occupy the adjoining squares, leaving a $2 \times (n - 2)$ rectangle to be covered freely; hence, the numbers of tilings satisfy the Fibonacci recurrence). So the number of domino tilings of a 2×8 grid is 34, and the number of pairs of such tilings is $34^2 = 1156$, the answer.

42. A tightrope walker stands in the center of a rope of length 32 meters. Every minute she walks forward one meter with probability $3/4$ and backward one meter with probability $1/4$. What is the probability that she reaches the end in front of her before the end behind her?

    Solution: $\boxed{3^{16}/(3^{16} + 1)}$

    After one minute, she is three times as likely to be one meter forward as one meter back. After two minutes she is either in the same place, two meters forward, or two meters back. The chance of being two meters forward is clearly $(3/4)^2$ and thus $3^2 = 9$ times greater than the chance of being two back ($(1/4)^2$). We can group the minutes into two-minute periods and ignore the periods of no net movement, so we can consider her to be moving 2 meters forward or backward each period, where forward movement is 3^2 times as likely as backward movement. Repeating the argument inductively, we eventually find that she is 3^{16} times more likely to move 16 meters forward than 16 meters backward, and thus the probability is $3^{16}/(3^{16}+1)$ that she will meet the front end of the rope first.

43. Write down an integer N between 0 and 10, inclusive. You will receive N points — unless some other team writes down the same N, in which case you receive nothing.

    Comments: Somewhat surprisingly, this game does not appear to be well-documented in the literature. A better-known relative is Undercut, described by Douglas Hofstadter in his book *Metamagical Themas: Questing for the Essence of Mind and Pattern*. In this game, two players each choose an integer in a specified interval and receive that many points — unless the two numbers differ by 1, in which case the lower player receives a number of points equal to the sum of the two numbers. See, for example, `http://mathforum.org/mam/96/undercut/` .

44. A *partition* of a number n is a sequence of positive integers, arranged in descending order, whose sum is n. For example, $n = 4$ has 5 partitions: $1+1+1+1 = 2+1+1 = 2+2 = 3+1 = 4$. Given two different partitions of the same number, $n = a_1 + a_2 + \cdots + a_k = b_1 + b_2 + \cdots + b_l$, where $k \le l$, the first partition is said to *dominate* the second if all of the following inequalities hold:

$$\begin{aligned} a_1 &\ge b_1; \\ a_1 + a_2 &\ge b_1 + b_2; \\ a_1 + a_2 + a_3 &\ge b_1 + b_2 + b_3; \\ &\vdots \\ a_1 + a_2 + \cdots + a_k &\ge b_1 + b_2 + \cdots + b_k. \end{aligned}$$

    Find as many partitions of the number $n = 20$ as possible such that none of the partitions dominates any other. Your score will be the number of partitions you find. If you make a mistake and one of your partitions does dominate another, your score is the largest m such that the first m partitions you list constitute a valid answer.

    Comments: Computer searches have found that the maximum possible number of partitions is 20, achieved e.g. by the following:

$9+1+1+1+1+1+1+1+1+1+1+1$
$8+2+2+1+1+1+1+1+1+1+1$
$7+4+1+1+1+1+1+1+1+1+1$
$7+3+3+1+1+1+1+1+1+1$
$7+3+2+2+2+1+1+1+1$
$7+2+2+2+2+2+2+1$
$6+5+2+1+1+1+1+1+1+1$
$6+4+3+2+1+1+1+1+1$
$6+4+2+2+2+2+1+1$
$6+3+3+3+2+1+1+1$
$6+3+3+2+2+2+2$
$5+5+4+1+1+1+1+1+1$
$5+5+3+2+2+1+1+1$
$5+5+2+2+2+2+2$
$5+4+4+3+1+1+1+1$
$5+4+4+2+2+2+1$
$5+4+3+3+3+1+1$
$5+3+3+3+3+3$
$4+4+4+4+2+1+1$
$4+4+4+3+3+2$

    In general, as shown in

`http://www-math.mit.edu/ẽfedula/chains.ps`
"Chain Lengths in the Dominance Lattice," by Edward Early, 2002

the maximum number of mutually nondominating partitions of n is roughly on the order of $e^{\pi\sqrt{2n/3}}$ (in fact, the ratio of the total number of partitions to n to the maximum number of nondominating partitions has polynomial order of magnitude), although a proof by explicit construction is not known.

45. Find a set S of positive integers such that no two distinct subsets of S have the same sum. Your score will be $\lfloor 20(2^n/r - 2)\rfloor$, where n is the number of elements in the set S, and r is the largest element of S (assuming, of course, that this number is nonnegative).

Comments: The obvious such sets are those of the form $\{1, 2, 4, \ldots, 2^{n-1}\}$, but these achieve a score of 0. A bit of experimentation finds that the set $\{3, 5, 6, 7\}$ also works; this answer achieves 5 points. The set $\{6, 9, 11, 12, 13\}$ achieves 9 points. Doing significantly better than this by hand computation becomes difficult. The reference

`http://www.combinatorics.org/Volume_5/PDF/v5i1r3.pdf`
"A Construction for Sets of Integers with Distinct Subset Sums,"
by Tom Bohman, 1997

(from whence the above examples) provides a construction that achieves a score of 50. Erdős conjectured in effect that a maximum possible score exists, but this conjecture remains open. The strongest known result in this direction is that, if S has n elements, the largest element is at least a constant times $2^n/\sqrt{n}$ (proven by Erdős and Moser in 1955, with an improvement on the constant by Elkies in 1986).

Hej då!