Graphing Calculator Intensive Calculus: A First Step in Calculus Reform

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Computer generated numerical, visual, and symbolic mathematics is revolutionizing the teaching and learning of calculus. The computer can be a desktop computer with computer algebra and graphing software or a pocket computer with built-in software (graphing calculator). The content of calculus is changing—less time is spent on paper and pencil methods and more time is spent on applications, problem solving, and concept development (MAA, 1986, 1987, and 1990). And teaching methods are also dramatically changing—moving toward an investigative, exploratory approach.

We believe that graphing calculators are the appropriate computer tools for most students today (1993) because they are inexpensive (some less than $50), user-friendly, powerful (some built-in software on newer graphing calculators like the TI-85 and HP-48 is phenomenal), small, and personal. In short, a graphing calculator intensive approach is *implementable* for *all* students. Expensive and logistically complex computer laboratories are not necessary to teach a computer intensive calculus course. *Any* classroom today can become a computer laboratory with student use of graphing calculators (Demana and Waits, 1992a).

The Calculator and Computer Enhanced Calculus (C^3E) calculus reform project

We approach the incorporation of hand-held computer technology in calculus as a natural evolution of our positive experience in large scale implementation of hand-held technology with *all* students in two projects. First, with calculators at Ohio State in the seventies [Leitzel and Waits, 1976] and then with graphing calculators in our highly regarded C^2PC project in the eighties (See The Calculator and Computer PreCalculs Project (C^2PC): What Have We Learned in Ten Years?, in press). Our C^2PC textbook, *Precalculus Mathematics, A Graphing Approach* is recognized as being the first widely adopted high school and college textbook to *require* graphing technology and is now in the third edition (Demana, Waits, and Clemens, 1994).

Our C^3E calculus reform project is based on what we learned in our many years with the C^2PC project. Fundamentally we learned that the principal of incremental change should guide our approach to calculus curriculum reform and the related integration of computer technology. We take a familiar body of calculus material and make the assumption that *every* student has an inexpensive, user-friendly *graphing calculator* for both in-class activities and for homework. We use graphing calculators as scientific calculators (they are the best we have every used), as "tools" for computing derivatives and integrals numerically, as computers for programming (certain "tool box" programs like Simpson's method and Euler's method), as numerical "solvers" (e.g. root and intersection finders), and for computer visualization using their built-in graphing software (for example graphing derivatives, functions defined by integrals, and power series).

We believe technology will not be routinely used by all calculus students (or required by professors) until it costs less than $100, is user-friendly, and fits in a backpack or purse. Our project *assumes* that every student has an inexpensive graphing calculator. The C^3E materials are reflected in a new textbook, *Calculus, A Graphing Approach* (Finney, Thomas, Demana, and Waits, 1994) which *requires* graphing calculators. Our project does not assume that every student has a computer algebra system (like DERIVE™ or Mathematica®). However, in a few years, computer algebra will no doubt become much cheaper and they may be a reasonable assumption. Colleagues who become comfortable with graphing calculators today will easily make the transition to the powerful and no doubt inexpensive computer algebra systems of the future (Demana and Waits, 1992b).

Our philosophy of using graphing calculator numerical and visual methods to enhance the teaching and learning of calculus can be summarized by the following three points.

I. Do analytically (paper and pencil), then SUPPORT numerically and graphically (with a graphing calculator)
II. Do numerically and graphically (with a graphing calculator), then CONFIRM analytically (with paper and pencil)
III. Do numerically and graphically, because other methods are IMPRACTICAL or IMPOSSIBLE!

In several years, we will certainly include computer algebra systems in our required technology "tool box" as the cost of these powerful systems come down. We are also convinced that required student use of graphing calculators today promotes a cooperative learning environment where calculus can be presented as an exciting, lively subject where student investigations (we call them EXPLORES) become routine.

We illustrate our three point philosophy with four examples.

Point I: Use graphing calculators to visually support results first obtained by analytic calculus paper and pencil manipulations.

These "support graphically" activities are part of the "bread and butter" of a *first step* towards calculus reform. Here we take old familiar topics and *support* them with technology and, at the same time, we add to students intuitive understanding of calculus concepts.

Problem 1. Use the limit definition and show that $\lim_{x\to 1} f(x) = -1$, where $f(x) = x^2 - 2x$.

The formal limit definition has remained a very mysterious concept to students. In fact, it is not commonly taught to freshman calculus students in many universities. The limit concept can be dramatically enhanced by computer graphing and related numerical analysis. Analytically it can be determined that given any $\varepsilon > 0$, choosing $\delta = +\varepsilon$ will satisfy the usual limit definition when applied to Problem 1. Here a graph can be much more instructive.

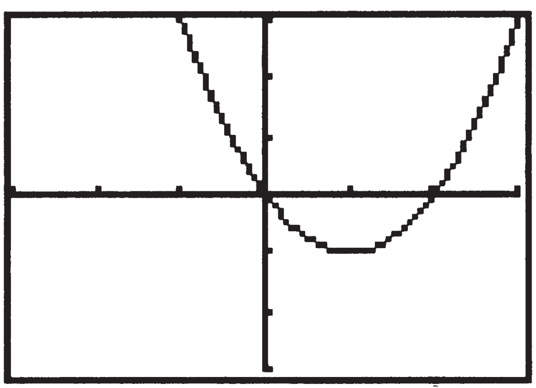

Figure 1. The graph of $f(x) = x^2 - 2x$.

The graph of $f(x) = x^2 - 2x$ (Figure 1) clearly indicates the continuous nature of the function (so the limit at x = 1 can be calculated by evaluating f(1)). However, the analytic "limit proof" of this fact is not so clear. A computer generated "magnified" graph is very valuable. In Figure 2, we illustrate the limit definition for a "given ε" of ε = 0.01. The student adds the "target" lines $y = -1 \pm \varepsilon$ or, in this case, $y = -1 - .01 = -1.01$ and $y = -1 + .01 = -0.99$. Then a graphing ZOOM procedure is used to obtain the graph shown.

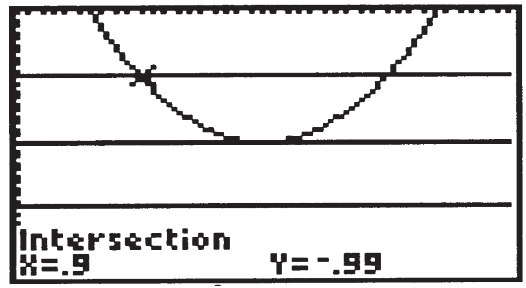

Figure 2. The graph of $f(x) = x^2 - 2x$, and lines $y = -1$, $y = -1 - .01$, $y = -1 + .01$ for $0.8 \le x \le 1.2$ and $-1.02 \le y \le -0.98$

It becomes clear that if x is kept between 0.9 and 1.1, the function values $f(x)$ are always between -1.01 and -0.99. That is, if $|x-1| < \delta = 0.1$, then $|f(x)-(-1)| < \varepsilon = 0.01$. This fact strongly suggests that $\delta = 0.1 = \sqrt{0.01} = \sqrt{\varepsilon}$ is the required delta value in terms of epsilon in the analytic limit analysis.

This example is very easy to deal with because the limit point was at the minimum value of the function. The *principal of local linearity* will help for other values. For example, suppose the problem is changed to "Use the limit definition to show that $\lim_{x\to 1.5} f(x) = -0.75$."

Figure 3 shows a magnified computer ZOOM-IN view of the graph in Figure 1 at the point (1.5, f(1.5)) and at another nearby point.

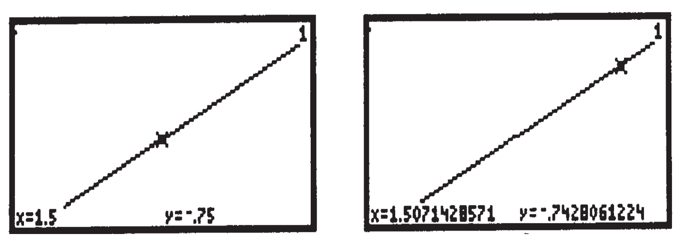

Figure 3. Zoom-in views of the graph of $y = x^2 - 2x$ near the point (1.5, -0.75).

Notice that the graph of $f(x) = x^2 - 2x$, for all practical purposes, is *a straight line* with slope m, where $m = (-0.75 - (-0.7428061224)) / (1.5 - 1.5071428571)$ which is very close to 1 (actually 1.00714285715). That is, the function $y + 0.75 = x - 1.5$ (or $y = x - 2.25$) closely approximates the quadratic function near $x = 1.5$. The fact that the slope is 1 strongly suggests that for any e>0, sufficiently small, then for all practical purposes, we can choose δ=e in the limit analysis of this example. Computer graphing can make the limit definition far more meaningful than past analytic paper and pencil "hocus-pocus." One can go on and complete the analytic analysis if desired. This could be an example of "do graphically, confirm analytically."

Point II: Use graphing calculators as <u>tools</u> to actually <u>do</u> calculus "manipulations" then confirm the results using analytic methods of calculus. Thus making the need for paper and pencil calculus manipulations less important.

Problem 2. The "brief case" box with lid problem

A box with lid is constructed from a 20 by 30 inch sheet of material in the following manner. First the material is folded in half forming a 20 by 15 inch double sheet. Then four equal squares of side-length x are removed from each corner of the folded sheet. The material is then unfolded and a box with sides *and a lid* are formed by folding along the dotted lines shown in Figure.4.

The following questions are typical of the investigations we ask students to deal with routinely.

a. Determine an algebraic representation of the volume of the "brief case" box with lid in terms of x.
b. What values of x make sense in this problem situation?
c. Draw a complete graph of the volume of the box in terms of x.
d. Find the maximum volume of the box. What is the associated side length of the removed square? Discuss the accuracy of your solution?
e. Confirm your results using paper and pencil analytic methods of calculus.

Figure 5A displays the graphs of the function $y = V(x) = 2x(15 - 2x)(20 - 2x)$ and its first derivative (using the numerical derivative feature). The graph of $y = V(x)$ for $0 < x < 7.5$ is the graph of the *box volume problem situation*. The figure also shows that a "solver" (for example, ROOT on the TI-82 or TI-85) has been applied to the derivative graph to find the zero of the derivative. Figure 5B shows the function value at the "root" of the derivative (which is the local maximum value of $y = V(x)$).

Thus the student can apply the theory of calculus ("look for possible local extrema where the derivative is zero") and solve this problem to a very high degree of accuracy using a graphing calculator. Students can then be required to CONFIRM the result analytically by computing the derivative and solving the resulting equation using ordinary paper and pencil calculus. A lively discussion will ensue when students are asked to write about and contrast both solution methods.

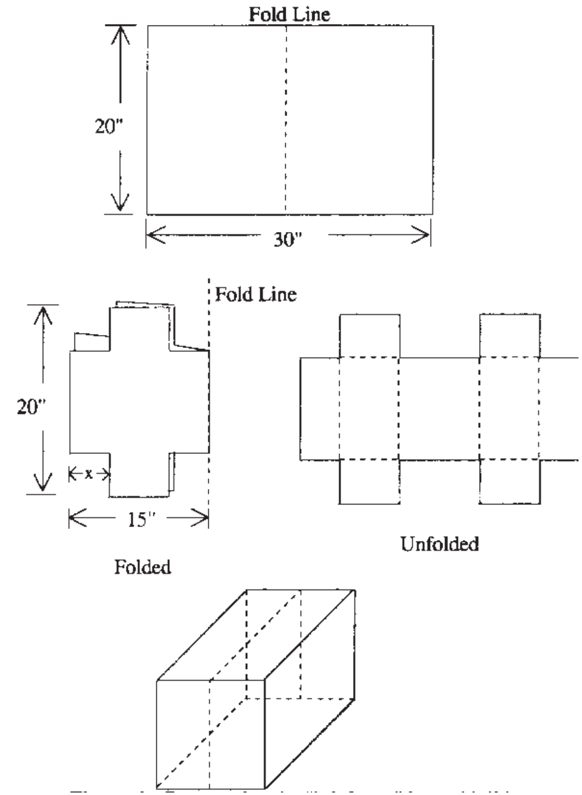

Figure 4. Constructing the "brief case" box with lid

An interesting related exercise involving a "simple" equation is given by the following variation of problem 2: Find the side length of a removed square to obtain a brief case box with volume 455 cubic inches. Can you confirm your results using analytic methods? If so, do it! Can you find the *exact* solution? If so, do it! Write a paragraph outlining the issues involved in applying an analytic method versus applying a numerical / graphical method.

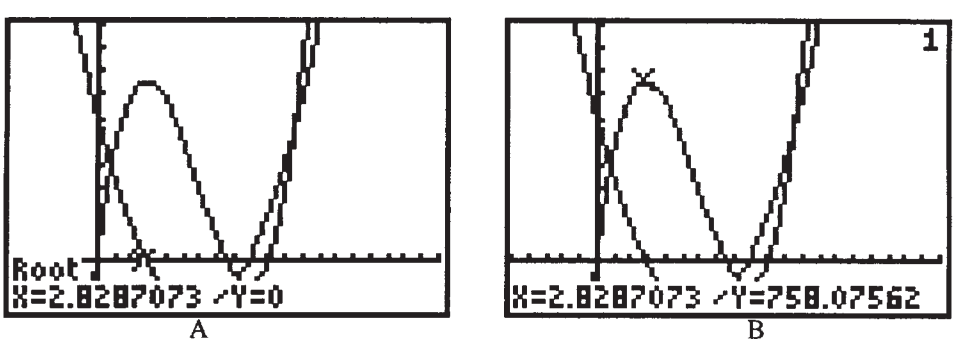

Figure 5. Part A. The graphs of $y = V(x)$ and $y = V'(x)$ = NDer (V, x) Part B. The (local) maximum value of $y = V(x)$ is $V = 758.07562$ (accurate to all digits shown)

Point III: Solve easily stated and understood problems that calculus students can't solve with paper and pencil analytic methods. And some that have <u>no</u> analytic solution. Illustrate mathematical ideas and applications in concrete geometric settings. We explore, investigate, and make and test mathematical conjectures.

Problem 3: Visually illustrate the Fundamental Theorem of Calculus. (Demana & Waits, in press)

Background: Consider a continuous function f defined on an interval [a, b] (*any* continuous function even those without closed form antiderivatives). The Fundamental Theorem of Calculus guarantees the existence of a function F, namely $F(x)=\int_a^x f(t)\,dt$ with the property that $F'(x)=f(x)$. The problem is that, until today, students could "find" these antiderivatives that we know exist for only a relative few *contrived* functions f. And these contrived functions are what make up typical calculus textbook integration problems! However, today with graphing calculators all students can "see" the antiderivative F easily for any continuous function (and those with continuous extensions) even if we can't write the explicit "closed form" analytic expression. All that is needed is a way of graphing $F(x)=\int_a^x f(t)\,dt$. The TI-82 and TI-85 have this as a built-in feature. Other graphing calculators can be programmed with this feature as a "tool box" item. See the *Graphing Calculator and Computer Algebra System Resource Manual for Calculus* that accompanies our textbook for programs for the TI-81, Sharp 9200 and 9300, Casio, and Hewlett Packard graphing calculators (Demana & Waits, 1994).

Solution: The Fundamental Theorem of Calculus implies that

$$D_x\,[F(x)]=D_x\left[\int_a^x f(t)\,dt\right]=f(x)$$

This is a TI-82 or TI-85 activity. Graph the function y = FnInt (t^2, t, {-2,0,2,3}, x) in the [-5, 5] by [-10, 10] window ($-5 \le x \le 5$, $-10 \le y \le 10$). This produces four graphs of $F(x)=\int_a^x t^2\,dt$ for a = -2, 0, 2, 3. Note the use of a *list* in the lower limit of integration position. Students can conjecture about what are the analytic forms of the antiderivatives that they "see." And they can test their conjecture by "overlaying" the analytic expression (eg DRAWF $x^3/3+C$). EXPLORE: determine C for the above four antiderivatives and explain how C and a are related, etc.

Figure 6 makes the constant of integration and family of antiderivatives concept come alive for students. Here are "connected" slope fields! Static figures do not do this activity justice. This dynamic activity must be "experienced" by the student.

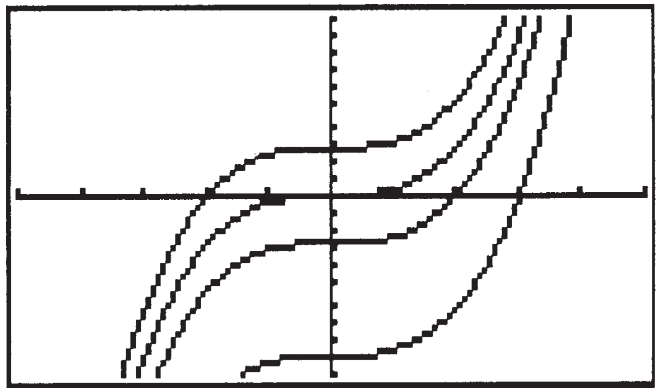

Figure 6. A graph of a family of antiderivatives of x^2 in [-5, 5] by [-10, 10]

Next we graph the numerical derivative of this function defined by the integral to visually illustrate the fundamental relationship that the derivative of F(x) is f(x) as claimed by the Fundamental Theorem of Calculus. That is, we show that

$$D_x\left[\int_a^x f(t)\,dt\right]=f(x)$$ for *any* continuous function f.

The graph of NDer (FnInt (sin(t) / t, t, 0, x), x) on the TI-82 or TI-85 is the graph of $y=D_x\left[\int_0^t \frac{\sin t}{t}\,dt\right]$ which should be $y=\frac{\sin(x)}{x}$. Figure 7 shows this is indeed the case! Note: we regard the integrand to be the continuous extension of sin(x) / x. TRACE can be used to compare the two function values for supporting numerical evidence. This activity is a powerful visualization!

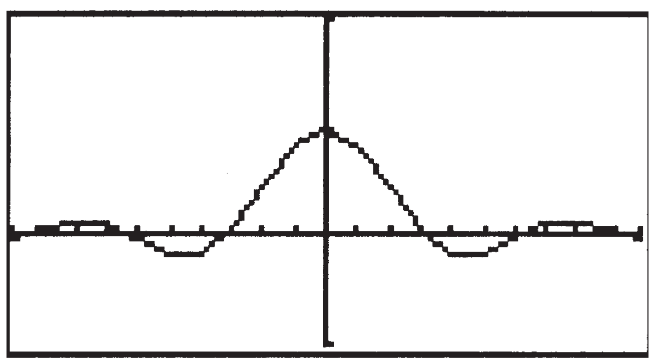

Figure 7. Graphs of $y=D_x\left[\int_0^t \frac{\sin t}{t}\,dt\right]$ and $y=\frac{\sin(x)}{x}$ in [-10,10] by [-1,2] appear to be the same.

The predator-prey problem

The classic Volterra predator-prey problem becomes a routine exercise using a graphing calculator like the TI-85. The model assumes the rates of population growth of predator-prey populations (foxes and rabbits) are related by the "highly coupled" first order differential equation system given by

$dF/dt = (-0.5 + 0.02R)F$

$dR/dt = (1 - 0.1F)R$

where $y = F(t)$ is the population of foxes at time t and $y = R(t)$ is the population of rabbits at time t (t measured in years).

Problem 4. Suppose there are 10 foxes and 50 rabbits at time $t = 0$ (today). What are the population graphs? How are the populations related over time? (Here a picture is as good as an analytic result!) Suppose the initial populations are changed. How do the population graphs change?

Solution: This simple system has no closed form solution. *Numerical methods are necessary*. Here is a TI-85 solution using its amazing built-in differential equation solver with graphics interface.

The graphs in Figures 8 and 9 show the population graphs over a 40 year time period with the given initial populations and with a change in the initial populations to 4 foxes and 20 rabbits.

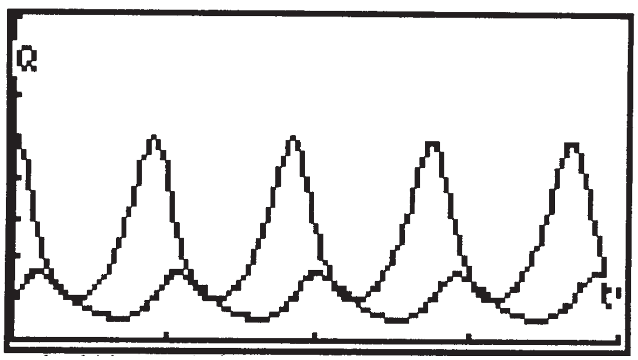

Figure 8. The fox and rabbit populations for 40 years starting with 10 foxes and 50 rabbits

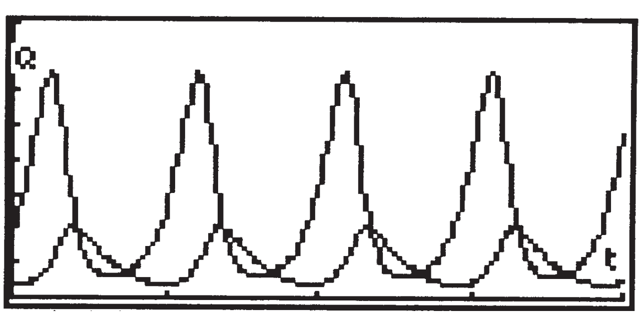

Figure 9. The fox and rabbit populations for 40 years starting with 4 foxes and 20 rabbits

Figure 10 shows the population patterns for various initial conditions in one view (orbits - the phase plane solution). These orbits are found by plotting the points $(F(t), R(t))$ for 6 different initial conditions (the beginning populations) which the TI-85 does automatically with an axes change selection.

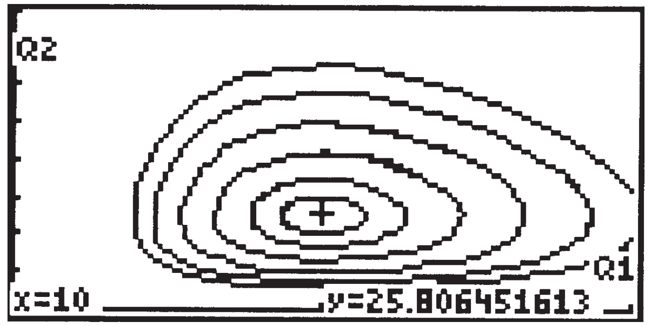

Figure 10. A phase plane solution to the fox-rabbit predator-prey problem

The student can be led to conjecture that perhaps there is a set of initial conditions that result in stable (constant) populations over time. The graphs suggest that if the starting population is 10 foxes and 25 rabbits then the populations will be stable. This result can then be confirmed analytically using paper and pencil calculus (when does $dF/dt = 0$?, etc.) and supported graphically.

Summary

Hopefully we have made the case that the types of graphing calculator activities represented by the four problems in this paper provide insight and understanding for *all* students in ways not possible with paper and pencil methods alone. These types of activities will also empower *and excite* calculus students in ways not possible with paper and pencil methods alone.

We readily admit that we have not yet made all the hard decisions regarding what content should be modified or deleted from the current calculus curriculum. The "jury" is still out! However, we believe our C^3E project is an important first step. Indeed, a necessary first step. A step of *enhancing* the traditional calculus curriculum with computer numerical and visual methods delivered by student use of inexpensive graphing calculators. We are also able to include many rich examples and illustrations that are possible for *all* students only with graphing calculators. Furthermore, graphing calculators promote sound mathematics teaching methods including cooperative learning, student investigations, and writing about mathematics.

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