Chapter 2

The Special Theory of Relativity

Read Chapter 2 of the hand-written notes

2.1 *Classical Relativity

Consider an observer, named O, who measures the position of an object in his coordinate system as $\vec{x} = (x, y, z)$, at time t. A second observer, named O', is in an inertial frame (no forces acting on the observer), but moving at linear velocity $\vec{u}$ with respect to O. This second observer measures the position of the object in his coordinate system to be $\vec{x}' = (x', y', z')$, at time t'. Now, we impose some constraints on our two observers to simplify the discussion. (It's not necessary to impose these constraints, it just makes the discussion much simpler.)

1. Both O and O' measure time in the same way, with a clock of *identical* design and function. Thus, any time interval between two events measured by O to be Δt is identical to the time interval $\Delta t'$ observed by O' for the same two events. In other words, the time difference between two events measured by both is the same. That is, $\Delta t = \Delta t'$.

2. At $t = 0$ and $t' = 0$, the coordinate systems of the two observers are perfectly aligned, with $\vec{x} = \vec{x}'$, or equivalently, $(x, y, z) = (x', y', z')$ at $t = 0$ and $t' = 0$. That is, the axes are all perfectly aligned, and the coordinate systems coincident.

3. Both O and O' measure distance (displacements) in the same way, with an instrument of *identical* design and function.

4. Both O and O' measure mass in the same way, with an instrument of *identical* design and function. The mass of the object seen by O and O' are identical.

Consequences:

1. Observer O sees observer O' moving with velocity $\vec{u}$, while observer O' sees observer O moving with velocity $-\vec{u}$.

2. Simultaneous events seen by observer O at time t are also seen by observer O' as simultaneous.

3. A displacement between two simultaneous events measured by O to be $\Delta\vec{x}$ is identical to the displacement $\Delta\vec{x}'$ observed by O' for the same two simultaneous events. This is illustrated in Figure 2.1.

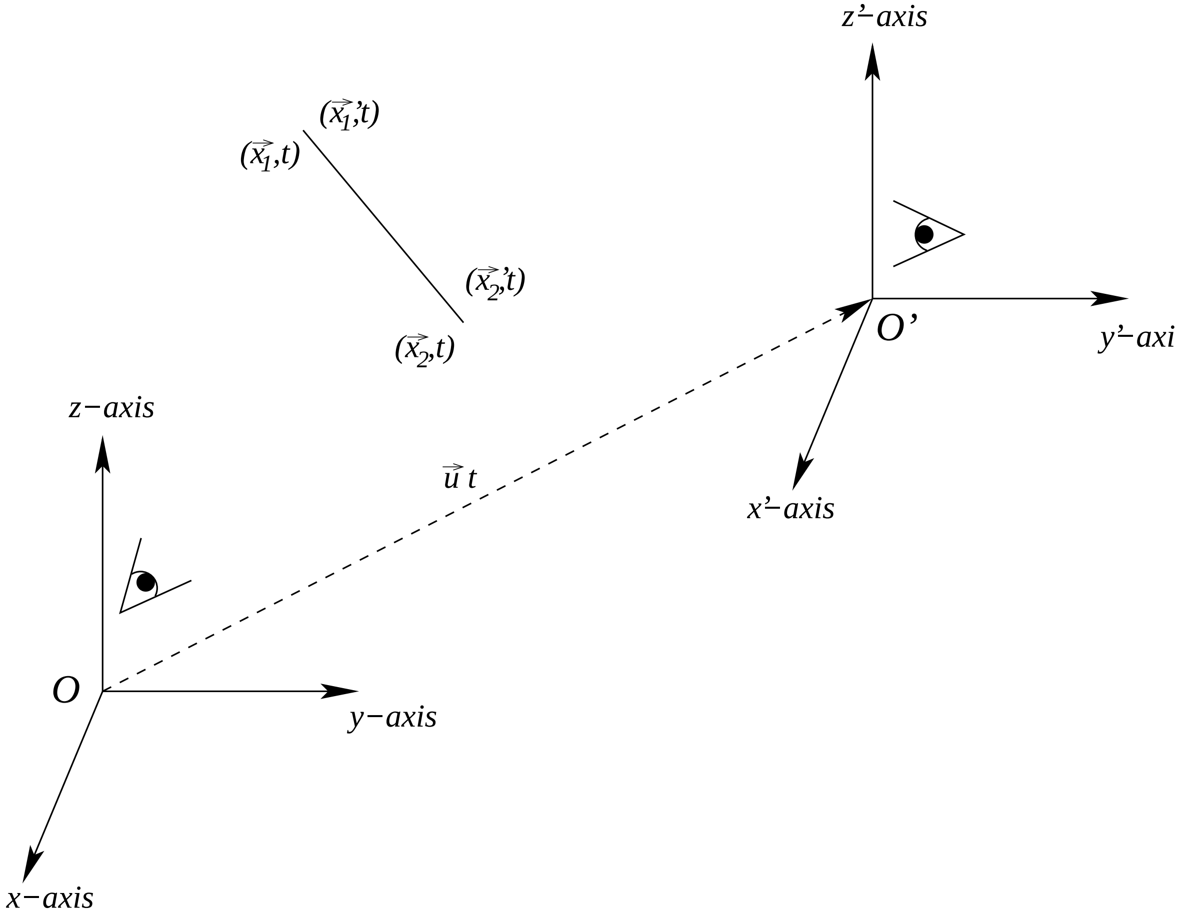

Figure 2.1: Galilean transformation of a displacement.

4. The coordinates of an event in one coordinate system may be related to the other by means of the *Galilean Transformation*:

$$\vec{x}'(t) = \vec{x}(t) - \vec{u}t$$

5. If $\vec{x}(t)$ represents the trajectory of a particle observed by observer O, then it is easily shown that the velocities are connected by:

$$\vec{v}'(t) = \vec{v}(t) - \vec{u} \ ,$$

since $\vec{u}$ is a vector that is constant in time. This shows that Newton's First Law of Motion holds in both frames. If $\vec{v}(t)$ is constant in time, then so is $\vec{v}'(t)$, and the motion (velocity) of the object persists in both frames unless acted on by another force.

6. The accelerations observed by the two observers, is easily shown to be constant,

$$\vec{a}'(t) = \vec{a}(t) \ .$$

Since $\vec{F} = ma$, the force on the object measured by both observers is the same, and Newton's Second Law of Motion applies equally in both frames.

If there are equal and opposing forces on the object in one frame, they are measured to be the same in the moving frame. Newton's Third Law of Motion is preserved.

Hence, the Classical Laws of Physics are the same in both frames.

2.2 *The Michelson-Morley Experiment

Oddly enough, the speed of light is not measured anymore, it has been assigned, via definition in 1983, that of an exact constant:

$$c = 299\ 792\ 458 \text{ m/s} \ .$$

You would only be wrong by about 0.1% if you said it was 3×10^8 m/s, and this approximation is often used in quick numerical calculations.

It is truly remarkable how slow the speed of light is! It takes light about $1/7^{\text{th}}$ of a second to circumnavigate the earth at the equator, and goes about a foot in a nanosecond.

So, if light really has a finite speed, and Galilean transformations are true, we should be able to measure how fast we are moving relative to it. That was the purpose of the Michelson-Morley experiment, the most famous failed experiment in physics[1].

Their experiment showed conclusively, that the speed of light is constant, no matter what direction the earth was moving. Albert Einstein's leap of imagination was to say that it is a fundamental property of light.

[1] That failed experiment got Albert Michelson a Nobel Prize in 1907.

*How the Michelson-Morley Experiment Works

Referring to the figure (it's missing!), let's imagine that the speed of the earth through the ether is u, directed from right to left, along the length $\overline{\mathrm{AC}}$. A pulse of radiation leaves S, the source, and travels upstream against the ether, and is split into two pulses by a "half-mirror" at A. One of the pulses goes along the length $\overline{\mathrm{AC}}$, gets reflected by a mirror at C, back again to the "half-mirror" at A and up to the detection apparatus, that is labelled by D. The half of the pulse that gets split off at A, goes along the length $\overline{\mathrm{AB}}$, gets reflected by the mirror at B, back again to the "half-mirror" at A and up to the detection apparatus at D.

The first pulse, once it leaves the splitter, takes total time

$$t_{||} = \frac{2\overline{\mathrm{AC}}}{c}\frac{1}{1-u^2/c^2} + \frac{\overline{\mathrm{AD}}}{c}\frac{1}{\sqrt{1-u^2/c^2}} \ , \tag{2.1}$$

to complete the journey. Here c is the speed of light (assuming the ether is at rest!).

Since we expect that $u \ll c$, we can do a series expansion of (2.1) in u^2/c^2 and find that, to lowest surviving order in u^2/c^2, that

$$t_{||} \approx \frac{2\overline{\mathrm{AC}}}{c}(1 + u^2/c^2) + \frac{\overline{\mathrm{AD}}}{c}(1 + \frac{u^2/c^2}{2}) \ , \tag{2.2}$$

which is an excellent approximation, when u is very small compared to c.

Using the same method, the second pulse takes time,

$$t_{\perp} = \frac{2\overline{\mathrm{AB}}}{c}\frac{1}{\sqrt{1-u^2/c^2}} + \frac{\overline{\mathrm{AD}}}{c}\frac{1}{\sqrt{1-u^2/c^2}} \ , \tag{2.3}$$

which, in the $u \ll c$ approximation is

$$t_{\perp} \approx \frac{2\overline{\mathrm{AB}}}{c}(1 + \frac{u^2/c^2}{2}) + \frac{\overline{\mathrm{AD}}}{c}(1 + \frac{u^2/c^2}{2}) \ . \tag{2.4}$$

The difference between them is

$$\Delta t_0 = t_{||} - t_{\perp} = 2\frac{\overline{\mathrm{AC}} - \overline{\mathrm{AB}}}{c} + \frac{\overline{\mathrm{AC}}}{c}\frac{2u^2}{c^2} - \frac{\overline{\mathrm{AB}}}{c}\frac{u^2}{c^2} \ . \tag{2.5}$$

That difference eliminates the contribution from $\overline{\mathrm{AD}}$. If $\overline{\mathrm{AC}}$ were exactly equal to $\overline{\mathrm{AB}}$, then the experiment would give a direct measurement of the speed of the earth through the ether. But that is impossible to do, because the difference would have to be immeasurably small.

So, they flipped the apparatus by $\pi/2$, so that S and D were in the same place, but B and C changed positions. This results in a time difference of,

$$\Delta t_{\pi/2} = t_{||} - t_{\perp} = 2\frac{\overline{\mathrm{AC}} - \overline{\mathrm{AB}}}{c} + \frac{\overline{\mathrm{AC}}}{c}\frac{u^2}{c^2} - \frac{\overline{\mathrm{AB}}}{c}\frac{2u^2}{c^2} \ , \tag{2.6}$$

and the difference between these two, is

$$\Delta t = \Delta t_0 - \Delta t_{\pi/2} = \frac{\overline{\mathrm{AC}} + \overline{\mathrm{AB}}}{2c}\frac{u^2}{c^2} \ , \tag{2.7}$$

thus ridding the experiment of the differences in distances between the half-mirror and the mirrors, and yielding a direct measure of the speed of the earth through the ether.

As the story unfolded, u was measured to be zero, no matter where the earth was in its yearly revolution around the sun, and the only conclusion that made sense was that light could propagate through a vacuum. That begged the question, "If there is no ether, then what can we say about the speed of light in different inertial frames?" That's exactly what Einstein had something to say about ...

2.3 *Einstein's Postulates

Einstein formulated his Special Theory of Relativity on two postulates, one of them a genuinely new idea:

1. The principle of relativity: *The laws of physics are the same in all inertial reference frames.*
2. Constant speed of light: *The speed of light has the same value in all inertial reference frames.*

The consequences of this idea are remarkable. All three of the constraints we applied to Galilean transformations: time, length and mass equivalence, must be undone. We will investigate these in the next section.

2.4 *The Lorentz Transformation

The Lorentz transformation relating two observers, O and O', where O' is moving along the positive x-axis with respect to O at speed u is:

$$
\begin{aligned}
x' &= \frac{x - ut}{\sqrt{1 - u^2/c^2}} \\
y' &= y \\
z' &= z \\
t' &= \frac{t - (u/c^2)x}{\sqrt{1 - u^2/c^2}} \; .
\end{aligned}
\tag{2.8}
$$

I'd like to introduce a more compact notation. Factors like u/c and $\sqrt{1 - u^2/c^2}$ occur so frequently that the following convenient shorthand notation is often used:

$$
\begin{aligned}
\beta_u &= u/c \\
\gamma_u &= \frac{1}{\sqrt{1 - \beta^2}} \; .
\end{aligned}
\tag{2.9}
$$

Frequently, when there is only one velocity in the discussion, the subscript u is dropped. β is the ratio of the velocity in question to the speed of light, while γ is related (as we shall see shortly) to the energy and momentum of a particle with mass.

The following property, a consequence of (2.8) is often employed to simplify expressions:

$$
\gamma^2 - \beta^2\gamma^2 = 1 \; . \tag{2.10}
$$

With this shorthand, the Lorentz transformation may be written:

$$
\begin{aligned}
x' &= \gamma(x - \beta ct) \\
y' &= y \\
z' &= z \\
ct' &= \gamma(ct - \beta x) \; .
\end{aligned}
\tag{2.11}
$$

***Lorentz Transformation of Position and Time with Arbitrary Velocity**

$O'(x', y', z', t') \to O(x, y, z, t)$, where the relative motion of O' with respect to O is along the positive $\vec{u}$-axis with speed u and direction $\hat{n}_u$:

$$
\begin{aligned}
ct' &= \gamma_u(ct - \vec{\beta}_u \cdot \vec{x}) \\
\vec{x}' &= \vec{x} - \gamma_u \vec{\beta}_u ct + (\gamma_u - 1)(\hat{n}_u \cdot \vec{x})\hat{n}_u \; .
\end{aligned}
\tag{2.12}
$$

*Length Contraction

With the Lorentz transformation, we are now in a position to obtain the formulae for length contraction and time dilation more simply.

Suppose that O measures a space and temporal displacement with coordinates at (x_1, t_1) and (x_2, t_2). In O''s frame, these coordinates correspond to (x'_1, t'_1) and (x'_2, t'_2). Thus, $\Delta x' = x'_2 - x'_1$ and $\Delta x = x_2 - x_1$ are related by (2.11) and found to be:

$$\Delta x' = \gamma(\Delta x - u\Delta t) \ , \tag{2.13}$$

where $\Delta t = t_2 - t_1$. Now, both O and O' make *simultaneous* measurements of the length of an object aligned along the direction of motion. O' measures the "proper" length, since the object is at rest in his frame. (By definition, the "proper" length is the length of an object as measured in its rest frame. It is always measured to be shorter if in motion relative to the frame in which the measurement is made.) However, O measures a different length L, given from (2.13) as:

$$L_0 = \gamma L, \text{ or } L = L_0/\gamma \ . \tag{2.14}$$

Thus, O measures the object as being "shorter".

*Length Contraction with Arbitrary Velocity

$$\begin{aligned} \vec{L} &= \vec{L}_0 - (\gamma_u - 1)\hat{n}_u(\hat{n}_u \cdot \vec{L}_0)/\gamma_u \\ \hat{n}_u \cdot \vec{L} &= (\hat{n}_u \cdot \vec{L}_0)/\gamma_u \\ \hat{n}_u \times \vec{L} &= \hat{n}_u \times \vec{L}_0 \end{aligned} \tag{2.15}$$

*Time Dilation

Now, consider the same situation in terms of time. According to (2.13),

$$c\Delta t' = \gamma(c\Delta t - \beta\Delta x) \ . \tag{2.16}$$

From (2.13),

$$\Delta x' = \gamma(\Delta x - \beta c\Delta t) \ , \tag{2.17}$$

or,

From (2.13),

$$\Delta x = \frac{\Delta x'}{\gamma} + \beta c \Delta t \ . \tag{2.18}$$

(2.18) $\rightarrow$ (2.16) $\Rightarrow$

$$c\Delta t' = \gamma(c\Delta t - \beta[\Delta x'/\gamma + \beta c\Delta t]) \ . \tag{2.19}$$

Observer O''s clock is stationary, so that $\Delta x' = 0$ and he measures the "proper" time Δt_0. (By definition, the "proper" time is the time as measured in the rest frame of the clock taking the measurement. If the clock is in motion, a stationary observer, comparing a time interval with an identical clock in his time frame, observes the moving clock to run slower.) After a little rearrangement, we obtain:

$$\Delta t_0 = \Delta t/\gamma, \text{ or } \Delta t = \gamma \Delta t_0 \ . \tag{2.20}$$

That it, a moving clock is always observed to run at a slower rate than that measured in a frame where the clock is stationary.

*Velocity Transformation

Just as distance and time differences vary according to the frame of reference, so do the velocities, as measured by two observers in different frames. We'll start with (2.11), and differentiate both sides with respect to t'.

$$\begin{aligned}
\frac{\mathrm{d}x'}{\mathrm{d}t'} &= \gamma_u \left(\frac{\mathrm{d}x}{\mathrm{d}t'} - u\frac{\mathrm{d}t}{\mathrm{d}t'} \right) \\
\frac{\mathrm{d}y'}{\mathrm{d}t'} &= \frac{\mathrm{d}y}{\mathrm{d}t'} \\
\frac{\mathrm{d}z'}{\mathrm{d}t'} &= \frac{\mathrm{d}z}{\mathrm{d}t'} \\
c &= \gamma_u (c\frac{\mathrm{d}t}{\mathrm{d}t'} - \beta_u \frac{\mathrm{d}x}{\mathrm{d}t'}) \ . \qquad (2.21)
\end{aligned}$$

Extracting 3 common factors results in:

$$v'_x = \gamma_u \left(\frac{\mathrm{d}t}{\mathrm{d}t'} \right) (v_x - u)$$

$$\begin{aligned} v'_y &= \left(\frac{\mathrm{d}t}{\mathrm{d}t'}\right) v_y \\ v'_z &= \left(\frac{\mathrm{d}t}{\mathrm{d}t'}\right) v_z \\ \frac{\mathrm{d}t}{\mathrm{d}t'} &= \frac{1}{\gamma_u(1-\beta_u v_x/c)} \ . \end{aligned} \tag{2.22}$$

Using the last of the above equations to replace for $\mathrm{d}t/\mathrm{d}t'$ in the previous 3, we obtain:

$$\begin{aligned} v'_x &= \frac{v_x - u}{1-\beta_u v_x/c} \\ v'_y &= \frac{v_y}{\gamma_u(1-\beta_u v_x/c)} \\ v'_z &= \frac{v_z}{\gamma_u(1-\beta_u v_x/c)} \ . \end{aligned} \tag{2.23}$$

***Lorentz Transformation of Velocity with Arbitrary Direction**

$\vec{v}'$ related to $\vec{v}$, where the relative motion of the inertial frame measuring $\vec{v}'$ with respect to the inertial frame measuring $\vec{v}$, is along the positive $\vec{u}$-axis with speed u:

$$\vec{v}' = \frac{\vec{v} - \vec{u}\gamma_u + (\gamma_u - 1)(\vec{u}\cdot\vec{v})\vec{u}/u^2}{\gamma_u(1-\vec{u}\cdot\vec{v}/c^2)} \tag{2.24}$$

***Lorentz Transformation of Dilation Factor**

$$\gamma_{v'} = \gamma_u\gamma_v(1-\vec{u}\cdot\vec{v}/c^2) \tag{2.25}$$

***Lorentz Transformation of Velocity and Dilation Factor**

$$\vec{v}'\gamma_{v'} = \gamma_v(\vec{v} - \vec{u}\gamma_u + (\gamma_u - 1)(\vec{u}\cdot\vec{v})\vec{u}/u^2) \tag{2.26}$$

Relativistic Energy and Momentum

The relativistic energy and momentum of an object with mass m and velocity $\vec{v}$:

$$\begin{aligned} E &= mc^2\gamma_v \\ \vec{p} &= m\gamma_v\vec{v} \\ (mc^2)^2 &= E^2 - (c\vec{p})^2 \end{aligned} \tag{2.27}$$

*Lorentz Transformation Energy and Momentum

The Lorentz transformation energy and momentum of an object with mass m and velocity $\vec{v}$ to an inertial frame that is moving along the positive x-axis with speed u:

$$\begin{aligned} E' &= \gamma_u(E - up_x) \\ \vec{p}'_x &= \gamma_u(p_x - uE/c^2) \\ \vec{p}'_y &= p_y \\ \vec{p}'_z &= p_z \end{aligned} \tag{2.28}$$

*Lorentz Transformation Energy and Momentum with Arbitrary Direction

The Lorentz transformation energy and momentum of an object with mass m and velocity $\vec{v}$ to an inertial frame that is moving along the positive $\vec{u}$-axis with speed u:

$$\begin{aligned} E' &= \gamma_u(E - \vec{u}\cdot\vec{p}) \\ \vec{p}' &= \vec{p} - \vec{u}\gamma_u E/c^2 + (\gamma_u - 1)(\vec{u}\cdot\vec{p})\vec{u}/u^2 \\ (mc^2)^2 &= E'^2 - (c\vec{p}')^2 \end{aligned} \tag{2.29}$$

*The Relativistic Doppler Effect

The relativistic Doppler effect, measured along the relative velocity vector between two objects, is given by:

$$\frac{\nu}{\nu_0} = \sqrt{\frac{1 \pm \beta}{1 \mp \beta}}\ , \tag{2.30}$$

where ν_0 is the frequency of the light source in the rest frame of the source. The top signs in the numerator and denominator of (2.30) signify that the source and observer are approaching each other, while the bottom signs signify that the source and observer are receding from each other. (Here, β is the relative speed of the source as seen by the observer.)

If the relative line of motion is different from the direction of observation, one can show that:

$$\frac{\nu}{\nu_0} = \gamma(1 - \hat{n}\cdot\vec{\beta})\ , \tag{2.31}$$

where $\vec{\beta} = \vec{u}/c$, and $\hat{n}$ is a unit vector along the line from the observer to the source.

2.5 Relativistic Dynamics

In this section we concern ourselves, primarily, with two-body scattering of relativistic particles, including photons. We start with some review of kinematic variables.

Symbol	= Expression	Interpretation
v		speed of a particle with mass
c		speed of light, speed of massless particles
β, or β_v	v/c	speed (in units of c) of a particle with mass $0 \leq \beta < 1$
γ or γ_v	$(1-\beta^2)^{-1/2}$	"energy factor", "dilation factor", "contraction factor"
		γ is often used as a symbol to represent a photon
	mc^2	rest mass energy of a particle with mass
E	$mc^2\gamma$	total energy of a particle with mass (rest + kinetic energy)
E_γ		total (or kinetic) energy of a photon (or massless particle)
K	$mc^2(\gamma-1)$	kinetic energy of a particle with mass
$\vec{p}$	$mc\vec{\beta}\gamma$, $m\vec{v}\gamma$	momentum of a particle with mass
$\vec{p}_\gamma$	E_γ/c	magnitude of momentum of a photon, or particle without mass
$(mc^2)^2$	$E^2-(pc)^2$	fundamental relation linking m, E, and $\lvert\vec{p}\rvert^2$
1	$\gamma^2(1-\beta^2)$	useful property of γ and β

Table 2.1: Relativistic kinematic variables

Non-relativistic limit

When performing relativistic calculations, one technique for verifying your result is to determine the non-relativistic limit. Generally, this is done by making a Taylor expansion in β, (See Chapter **??**.) and keep leading order expressions that express the non-relativistic limit. Factors of β should be replaces by cv, and the final result should resemble:

$$\lim_{\beta\to 0}(\text{Relativistic expression}) = (\text{Non}-\text{relativistic expression}) + O(1/c^n)\ , \tag{2.32}$$

where $n \geq 1$. Finally the non-relativistic limit is obtained by setting the $O(1/c^n)$ expressions to zero. Note that some expressions are intrinsically relativistic and not reducible to non-relativistic limits. For example, rest mass energy, and photon kinematic variables are some that we have encountered to far.

For example, the kinetic energy of a particle of mass m, in the limit that $\beta \to 0$ is:

$$\lim_{\beta\to 0} K = \lim_{\beta\to 0} mc^2(\gamma-1) = \frac{1}{2}mv^2 + O(1/c^2)\ , \tag{2.33}$$

while its momentum is

$$\lim_{\beta \to 0} mc\vec{\beta}\gamma = m\vec{v} + O(1/c^2) \ , \tag{2.34}$$

Relativistic Collision Kinematics

We now repeat the discussion of Section **??** but include the effect of relativistic speeds.

Consider the collision of two moving particles with masses m_1 and m_2, producing particles m_a and m_b following the collision. We conserve *total* energy and momentum, to obtain the following equations:

CoE⇒

$$m_1c^2\gamma_1 + m_2c^2\gamma_2 = m_ac^2\gamma_a + m_bc^2\gamma_b \ , \tag{2.35}$$

We note that Q is the zero-speed limit of (2.35) and is included automatically in the subsequent analysis.

Co$\vec{M}$ ⇒

$$m_1c\vec{\beta}_1\gamma_1 + m_2c\vec{\beta}_2\gamma_2 = m_ac\vec{\beta}_a\gamma_a + m_bc\vec{\beta}_b\gamma_b \ . \tag{2.36}$$

Solution Strategies

How we manipulate (2.35) and (2.36) depends on what information we know, and what information we wish to extract. We shall discuss the most common situation now, and leave some of the special cases to the examples and problems.

The most common situation involves the scattering of a known projectile from a known target, where initial masses and velocities are known, to a set of final particles whose masses are known, but only the lighter product particle leaves the collision area. (For example, a proton scattering from a stationary nucleus, with a transformed nucleus and a neutron in the final state.) Since the heavier product particle stays in the collision area, it is unobserved, hence its velocity is not measurable, and we strive to eliminate it. We proceed as follows.

Reorganize (2.35) and (2.36) as follows, to put the kinematics of the "b" particle on the right hand side (RHS) of the equations:

From the CoE equation:

$$m_1c^2\gamma_1 + m_2c^2\gamma_2 - m_ac^2\gamma_a = m_bc^2\gamma_b \ , \tag{2.37}$$

and the Co$\vec{M}$ equation:

$$m_1c\vec{\beta}_1\gamma_1 + m_2c\vec{\beta}_2\gamma_2 - m_ac\vec{\beta}_a\gamma_a = m_bc\vec{\beta}_b\gamma_b \ . \tag{2.38}$$

Dividing the square of (2.37) by c^4 and subtracting the square of (2.38) divided by c^2 gives us:

$(2.37)^2/c^4$ - $(2.38)^2/c^2 \Rightarrow$

$$(m_1\gamma_1 + m_2\gamma_2 - m_a\gamma_a)^2 - (m_1\vec{\beta_1}\gamma_1 + m_2\vec{\beta_2}\gamma_2 - m_a\vec{\beta_a}\gamma_a)^2 = m_b^2\gamma_b^2(1-\beta_b^2) \ . \tag{2.39}$$

The motivation for this arithmetical manipulation is now evident: no factors of c appear, and most importantly, we may exploit the $\beta\gamma$ relation, $\gamma^2(1-\beta^2)$ to great effect. Doing so results in:

$$m_1^2+m_2^2-m_a^2-m_b^2+2m_1m_2\gamma_1\gamma_2(1-\vec{\beta_1}\cdot\vec{\beta_2}) = 2m_1m_a\gamma_1\gamma_a(1-\vec{\beta_1}\cdot\vec{\beta_a})+2m_2m_a\gamma_2\gamma_a(1-\vec{\beta_2}\cdot\vec{\beta_a}) \ . \tag{2.40}$$

We see that we have isolated the only unknown quantity, $\vec{\beta_a}$, and by inference γ_a on the RHS of (2.40). We may further reduce this equation by noting that the mass term on the LHS may be rewritten as follows:

$$m_1^2 + m_2^2 - m_a^2 - m_b^2 = (m_1 + m_2 - m_a - m_b)(m_1 + m_2 + m_a + m_b) = (\Delta M)M \ , \tag{2.41}$$

where $M = M_i + M_f = m_1 + m_2 - m_a - m_b$ is the sum of the masses of the initial and final particles, while $\Delta M = M_i - M_f = m_1 + m_2 - m_a - m_b$ is the difference of the sum of the initial masses and the sum of the final masses of the particles participating in the reaction. We also note that ΔMc^2 is the reaction Q-value discussed previously. Note how it appears naturally in the analysis, while it has to be "tacked on" in an *ad hoc* fashion in the non-relativistic analysis.

So, finally we write:

$$(\Delta M)M+2m_1m_2\gamma_1\gamma_2(1-\vec{\beta_1}\cdot\vec{\beta_2}) = 2m_1m_a\gamma_1\gamma_a(1-\vec{\beta_1}\cdot\vec{\beta_a})+2m_2m_a\gamma_2\gamma_a(1-\vec{\beta_2}\cdot\vec{\beta_a}). \tag{2.42}$$

Having derived a relativistic result, we should check that it gives the correct non-relativistic limit. To do this, we note that we can rewrite

(2.39) as:

$$\vec{P}^2 - 2m_b(K+Q) = (Q+K)^2/c^2 \ , \tag{2.43}$$

where

$$\vec{P} \equiv m_1c\vec{\beta_1}\gamma_1 + m_2c\vec{\beta_2}\gamma_2 - m_ac\vec{\beta_a}\gamma_a \ ,$$

and

$$K \equiv m_1c^2(\gamma_1 - 1) + m_2c^2(\gamma_2 - 1) - m_ac^2(\gamma_a - 1) \ .$$

(2.43) is fully relativistic. Obtaining the non-relativistic form is tantamount to replacing $\vec{P}$ and K with their non-relativistic counterparts (given below (**??**) and setting the $1/c^2$ on the RHS of (2.43) to zero. This agrees with the non-relativistic form given in (**??**), and we have verified the non-relativistic limit of our relativistic expression. It is not absolutely foolproof, however, verifying non-relativistic limits is a very important verification tool.

2D relativistic elastic collision, equal masses

Problem: Find the opening angle of the resultant particles, when a relativistic particle of mass m, collides with an equal mass, at rest. Show explicitly the transition to the well-known non-relativistic limit?

Solution:

1. *Set up the CoE and Co$\vec{M}$ equations assuming "2" is at rest:*

$$mc^2\gamma_0 + mc^2 = mc^2\gamma_1 + mc^2\gamma_2 \tag{2.44}$$

$$mc\vec{\beta}_0\gamma_0 = mc\vec{\beta}_1\gamma_1 + mc\vec{\beta}_2\gamma_2 \tag{2.45}$$

2. *We require the angle between the resultant trajectories. The cosine of this angle is obtained by $\vec{\beta}_1 \cdot \vec{\beta}_2$. To isolate this: $(2.44)^2/(mc^2)^2$ - $(2.45)^2/(mc)^2 \Rightarrow$*

$$\gamma_1\gamma_2\vec{\beta}_1 \cdot \vec{\beta}_2 = \gamma_1\gamma_2 - \gamma_0 \ .$$

If we let α represent the opening angle of the outgoing particles, we may manipulate the above (Show this!) equation to be:

$$\cos\theta = \sqrt{\frac{K_1K_2}{(K_1 + 2mc^2)(K_2 + 2mc^2)}} \ , \tag{2.46}$$

explicitly showing the dependence on the outgoing kinetic energies.

This relationshop is plotted in Figure 2.2 for a logarithmic spacing of K_1/K_0 between 0.01 and 1000.

Taking $c \to \infty$ yields the expected result, that the opening angle is $\pi/2$, in a non-relativistic analysis. This is tantamount to saying that $K_a << mc^2$ <u>and</u> $K_b << mc^2$. However, (2.46) contains even more information. It says that if either outgoing particle is non-relativistic, that is, $K_1 << mc^2$ <u>or</u> $K_2 << mc^2$, the opening angle tends to $\pi/2$. Finally, if <u>either</u> outgoing particle is at rest, the opening angle is $\pi/2$, exactly as in the non-relativistic case, and also true for the relativistic case. It is a consequence of the conservation of energy and momentum in both non-relativistic and relativistic formalisms.

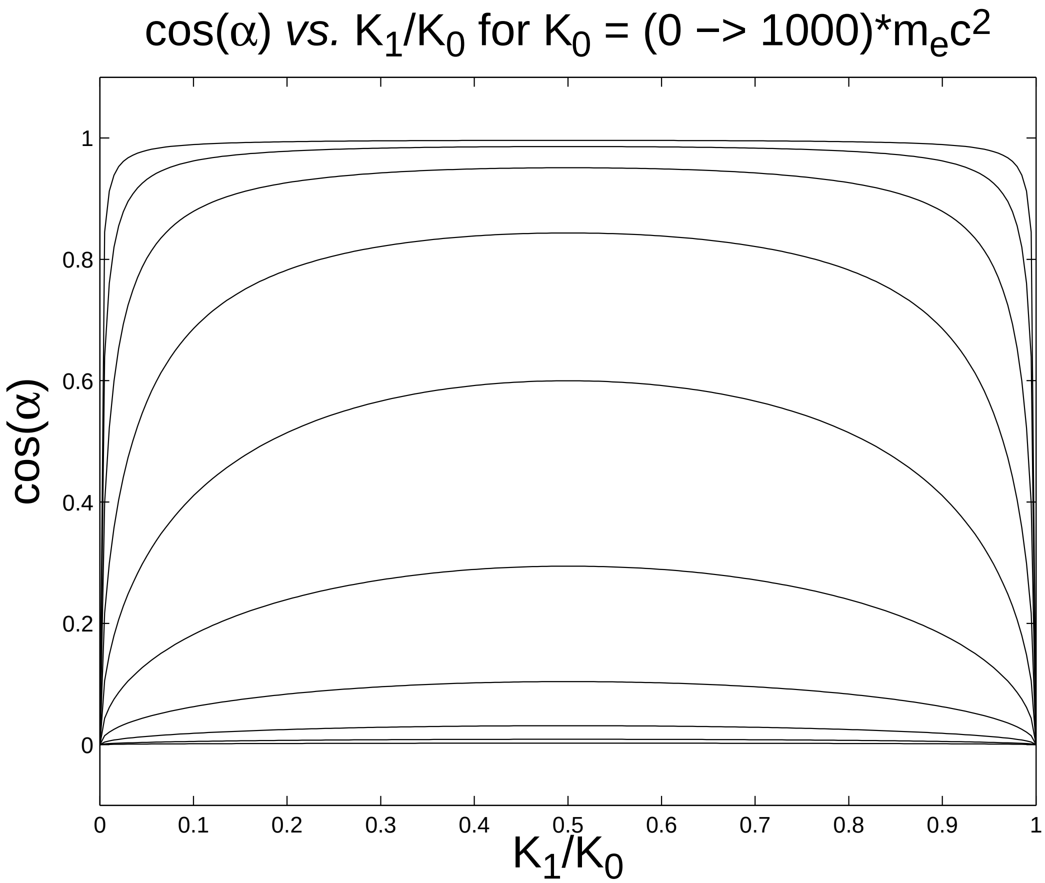

Figure 2.2: $\cos\theta$ *vs.* K_1/K_0

Interpretation: In the case that both outgoing particles are relativistic, (2.46) demonstrates that the opening angle is less than $\pi/2$*. Since* $K_1 = K_0 - K_2$*, it also follows that there must be a particular sharing of the initial kinetic energy,* K_0*, with that of the outgoing particles, that minimizes the opening angle. Since (2.46) is symmetric under the interchange* $1 \leftrightarrow 2$*, the sharing of kinetic energy can not be asymmetric. Therefore the only symmetric way of sharing the energy is to give each outgoing particle half. You can prove this mathematically*[2]*, but making the argument this way is more fun. Therefore at the midpoint,* $K_1 = K + 2 = K_0/2$*, and the minimum opening angle can be shown to be given by:*

[2]The mathematician within me is prone to say that a mathematical proof is always more general and powerful than a physical one. Historically, though, it is well known that physicists, even theoretical ones, are better socially-adjusted than mathematicians. Then again, mathematicians ... (I digress). Here's the proof. Consider a function, $f(x)$, that is symmetric about the origin. (Any different point of symmetry may be used, but it can be translated back to the origin.) It follows that the function's first derivative is antisymmetric, and must pass through the origin. Hence, the origin is the location of an extremum of $f(x)$. *Q.E.D.*

$$\cos\alpha_{\rm min} = \frac{K_0}{K_0 + 4mc^2} \ . \tag{2.47}$$

This relationshop is plotted in Figure 2.3 for a logarithmic spacing of K_1/K_0 between 0.01 and 1000.

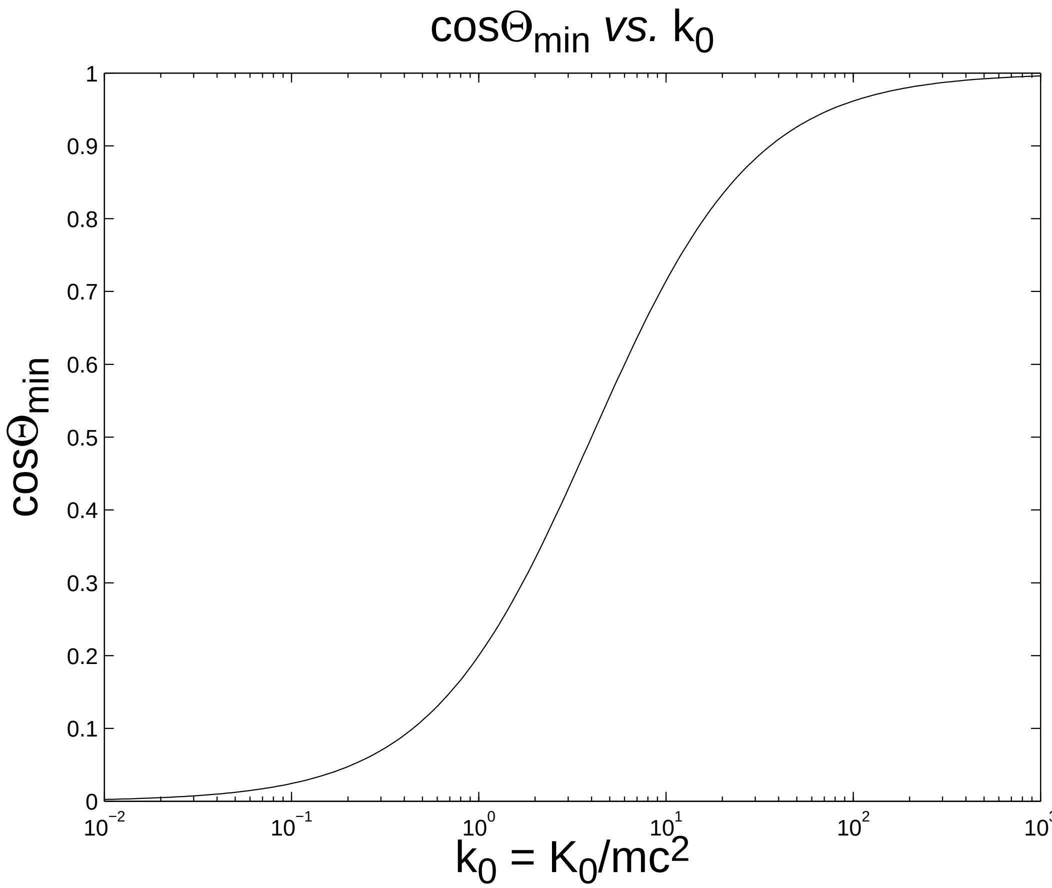

Figure 2.3: $\cos\theta_{\rm min}$ *vs.* K_1/K_0

We note from (2.47) that the expected non-relativistic limit is obtained again. However, as the incoming kinetic energy is extended upwards into the relativistic range, the energy is increasingly carried into the forward direction. This is the principle upon which particle ray-guns operate. It is also responsible for spectacular accidents when charged high energy particle beams are mistakenly steered into beam pipes and magnets.

Zero-Momentum Frame

We can also perform ant calculation in the zero-momentum frame. in these set of notes, we don't exploit the zero-momentum frame extensively, since the laboratory frame makes more sense for nuclear engineering and radiological applications (fixed targets, β- and γ-decay). High-energy physics exploit the zero-momentum frame extensively, since particle-antiparticle colliders operate in the zero-momentum frame. We shall exploit it, however, for two important illustrations:

1. Particle/antiparticle creation with mass

Consider a collision of two photons, going in exact opposite directions, each with energy E_0. E_0 is arranged so that after the collision, a particle and antiparticle, each with mass m, is at rest. Thus, by CoE, $E_0 = 2mc^2$.

Now consider that a different observer, moving along the direction of one of the photons, observes the event. In his frame of motion, the pair of particles is moving in the direction opposite to him. In his frame of motion, his expressions for CoE and Co$\vec{M}$ are:

$$2mc^2\gamma = E_+ + E_- \tag{2.48}$$
$$2mc\beta\gamma = (E_+ + E_-)/c\ , \tag{2.49}$$

where β is the observer's velocity with respect to the zero-momentum frame, E_+ is the higher energy photon he observes, with E_- is the lesser energy photon in his frame. By manipulating the equations in the now familiar way, we may relate the energy of the photon in the moving frame, relative to the rest frame. The result is:

$$E_\pm = E_0\sqrt{\frac{1\pm\beta}{1\mp\beta}}\ , \tag{2.50}$$

where we consistently use only the upper or lower signs in the expressions involving $\pm$ or $\mp$.

Thus, we have derived the Doppler effect stated in (2.30), whereby motion toward a photon increases its energy, and motion away decreases its energy.

2. Particle/antiparticle decay into photons

Here we consider a particle with mass m, β_0, γ_0 on a collision course with its antiparticle, moving in the exact opposite direction. They annihilate, producing two photons, each with energy E_0, moving in exact opposite directions, along the original direction of motion. An observer, moving with parameters β and γ, along the original direction of motion, observes the same annihilation, and his CoE and Co$\vec{M}$ equations take the forms:

$$mc^2\gamma_+ + mc^2\gamma_- = E_+ + E_- \tag{2.51}$$
$$mc\gamma_+ + mc\gamma_- = (E_+ + E_-)/c\ . \tag{2.52}$$

Here, the "+" refers to the more energetic particle and photon in the frame of the observer. The arithmetic is a little more involved than in the previous example, but, after some work, we can conclude that:

$$\gamma_\pm = \gamma_0\gamma(1 \pm \beta\beta_0) \ , \tag{2.53}$$

which expresses a "Doppler shift", but for particles with mass.

Applying (2.53), imagine that the observer is travelling at exactly β_0, putting one of the charged particles in the rest frame. The higher energy electron will have a "γ-shift" of approximately $2\gamma_0^2$. For example, the Stanford Linear Accelerator produces electrons and electrons with energies of about 50 GeV, a γ-shift of about 10^5. The collision of these particles in the zero-momentum frame, is equivalent to a fixed target γ-shift of 2×10^5. It is no wonder that particle-antiparticle colliders are such an important research tool.

Sticky collisions/exploding masses

Finally, we end this section with a discussion on inelastic collisions.

In the last chapter, we inferred the Q-value of a sticky collision. Let's reformulate the problem in a relativistic framework. Imagine that a particle of mass m_0, with speed v_0, strikes an identical particle at rest, and they fuse. You can not balance the CoE and Co$\vec{M}$ equations if the masses are allowed to remain unchanged. One finds, in this case that the fused mass is

$$m = 2m_0\sqrt{\frac{1+\gamma_0}{2}} \ .$$

The increase in mass is due to the increase in internal energy of the mass m.

Similarly, if a mass m explodes into two equal masses, m_0, you may show that

$$m = 2m\gamma_0 \ .$$

In other words, internal energy is converted into kinetic energy of the resultant particles.

2.6 Questions

Answer these questions (on paper or in your head). If you can't find a good answer, re-read the relevant sections of Chapter 2. If you still can't find a good answer, ask a colleague, a TA, or a Prof.

Some of these questions may appear on assignments or exams.

2.7 Problems

If you find some of these problems interesting, attempt them on paper. If you can't find a good answer, re-read the relevant sections of Chapter 2. If you still can't find a good answer, try working it out with a colleague, ask a TA, or a Prof.

Some of these problems may appear on assignments or exams.