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Networks with small stretch number

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Abstract

In a previous work, the authors introduced the class of *graphs with bounded induced distance of order k* (BID(k) for short), to model non-reliable interconnection networks. A network modeled as a graph in BID(k) can be characterized as follows: if some nodes have failed, as long as two nodes remain connected, the distance between these nodes in the faulty graph is at most k times the distance in the non-faulty graph. The smallest k such that $G \in \mathrm{BID}(k)$ is called *stretch number* of G. We show an odd characteristic of the stretch numbers: every rational number greater or equal 2 is a stretch number, but only discrete values are admissible for smaller stretch numbers. Moreover, we give a new characterization of classes $\mathrm{BID}(2-1/i)$, $i \geqslant 1$, based on forbidden induced subgraphs. By using this characterization, we provide a polynomial time recognition algorithm for graphs belonging to these classes, while the general recognition problem is Co-NP-complete.

Keywords: Graph class characterization; Recognition problem; Stretch number; Distance-hereditary graphs; Hierarchy of graph classes

1. Introduction

The main function of a network is to provide connectivity between the sites. In many cases it is crucial that connectivity is preserved even in the case of (multiple) faults in sites. Even if the connectivity between nodes is preserved, distances usually increase in case of faults because shortest paths could be no longer available.

In this work, that concerns bounded distances, our goal is to investigate about networks in which distances between sites remain *small* in the case of multiple faulty sites. As the underlying model, we use unweighted graphs, and measure a distance between two nodes

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by the number of arcs of a shortest path connecting them. We model a network in which node faults have occurred by the subnetwork induced by the non-faulty components. Using this model, in [7] we have introduced the class $\mathrm{BID}(k)$ of *graphs with bounded induced distance* of order k. A network modeled as a graph in $\mathrm{BID}(k)$ can be characterized as follows: if some nodes have failed, as long as two nodes remain connected, the distance between these nodes in the faulty graph is at most k times the distance in the non-faulty graph.

Some characterization, complexity, and structural results about $\mathrm{BID}(k)$ are given in [7]. In particular, the concept of *stretch number* has been introduced: the stretch number $s(G)$ of a given graph G is the smallest rational number k such that G belongs to $\mathrm{BID}(k)$. Given the relevance of graphs in $\mathrm{BID}(k)$ in the area of communication networks, our purpose is to provide characterization, algorithmic, and existence results about graphs having small stretch number.

Results: We first investigate graphs having stretch number at most 2. In this context we show that: (i) there is no graph G with stretch number $s(G)$ such that $2 - 1/i < s(G) < 2 - 1/(i+1)$, for each integer $i \geqslant 1$ (this fact was conjectured in [7]); (ii) there exists a graph G such that $s(G) = 2 - 1/i$, for each integer $i \geqslant 1$. These results give a partial solution to the following more general problem: Given a rational number k, is k an *admissible stretch number*, i.e., is there a graph G such that $s(G) = k$? We complete the solution to this problem by showing that every rational number $k \geqslant 2$ is an admissible stretch number (note that an irrational number cannot be a stretch number). Finally, we give a characterization result in term of forbidden subgraphs for the class $\mathrm{BID}(2 - 1/i)$, for each integer $i > 1$. This result has been obtained by extending the technique used in [7] to show a similar characterization for the class $\mathrm{BID}(3/2)$. In turn, this new result allows us to design a polynomial time algorithm to solve the recognition problem for the class $\mathrm{BID}(2 - 1/i)$, for each $i \geqslant 1$ (if k is not fixed, this problem is Co-NP-complete for the class $\mathrm{BID}(k)$ [7]). Unfortunately, the running time of this algorithm is exponential in i (more precisely, it is bounded by $\mathrm{O}(n^{3i+2})$). We conclude the paper by showing that such an algorithmic approach cannot be used for class $\mathrm{BID}(k)$, for each integer $k \geqslant 2$.

Related works: In literature there are several papers devoted to fault-tolerant network design, mainly starting from a given desired topology and introducing fault-tolerance to it (e.g., see [4,16,20]). The approach used in this paper if followed by other works.

In [15], authors give characterizations for graphs in which *no delay* occurs in the case that a *single* node fails. These graphs are called *self-repairing*. In [9], authors introduce and characterize new classes of graphs that, even when a multiple number of *edges* have failed, guarantee constant stretch factors k between nodes which remain connected. In a first step, they do not limit the number of edge faults at all, allowing for *unlimited* edge faults. Secondly, they examine the case where the number of edge faults is *bounded* by a value ℓ. The corresponding graphs are called k-self-spanners and (k, ℓ)-self-spanners, respectively. In both cases, the names are motivated by strong relationships to the concept of *k-spanners* [22]. Related works are also those concerning distance-hereditary graphs [19]. In fact, the class of distance-hereditary graphs is the class $\mathrm{BID}(1)$, and graphs with bounded induced distance can be also viewed as a their parametric extension (in fact, BID(k) graphs are mentioned in the survey [2] as k-distance-hereditary graphs). Distance-hereditary graphs have been investigated to design interconnection network topologies [6,12,14], and several papers have been devoted to them (e.g., see [1,3,5,11,13,17,21,23]).

The remainder of this paper is organized as follows. Notations and basic concepts used in this work are given in Section 2. In Section 3 we recall definitions and results from [7]. Section 4 shows the new characterization results, and in Section 5 we answer the question about admissible stretch numbers. In Section 6 we give the complexity result for the recognition problem for the class BID($2 - 1/i$), for every integer $i \geqslant 1$, by showing a polynomial time recognition algorithm, and in Section 7 we give some final remarks.

2. Notation

In this work we consider finite, simple, loop-less, undirected and unweighted graphs $G = (V, E)$ with node set V and edge set E. We use standard terminologies from [2,18], some of which are briefly reviewed here.

A *subgraph* of G is a graph having all its nodes and edges in G. Given a subset S of V, the *induced subgraph* $\langle S\rangle$ of G is the maximal subgraph of G with node set S. $|G|$ denotes the cardinality of V. If x is a node of G, by $N_G(x)$ we denote the *neighbors* of x in G, that is, the set of nodes in G that are adjacent to x. We write $N(x)$ when no ambiguity occurs. $G - S$ is the subgraph of G induced by $V \setminus S$.

A sequence of pairwise distinct nodes $(x_0, \ldots, x_n)$ is a *path* in G if $(x_i, x_{i+1}) \in E$ for $0 \leqslant i < n$, and is an *induced path* if $\langle\{x_0, \ldots, x_n\}\rangle$ has n edges. Two nodes x and y are *connected* in G if exist a path $(x, \ldots, y)$ subgraph of G. A graph is *connected* if every pair of nodes is connected.

A *cycle* in G is a path $(x_0, \ldots, x_{n-1})$ where also $(x_0, x_{n-1}) \in E$. We denote by C_n the class of cycles with n nodes; sometimes, when no ambiguity occurs, we use C_n to denote a specific instance of a cycle with n nodes. Two nodes x_i and x_j are *consecutive* in C_n if $j = (i + 1) \bmod n$ or $i = (j + 1) \bmod n$. A *chord* of a cycle is an edge joining two non-consecutive nodes in the cycle. H_n denotes a *hole*, i.e., a cycle C_n, $n \geqslant 5$, without chords. The *chord distance* of a cycle C_n is denoted by $cd(C_n)$, and it is defined as the minimum number of consecutive nodes in C_n such that every chord of C_n is incident to some of such nodes (see Fig. 1). We assume $cd(H_n) = 0$.

The length of a shortest path between two nodes x and y in a graph G is called *distance* and is denoted by $d_G(x, y)$. Moreover, the length of a longest induced path between them is denoted by $D_G(x, y)$. We use the symbols $p_G(x, y)$ and $P_G(x, y)$ to denote a shortest and a longest induced path between x and y, respectively. Sometimes, when no ambiguity occurs, we use $p_G(x, y)$ and $P_G(x, y)$ to denote the sets of nodes belonging to the corresponding paths. $I_G(x, y)$ denotes the set containing all the nodes (except x and y) that belong to a shortest path from x to y.

If x and y are two nodes of G such that $d_G(x, y) \geqslant 2$, then $\{x, y\}$ is a *cycle-pair* if there are two induced paths $p_G(x, y)$ and $P_G(x, y)$ such that $p_G(x, y) \cap P_G(x, y) = \{x, y\}$. In other words, if $\{x, y\}$ is a cycle-pair, then the set $p_G(x, y) \cup P_G(x, y)$ induces a cycle in G. In Fig. 1 there is no cycle-pair that induces the whole graph G, but, for example,

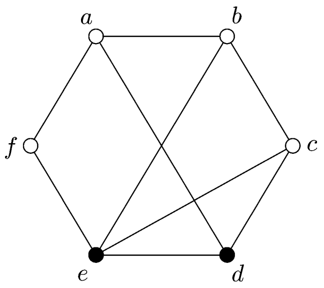

Fig. 1. The chord distance of this C_6 graph is 2 because nodes d and e are consecutive and every chord is incident to one of them. Moreover, there is no other set with less then 3 nodes with the same properties.

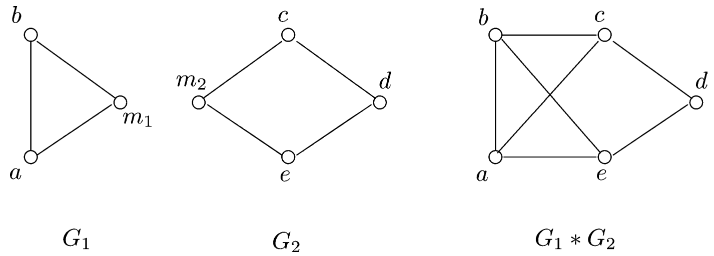

Fig. 2. The split composition $G_1 * G_2$ of G_1 and G_2 with respect to m_1 and m_2.

$\{c, f\}$ is a cycle-pair for the cycle $\langle\{a, b, c, e, f\}\rangle$ induced by $p_G(c, f) = (c, e, f)$ and $P_G(c, f) = (c, b, a, f)$.

Let G_1, G_2 be graphs having node sets $V_1 \cup \{m_1\}$, $V_2 \cup \{m_2\}$ and edge sets E_1, E_2, respectively, where $\{V_1, V_2\}$ is a partition of V and $m_1, m_2 \notin V$. The *split composition* [10] of G_1 and G_2 with respect to m_1 and m_2 is the graph $G = G_1 * G_2$ having node set V and edge set $E = E_1' \cup E_2' \cup \{(x, y) \mid x \in N(m_1), y \in N(m_2)\}$, where $E_i' = \{(x, y) \in E_i \mid x, y \in V_i\}$ for $i = 1, 2$ (see Fig. 2).

3. Basic definitions and results

In this section we recall from [7] some definitions and results useful in the remainder of the paper.

Definition 3.1 [7]. Let k be a real number. A graph $G = (V, E)$ is a *bounded induced distance graph of order* k if for each connected induced subgraph G' of G:

$$d_{G'}(x, y) \leqslant k \cdot d_G(x, y), \quad \text{for each } x, y \in G'.$$

The class of all the bounded induced distance graphs of order k is denoted by $\mathrm{BID}(k)$.

Note that the definition holds for both connected and disconnected graphs. The following facts hold:

– A graph G is distance-hereditary if and only if $G \in \mathrm{BID}(1)$;
– $\mathrm{BID}(k_1) \subseteq \mathrm{BID}(k_2)$, for each $k_1 \leqslant k_2$;
– Every class $\mathrm{BID}(k)$ is hereditary, i.e., if $G \in \mathrm{BID}(k)$, then $G' \in \mathrm{BID}(k)$ for every induced subgraph G' of G.

Definition 3.2 [7]. Let G be a graph, and $\{x, y\}$ be a pair of connected nodes in G. Then:

(1) the *stretch number* $s_G(x, y)$ *of the pair* $\{x, y\}$ is given by $s_G(x, y) = \frac{D_G(x,y)}{d_G(x,y)}$;
(2) the *stretch number* $s(G)$ of G is the maximum stretch number over all possible pairs of connected nodes, that is, $s(G) = \max_{\{x,y\}} s_G(x, y)$;
(3) $\mathcal{S}(G)$ is the set of all the pairs of nodes inducing the stretch number of G, that is, $\mathcal{S}(G) = \{\{x, y\} \mid s_G(x, y) = s(G)\}$.

The stretch number of a graph determines the minimum class which a given graph G belongs to since the shortest path between any pair of nodes in any induced subgraph is an induced path in the original graph. In fact, $s(G) = \min\{t\colon\ G \in \mathrm{BID}(t)\}$. As a consequence, $G \in \mathrm{BID}(k)$ if and only if $s(G) \leqslant k$.

Lemma 3.3 [7]. *Let* $G \in \mathrm{BID}(k)$, *and* $s(G) > 1$. *Then, there exists a cycle-pair* $\{x, y\}$ *that belongs to* $\mathcal{S}(G)$.

In Fig. 1, the represented graph G belongs to $\mathrm{BID}(3/2)$, moreover both $\{a, c\}$ and $\{c, f\}$ are cycle-pairs in $\mathcal{S}(G)$.

Theorem 3.4 [7]. *Let* G *be a graph and* $k \geqslant 1$ *a real number. Then,* $G \in \mathrm{BID}(k)$ *if and only if* $cd(C_n) > \lceil \frac{n}{k+1} \rceil - 2$ *for each cycle* C_n, $n > 2k + 2$, *of* G.

To find the class with minimum order which a graph belongs to, Lemma 3.3 and Theorem 3.4 assure that it is enough to study only chord distances of induced subgraphs forming cycles.

Theorem 3.5. *Let* G *be a graph such that* $s(G) < 2$. *Then,* G *does not contain a cycle* C_n, *with* $n \geqslant 6$ *and* $cd(C_n) \leqslant 1$, *as induced subgraph.*

Proof. Let $C_n = (u_1, u_2, \ldots, u_n)$ be a cycle with $n \geqslant 6$ and $cd(C_n) \leqslant 1$. Assuming u_2 be the only node incident to chords of C_n (if any), the stretch number of C_n is given by $s_{C_n}(u_1, u_3)$. Since $P_{C_n}(u_1, u_3) = (u_1, u_n, u_{n-1}, \ldots, u_3)$ and $p_{C_n}(u_1, u_3) = (u_1, u_2, u_3)$, then

$$s(C_n) = s_{C_n}(u_1, u_3) = \frac{n-2}{2} \geqslant 2.$$

Since $s(G) < 2$ and since $\mathrm{BID}(k)$ is hereditary, then G cannot contain C_n as induced subgraph. □

4. New characterization results

Graphs in BID(1) have been extensively studied and different characterizations have been provided. In particular, one of these characterizations is based on forbidden induced subgraphs [1], and in [7] this result has been extended to the class BID(3/2). In this section we further extend this characterization to the class BID$(2-1/i)$, for every integer $i \geqslant 2$.

Lemma 4.1. *Let G be a graph with $1 < s(G) < 2$, and let $\{x, y\} \in \mathcal{S}(G)$ be a cycle-pair. If C is the cycle induced by $p_G(x, y) \cup P_G(x, y)$, then every internal node of $p_G(x, y)$ is incident to a chord of C.*

Proof. Assume that paths $P_G(x, y)$ and $p_G(x, y)$ are equal to $(x, u_1, u_2, \ldots, u_p, y)$ and $(x, v_1, v_2, \ldots, v_q, y)$, respectively. By definition of induced path, since $s(G) > 1$ and $\{x, y\} \in \mathcal{S}(G)$, then $q \geqslant 1$. Since $(x, u_1, u_2, \ldots, u_p, y)$ and $(x, v_1, v_2, \ldots, v_q, y)$ are induced paths of G, every chord (w_1, w_2) of C fulfills $w_1 \in \{v_1, v_2, \ldots, v_q\}$ and $w_2 \in \{u_1, u_2, \ldots, u_p\}$ or vice versa. Moreover, $\{x, y\} \in \mathcal{S}(G)$ and $s(G) > 1$ imply that v_1 and v_q are incident to chords of C. In fact, if v_1 (v_q, respectively) would not be incident to some chord then $s_G(v_1, y) > s_G(x, y)$ ($s_G(x, v_q) > s_G(x, y)$, respectively), a contradiction. In the following we show that, for each $2 \leqslant i \leqslant q-1$, v_i is incident to a chord of C.

By contradiction, let us suppose that there exists a sequence of nodes

$$v_k, v_{k+1}, \ldots, v_{k+t}, v_{k+t+1}$$

such that the following conditions hold:

- $k \geqslant 1$,
- $t \geqslant 1$,
- $k + t + 1 \leqslant q$,
- v_k and v_{k+t+1} are incident to chords of C,
- every v_i, $k + 1 \leqslant i \leqslant k + t$, is not incident to chords of C.

Now, let v_l be a node such that $l \geqslant k + t + 2$, l is minimum, and v_l is incident to chords of C. Notice that, if v_l does not exist then $v_{k+t+1} = v_q$.

We analyze two major cases, according whether v_l exists or not, and some sub-cases. For each case we show a contradiction.

Let us now suppose that v_l exists. Let us consider the chord (v_k, u_{h_1}) such that $h_1 = \max\{h \mid (v_k, u_h)$ is a chord of $C\}$, and the chord (v_l, u_{h_2}) such that $h_2 = \min\{h \mid (v_l, u_h)$ is a chord of $C\}$.

According to the values of h_1 and h_2, we have three different sub-cases.

(1) $h_1 = h_2$.
In this case there is a shortcut from v_k to v_l through u_{h_1}. This implies that the path $(x, v_1, v_2, \ldots, v_k, u_{h_1}, v_l, v_{l+1}, \ldots, v_q, y)$ has a length less than $q + 1$. This contradicts $d_G(x, y) = q + 1$.

(2) $h_1 < h_2$.
In this case the cycle induced by the nodes $v_k, v_{k+1}, \ldots, v_l, u_{h_2}, u_{h_2-1}, \ldots, u_{h_1}$ is a

cycle with at least 6 nodes and chord distance at most 1. Since $s(G) < 2$, this is a contradiction of Lemma 3.5.

(3) $h_1 > h_2$.
Let us consider the chord $(v_l, u_{h'_2})$ such that

$$h'_2 = \max\{h \mid (v_l, u_h) \text{ is a chord of } C \text{ and } h_2 \leqslant h'_2 < h_1\}.$$

Notice that neither chord $(v_k, u_{h'_2})$ or (v_l, u_{h_1}) can exist, otherwise $d_G(x, y) < q + 1$. Then, the cycle induced by the nodes

$$v_k, v_{k+1}, \ldots, v_l, u_{h'_2}, u_{h'_2+1}, \ldots, u_{h_1}$$

is a cycle with at least 6 nodes and chord distance at most 1. Since $s(G) < 2$, this is a contradiction of Lemma 3.5.

Let us suppose that v_l does not exist. It follows that $v_{k+t+1} = v_q$. Moreover, $u_{h_1} = u_p$ otherwise the cycle induced by the nodes $v_k, v_{k+1}, \ldots, v_q, y, u_p, u_{p-1}, \ldots, u_{h_1}$ is a cycle with at least 6 nodes and chord distance at most 1 (a contradiction of Lemma 3.5). In this case the path $(x, v_1, v_2, \ldots, v_k, u_p, y)$ has a length less than $q + 1$. This contradicts $d_G(x, y) = q + 1$, and concludes the proof. □

Theorem 4.2. *Given a graph G and an integer $i \geqslant 2$, then $G \in \mathrm{BID}(2 - 1/i)$ if and only if the following graphs are not induced subgraphs of G:*

(1) *H_n, for each $n \geqslant 6$;*
(2) *cycles C_6 with $cd(C_6) = 1$;*
(3) *cycles C_7 with $cd(C_7) = 1$;*
(4) *cycles C_8 with $cd(C_8) = 1$;*
(5) *cycles C_{3i+2} with $cd(C_{3i+2}) = i$.*

Proof. ($\Rightarrow$) Holes H_n, $n \geqslant 6$, have stretch number at least 2. Cycles with 6, 7, or 8 nodes and chord distance 1 have stretch number equal to 2, 5/2, and 3, respectively. Let $C_{3i+2} = (v_0, v_1, \ldots, v_{3i+1})$ be a cycle with chord distance equal to i. If $v_1, \ldots, v_i$ are consecutive nodes incident to all the chords of C_{3i+2}, then

$$s_G(v_0, v_{i+1}) \geqslant \frac{2i+1}{i+1} = 2 - \frac{1}{i+1},$$

because $D_G(v_0, v_{i+1})$ is at least the length of the path $(v_{i+1}, v_{i+2}, \ldots, v_{3i+1}, v_0)$. Since the considered cycles have stretch number greater than $2 - 1/i$, then they are forbidden induced subgraphs for every graph belonging to $\mathrm{BID}(2 - 1/i)$.

($\Leftarrow$) Given an arbitrary integer $i \geqslant 2$, we prove that every graph $G \notin \mathrm{BID}(2 - 1/i)$ contains one of the forbidden subgraphs or a proper induced subgraph G' such that $G' \notin \mathrm{BID}(2 - 1/i)$. In the latter case, we can recursively apply to G' the following proof.

Let us assume $G \notin \mathrm{BID}(2 - 1/i)$. This implies $S(G) > 3/2$, and, by Lemma 3.3, there exists a cycle-pair $\{x, y\} \in \mathcal{S}(G)$. Assume that $P_G(x, y)$ and $p_G(x, y)$ are $(x, u_1, u_2, \ldots, u_p, y)$ and $(x, v_1, v_2, \ldots, v_q, y)$, respectively, such that $p + q + 2 = n$ and $C_n =$

$\langle P_G(x,y) \cup p_G(x,y)\rangle$. By construction, $cd(C_n) = q$, and, by Theorem 3.4, we can state that $n \geqslant 6$ and $0 \leqslant cd(C_n) \leqslant \lceil \frac{i \cdot n}{3i-1} \rceil - 2$.

If $q = 0$ then we obtain the holes H_n, $n \geqslant 6$. If $q = \lceil \frac{i \cdot n}{3i-1} \rceil - 2$ and $n = 6, 7, 8, 3i+2$, then, for each value of n, we obtain the following forbidden subgraphs: C_6 with $cd(C_6) = 1$; C_7 with $cd(C_7) = 1$; C_8 with $cd(C_8) = 2$ if $i = 2$ (case 5) and with $cd(C_8) = 1$ if $i > 2$ (case 4); C_{3i+2} with $cd(C_{3i+2}) = i$.

Now, we show that if $n \geqslant 9$, $n \neq 3i+2$, and q fulfills $1 \leqslant q \leqslant \lceil \frac{i \cdot n}{3i-1} \rceil - 2$, then C_n contains one of the given forbidden subgraphs or an induced subgraph G' such that $G' \notin \mathrm{BID}(2 - 1/i)$.

By Lemma 4.1, every node v_k, $1 \leqslant k \leqslant q$, must be incident to a chord of C_n, otherwise C_n has a stretch number greater or equal to 2 and hence it is a forbidden subgraph of G. As a consequence, we can denote by r_j the largest index j' such that v_j and $u_{j'}$ are connected by a chord of C_n, i.e., $r_j = \max\{j' \mid (v_j, u_{j'}) \text{ is a chord of } C_n\}$. Informally, r_j gives the *rightmost* chord connecting v_j to some node of $P_G(x,y)$.

Notice that, if $r_1 > 3$, then, by Lemma 3.5, the subgraph of C_n induced by $v_1, x, u_1, \ldots, u_{r_1}$ is forbidden, since it is a cycle with at least 6 nodes and chord distance at most 1. Hence, in the remainder of this proof we assume that $r_1 \leqslant 3$.

Let us now analyze two distinguished cases for C_n, according whether the chord distance q of C_n either (i) fulfills $1 \leqslant q < \lceil \frac{i \cdot n}{3i-1} \rceil - 2$, or (ii) is equal to $\lceil \frac{i \cdot n}{3i-1} \rceil - 2$.

(1) Consider C_n with $n \geqslant 9$ and chord distance q such that $1 \leqslant q < \lceil \frac{i \cdot n}{3i-1} \rceil - 2$.

If $C_{n'}$ denotes the subgraph induced by the nodes of C_n except $x, u_1, \ldots, u_{r_1-1}$, then $C_{n'}$ is a cycle with $n' \geqslant n-3$ nodes and chord distance at most $q-1$. To prove that $C_{n'}$ is forbidden, it is sufficient to show that $\lceil \frac{i \cdot n'}{3i-1} \rceil - 2 \geqslant q - 1$:

$$\left\lceil \frac{i \cdot n'}{3i-1} \right\rceil - 2 \geqslant \left\lceil \frac{i \cdot n - 3i}{3i-1} \right\rceil - 2 \geqslant q - 1,$$
$$\left\lceil \frac{i \cdot n - 3i}{3i-1} \right\rceil - 2 > q - 2,$$
$$\left\lceil \frac{i \cdot n - 3i}{3i-1} + 2 \right\rceil - 2 > q,$$
$$\left\lceil \frac{i \cdot n + 3i - 2}{3i-1} \right\rceil - 2 > q.$$

The last inequality holds because $3i - 2 \geqslant 0$ for each integer $i \geqslant 1$, and $\lceil \frac{i \cdot n}{3i-1} \rceil - 2 > q$.

(2) Consider C_n with $n \geqslant 9$ and chord distance q such that $q = \lceil \frac{i \cdot n}{3i-1} \rceil - 2$.

In this case q is given whenever a fixed value for n is chosen. In general, since $n \geqslant 9$, it follows that $q \geqslant 2$.

Let us analyze again the cycle $C_{n'}$. Recalling that $n' \geqslant n-3$ and $cd(C_{n'}) \leqslant q-1$, then

$$\left\lceil \frac{i \cdot n'}{3i-1} \right\rceil - 2 \geqslant \left\lceil \frac{i \cdot n - 3i}{3i-1} \right\rceil - 2 \geqslant q - 1$$

is equivalent to

$$\left\lceil \frac{i \cdot n - 1}{3i-1} \right\rceil - 2 \geqslant q.$$

In the following we show that, for every n such that $9 \leqslant n \leqslant 6i$, either this relation holds or n is equal to $3i+2$. This means that the cycle $C_{n'}$ is forbidden for each cycle C_n, $9 \leqslant n \leqslant 6i$.

Since $\lceil \frac{i \cdot n}{3i-1} \rceil - 2 = q$ holds by hypothesis, we have to study when $\lceil \frac{i \cdot n - 1}{3i-1} \rceil \geqslant \lceil \frac{i \cdot n}{3i-1} \rceil$. This relation does not hold if and only if there exists an integer m such that $\frac{i \cdot n - 1}{3i-1} \leqslant m < \frac{i \cdot n}{3i-1}$, that is $(3i-1)m < i \cdot n \leqslant (3i-1)m + 1$. Then $n = 3m - \frac{m-1}{i}$, and, as consequence, m can be equal to $\ell \cdot i + 1$ only, for each integer $\ell \geqslant 0$. Hence $n = 3m - \frac{m-1}{i} = 3(\ell \cdot i + 1) - \ell$, $\ell \geqslant 0$. For $\ell = 0$ we obtain $n = 3$ (but we are considering $n \geqslant 9$), for $\ell = 1$ and $\ell = 2$ the value of n is $3i+2$ and $n = 6i+1$, respectively. The cycle with $3i+2$ nodes is one of the forbidden cycles in the statement of the theorem. As a conclusion, the cycle $C_{n'}$ shows that C_n contains a forbidden induced subgraph when $9 \leqslant n \leqslant 6i$.

It remains to be considered the case when $n \geqslant 6i+1$. In this case $q = \lceil \frac{i \cdot n}{3i-1} \rceil - 2$ implies $q \geqslant 2i$, and hence we can compute r_i. If $r_i \geqslant 2i+1$ then the cycle induced by $v_i, v_{i-1}, \ldots, v_1, x, u_1, \ldots, u_{r_i}$ is forbidden. In fact, paths $(x, u_1, u_2, \ldots, u_{r_i})$ and $(x, v_1, v_2, \ldots, v_i, u_{r_i})$ give the following lower bound to $s_G(x, u_{r_i})$:

$$s_G(x, u_{r_i}) \geqslant \frac{r_i}{i+1} \geqslant \frac{2i+1}{i+1} = 2 - \frac{1}{i+1} > 2 - \frac{1}{i}.$$

Hence, $r_i \leqslant 2i$. The cycle $C_{n''}$, subgraph induced by the nodes of C_n except the nodes $v_{i-1}, \ldots, v_1, x, u_1, \ldots, u_{r_i-1}$, is a cycle with $n'' \geqslant n - 3i + 1$ nodes and chord distance at most $q - i$. To prove that $C_{n''}$ is forbidden, let us show that $\lceil \frac{i \cdot n''}{3i-1} \rceil - 2 \geqslant q - i$. The inequality

$$\left\lceil \frac{i \cdot n''}{3i-1} \right\rceil - 2 \geqslant \left\lceil \frac{i \cdot (n - 3i + 1)}{3i-1} \right\rceil - 2 \geqslant q - i$$

is equivalent to

$$\left\lceil \frac{i \cdot n}{3i-1} \right\rceil - 2 \geqslant q.$$

The last relation holds by hypothesis, and this concludes the proof. □

5. Admissible stretch numbers

In [7], it was conjectured that, for each integer $i \geqslant 1$, there exists no graph G such that $2 - 1/i < s(G) < 2 - 1/(i+1)$. In this section we show that such a conjecture is true. Moreover, we extend the result by showing that it is possible to answer to the following more general question: Given a rational number $t \geqslant 1$, is there a graph G such that $s(G) = t$? In other words, we can state when a given positive rational number is an *admissible* stretch number.

Definition 5.1. A positive rational number t is called *admissible stretch number* if there exists a graph G such that $s(G) = t$.

In the remainder of this section we first show that the conjecture recalled above is true, and then we show that each positive rational number greater or equal than 2 is an admissible stretch number.

Lemma 5.2. *If p and q are two positive integers such that*

$$2-\frac{1}{i}<\frac{p}{q}<2-\frac{1}{i+1},$$

for some integer $i \geqslant 1$, then $q > i$.

Proof. By contradiction, let us assume that $q \leqslant i$, and let us consider the cases $q = i$ and $q < i$.

If $q = i$ then $p/q > 2 - 1/i$ implies $p > 2i - 1$, that is $p \geqslant 2i$. Since $i \geqslant 1$ then $p/q \geqslant 2$, and this contradicts the relation $p/q < 2 - 1/(i + 1) < 2$.

If $q < i$ then both the relations $p > 2q - q/i$ and $p < 2q - q/(i + 1)$ hold. But these relations imply that $2q - 1 < p < 2q$, contradicting the hypothesis that p is an integer. □

Theorem 5.3. *If t is a rational number such that*

$$2-\frac{1}{i}<t<2-\frac{1}{i+1},$$

for some integer $i \geqslant 1$, then t is not an admissible stretch number.

Proof. We have to show that there exists no graph G such that

$$2-\frac{1}{i}<s(G)<2-\frac{1}{i+1},$$

for each integer $i \geqslant 1$.

By contradiction, let us assume that there exist an integer $i \geqslant 1$ and a graph G such that

$$2-\frac{1}{i}<s(G)<2-\frac{1}{i+1}.$$

By Lemma 3.3 there exists a cycle-pair $\{x, y\} \in \mathcal{S}(G)$. If we assume that $P_G(x, y)$ and $p_G(x, y)$ correspond to $(x, u_1, u_2, \ldots, u_{p-1}, y)$ and $(x, v_1, v_2, \ldots, v_{q-1}, y)$, respectively, then $p_G(x, y) \cup P_G(x, y)$ induces a cycle C, and $s(G) = p/q$. By Lemma 5.2, the relation $q > i$ holds; then, the node v_i exists in the path $p_G(x, y)$. By Lemma 4.1, the node v_i is incident to a chord of C, and hence, like in Theorem 4.2, we can define the integer r, $1 \leqslant r \leqslant q - 1$, such that

$$r = r_i = \max\bigl\{ j \mid (v_i, u_j) \text{ is a chord of } C \bigr\}.$$

Now, denote by C_L the cycle induced by the nodes $v_i, v_{i-1}, \ldots, v_1, x, u_1, u_2, \ldots, u_r$, and by C_R the cycle induced by the nodes $v_i, v_{i+1}, \ldots, v_{q-1}, y, u_{p-1}, u_{p-2}, \ldots, u_r$. In other words, the chord (v_i, u_r) divides C into the *left* cycle C_L, and the *right* cycle C_R.

First of all, let us compute the stretch number of the cycle C_R. Since $p_G(x, y) = (x, v_1, v_2, \ldots, v_{q-1}, y)$ then $p_{C_R}(v_i, y) = (v_i, v_{i+1}, \ldots, v_{q-1}, y)$. Moreover, since the path

$(v_i, u_r, u_{r+1}, \ldots, u_{p-1}, y)$ is induced in C, then $D_{C_R}(v_i, y) \geqslant p - r + 1$. Then

$$s(C_R) \geqslant s_{C_R}(v_i, y) \geqslant \frac{p-r+1}{q-i}.$$

Since C_R is an induced subgraph of G then

$$\frac{p-r+1}{q-i} \leqslant \frac{p}{q}.$$

This inequality is equivalent to

$$\frac{p}{q} \leqslant \frac{r-1}{i}.$$

From the relations

$$2 - \frac{1}{i} < \frac{p}{q} \leqslant \frac{r-1}{i}$$

we obtain that $r > 2i$, that is $r \geqslant 2i + 1$.

Let us now compute the stretch number of the cycle C_L when $r \geqslant 2i + 1$. In this case, $p_{C_L}(x, u_r) = (x, v_1, v_2, \ldots, v_i, u_r)$ and $P_{C_L}(x, u_r) = (x, u_1, u_2, \ldots, u_r)$. Then

$$s(C_L) \geqslant s_{C_L}(x, u_r) = \frac{r}{i+1} \geqslant \frac{2i+1}{i+1} \geqslant 2 - \frac{1}{i+1}.$$

The obtained relation implies that $s(C_L) > s(G)$. This is a contradiction since C_L is an induced subgraph of G. □

In order to show that each rational number equal or greater than 2 is an admissible stretch number, let us consider the graph $G(n_1, n_2, \ldots, n_t)$ obtained by composing t holes $H_{n_1}, H_{n_2}, \ldots, H_{n_t}$ by split composition, where $n_i \geqslant 5$ for $1 \leqslant i \leqslant t$. In detail, the holes correspond to the following chord-less cycles (as an example, see Fig. 3, where $t = 5$):

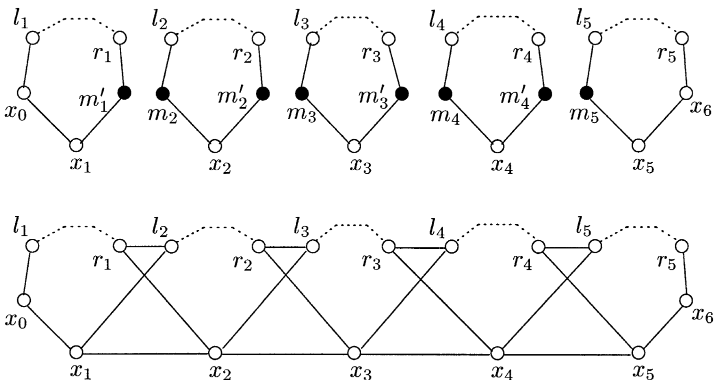

Fig. 3. The graph $G(n_1, n_2, n_3, n_4, n_5)$ obtained by the split composition of 5 holes. The ith hole has $n_i \geqslant 5$ nodes. Dotted lines between nodes l_i and r_i, $1 \leqslant i \leqslant 5$, represent induced paths.

- $H_{n_1} = (l_1, x_0, x_1, m'_1, r_1, \ldots)$;
- $H_{n_i} = (l_i, m_i, x_i, m'_i, r_i, \ldots)$, for each i such that $1 < i < t$;
- $H_{n_t} = (l_t, m_t, x_t, x_{t+1}, r_t, \ldots)$.

These holes are composed by means of the split composition as follows:

$$G(n_1, n_2, \ldots, n_t) = H_{n_1} * H_{n_2} * \cdots * H_{n_t},$$

where the marked nodes between H_{n_i} and $H_{n_{i+1}}$ are m'_i and m_{i+1}, $1 \leqslant i < t$, respectively.

In the following, we denote by V_1 (V_t, respectively) the set containing all the nodes of the hole H_{n_1} (H_{n_t}, respectively) but x_0, x_1, and m'_1 (m_t, x_t, and x_{t+1}, respectively); we denote by V_i the set containing all the nodes of the hole H_{n_i} but m_i, x_i, and m'_i, $1 < i < t$. Finally, we denote by X the set $\{x_0, x_1, \ldots, x_{t+1}\}$.

Lemma 5.4. *Given the graph $G = G(n_1, n_2, \ldots, n_t)$, the following facts hold*:

(1) $s_G(x_0, x_{t+1}) = (\sum_{i=1}^{t} n_i - 3t + 1)/(t + 1)$;
(2) *if $j - i \geqslant 2$, then $p_G(x_i, x_j) \cup P_G(x_i, x_j)$ induces a subgraph isomorphic to $G(n_{i+1}, n_{i+2}, \ldots, n_{j-1})$*;
(3) *there exists a pair $\{u, v\} \in \mathcal{S}(G)$ such that $u \in X$, $v \in X$, and $d_G(u, v) \geqslant 2$*;
(4) *if $n_t \geqslant \max\{n_i \mid 1 \leqslant i \leqslant t - 1\}$ then $s_G(x_0, x_{t+1}) > s_G(x_0, x_t)$*;
(5) *if $n_i = n$ for some fixed integer n and for each $1 \leqslant i \leqslant t$, then $s(G) = s_G(x_0, x_{t+1}) = (nt - 3t + 1)/(t + 1)$*;
(6) *let k, $1 \leqslant k < t$, be an integer such that $n_i = n$, for each $k < i \leqslant t$ and for a fixed integer n. Then, one of the following relationships holds*:
 (a) *$s_G(x_0, x_j) \geqslant s_G(x_0, x_{j+1})$, for each $k \leqslant j < t$*;
 (b) *$s_G(x_0, x_j) < s_G(x_0, x_{j+1})$, for each $k \leqslant j < t$.*

Proof. We prove each fact separately.

(1) Here $p_G(x_0, x_{t+1})$ and $P_G(x_0, x_{t+1})$ coincide with $(x_0, x_1, x_2, \ldots, x_t, x_{t+1})$ and $(x_0, l_1, \ldots, r_1, l_2, \ldots, r_2, \ldots, r_{i-1}, l_i, \ldots, r_i, l_{i+1}, \ldots, r_{t-1}, l_t, \ldots, r_t, x_{t+1})$, respectively. In particular, $P_G(x_0, x_{t+1})$ coincides with the induced path obtained from G by removing $x_1, x_2, \ldots, x_t$. Notice that $d_G(x_0, x_{t+1}) = t + 1$, while $P_G(x_0, x_{t+1})$ contains one edge connecting x_0 to l_1; $n_i - 4$ edges from l_i to r_i, $1 \leqslant i \leqslant t$; one edge connecting r_i to l_{i+1}, $1 \leqslant i < t$; and, finally, one edge connecting r_t to x_{t+1}. Hence, the length of $P_G(x_0, x_{t+1})$ is $1 + \sum_{i=1}^{t}(n_i - 4) + (t - 1) + 1 = \sum_{i=1}^{t} n_i - 3t + 1$.
(2) This fact simply follows from the proof of the first one and from the observation that, by definition of split composition and of induced paths, neither $p_G(x_i, x_j)$ nor $P_G(x_i, x_j)$ may contain nodes of $H_{n_1} \cup H_{n_2} \cup \cdots \cup H_{n_i} \cup H_{n_j} \cup H_{n_{j+2}} \cup \cdots \cup H_{n_t}$.
(3) First notice that, since $s(G) \geqslant 3/2$ (because G contains a hole H_n, $n \geqslant 5$, as induced subgraph) and since $s_G(u, v) = 1$ when $d_G(u, v) = 1$, then every pair of nodes in $\mathcal{S}(G)$ has distance in G at least 2. Now, let $\{u, v\} \in \mathcal{S}(G)$ and $v \notin X$.
 Because of the symmetry of G, without loss of generality we can assume that $u \in V_i \cup \{x_i\}$, $v \in V_j$, and $i \leqslant j$.

If $j = i$ then $p_G(u, v) \cup P_G(u, v)$ induces an hole H isomorphic to H_{n_i}; it follows that $d_G(u, v) = 2$ and that every pair of nodes at distance 2 in H gives the same stretch number of u and v. Then, $\{x_{i-1}, x_{i+1}\} \in \mathcal{S}(G)$.

If $j \geqslant i + 1$, then we prove that a node $x \in X$ exists such that $s_G(u, v) \leqslant s_G(u, x)$ (in particular $x \in \{x_j, x_{j+1}\}$). Since $\{u, v\} \in \mathcal{S}(G)$, then $\{u, x\} \in \mathcal{S}(G)$.

If $v = l_j$, then $u \neq \{x_{j-1}, r_{j-1}\}$ (otherwise $d_G(u, v) = 1$), and $D_G(u, v) = D_G(u, x_j)$, because $x_j \notin P_G(u, v)$ otherwise $P_G(u, v)$ is not an induced path. Since $d_G(u, v) = d_G(u, x_j)$, then $s_G(u, v) = s_G(u, x_j)$.

If $v \neq l_j$, then either (i) $l_j \in p_G(u, v)$ and $r_j \in P_G(u, v)$ or (ii) $r_j \in p_G(u, v)$ and $l_j \in P_G(u, v)$.

(i) In this case $p_G(u, v) = (u, \ldots, l_j, \ldots, v)$. Since $d_G(u, l_j) = d_G(u, x_j)$, then $d_G(u, v) \geqslant d_G(u, x_{j+1})$. On the other hand, $P_G(u, v)$ corresponds to the path $(u, \ldots, w, x_j, z, r_j, \ldots, v)$, where $w \in \{x_{j-1}, r_{j-1}\}$ and $z \in \{x_{j+1}, l_{j+1}\}$. Let $p = (u, \ldots, w, l_j, \ldots, v, \ldots, r_j, x_{j+1})$ be a path which coincides with $P_G(u, v)$ from u to w. By construction of G, this path is induced. Then $D_G(u, v) \leqslant |p| \leqslant D_G(u, x_{j+1})$. As consequence,

$$s_G(u, v) = \frac{D_G(u, v)}{d_G(u, v)} \leqslant \frac{D_G(u, x_{j+1})}{d_G(u, x_{j+1})} = s_G(u, x_{j+1}).$$

(ii) In this case $p_G(u, v) = (u, \ldots, x_j, z, r_j, \ldots, v)$, where $z \in \{x_{j+1}, l_{j+1}\}$. As consequence, $d_G(u, v) \geqslant d_G(u, z) = d_G(u, x_{j+1})$. By construction of G, $p = \langle P_G(u, v) \cup p_G(v, x_{j+1})\rangle$ is a path and $D_G(u, v) \leqslant |p| \leqslant D_G(u, x_{j+1})$. Hence, $s_G(u, v) \leqslant s_G(u, x_{j+1})$. This implies $\{u, x_{j+1}\} \in \mathcal{S}(G)$.

Now, if $u \in X$ we are done. Otherwise, because of the symmetry of G, we can apply the same technique used above to find a node $x' \in X$ such that $\{x', v\} \in \mathcal{S}(G)$, and hence $\{x', x\} \in \mathcal{S}(G)$.

(4) By the first two facts, it follows that

$$s_G(x_0, x_t) = \frac{\sum_{i=1}^{t-1} n_i - 3(t-1) + 1}{t}.$$

Then,

$$\begin{aligned} s_G(x_0, x_{t+1}) &= \frac{\sum_{i=1}^{t} n_i - 3t + 1}{t+1} \\ &= \frac{\sum_{i=1}^{t-1} n_i - 3(t-1) + 1}{t+1} + \frac{n_t - 3}{t+1} \\ &= s_G(x_0, x_t) - \frac{s_G(x_0, x_t)}{t+1} + \frac{n_t - 3}{t+1}. \end{aligned}$$

In order to prove that $s_G(x_0, x_{t+1}) > s_G(x_0, x_t)$ it is sufficient to show that

$$-\frac{s_G(x_0, x_t)}{t+1} + \frac{n_t - 3}{t+1} > 0$$

that is

$$s_G(x_0, x_t) < n_t - 3.$$

The latter inequality is equivalent to

$$\frac{\sum_{i=1}^{t-1} n_i - 3(t-1) + 1}{t} < n_t - 3,$$

and, in turn, to $\sum_{i=1}^{t-1} n_i + 4 < t \cdot n_t$. It holds since, by hypothesis, $n_i \geqslant 5$, $1 \leqslant i \leqslant t$.

(5) This fact is an immediate consequence of the previous Facts 1 and 4.

(6) Denoting $p' = D_G(x_0, x_{k+1})$, then, by Fact 1, $s_G(x_0, x_{k+1}) = p'/(k+1)$. Assuming $j = k + h$, $h \geqslant 0$, since $n_i = n$ for each $i > k$, then $D_G(x_0, x_j)$ is equal to $D_G(x_0, x_{k+1}) + (n-3)h$. This implies that the inequality $s_G(x_0, x_j) \geqslant s_G(x_0, x_{j+1})$ can be rewritten as:

$$\frac{p' + (n-3)h}{j+1} \geqslant \frac{p' + (n-3)(h+1)}{j+2}$$

and can be further simplified to the following inequality:

$$\frac{p'}{k+1} \geqslant n - 3. \tag{5.1}$$

Since inequality (5.1) does not depend on j, according whether it is true or not then one of two relationships of the statement holds.

This concludes the proof. □

Notice that the stretch number of nodes x_0 and x_{t+1} in $G(n_1, n_2, \ldots, n_t)$ does not depend on how many nodes are in each hole; it depends only on the total number of nodes in $G(n_1, n_2, \ldots, n_t)$ and on the number t of used holes.

Corollary 5.5. *For each integer $i \geqslant 1$, $2 - 1/i$ is an admissible stretch number.*

Proof. Every distance-hereditary graph has stretch number equal to 1. If $i > 1$, from Fact 5 of Lemma 5.4, it follows that the graph $G = G(n_1, n_2, \ldots, n_{i-1})$ such that $n_j = 5$ for each $1 \leqslant j \leqslant i - 1$, has stretch equal to $2 - 1/i$. □

Theorem 5.6. *If t is a rational number such that $t \geqslant 2$, then t is an admissible stretch number.*

Proof. Let us suppose that $t = p/q$ for two positive integers p and q without common divisors greater than 1. If $q = 1$ (i.e., the only case in which t is an integer) then $G = H_{2p+2}$, and if $q = 2$ then $G = H_{p+2}$. In the remainder of the proof we show that if $q \geqslant 3$ then the requested graph G is equal to $G(n_1, n_2, \ldots, n_{q-1})$, for suitable integers $n_1, n_2, \ldots, n_{q-1}$. We now determine such integers, that is, the size of each hole H_{n_i}, $1 \leqslant i \leqslant q - 1$, we use to compose G.

Let $b = 3 + \lfloor \frac{p-1}{q-1} \rfloor$ and $r = (p-1) \bmod (q-1)$. The sizes of the holes $H_{n_1}, H_{n_2}, \ldots, H_{n_{q-1}}$ are defined according to the following strategy: r holes contain $b + 1$ nodes, while the remaining $q-1-r$ contain b nodes. By Fact 1 of Lemma 5.4 it follows that

$$s_G(x_0, x_q) = \frac{\sum_{i=1}^{q-1} n_i - 3(q-1) + 1}{q} = \frac{p}{q}.$$

Notice that this result does not depend on which holes contain b or $b+1$ nodes. In particular, if $r = 0$ then, by Fact 5 of Lemma 5.4, the proof is concluded. So, in the remainder of the proof we assume $r \geqslant 1$. According to Fact 3 of Lemma 5.4, to prove that $S(G) = p/q$ it is sufficient to show that $s_G(x_i, x_j) \leqslant p/q$, $0 \leqslant i < j \leqslant q-1$. Since this property depends on which holes contain b or $b+1$ nodes, to conclude the proof we have to fix the size of each hole H_{n_k}, $1 \leqslant k \leqslant q-1$.

For sake of convenience, assume that each hole with $b+1$ nodes is arbitrarily numbered from 1 to r. To fix the size of each hole, we introduce an injective function $pos: \{1, 2, \ldots, r\} \to \{1, 2, \ldots, q-1\}$ having the following meaning: H_k, $1 \leqslant k \leqslant q-1$, contains $b+1$ nodes if and only if $pos(x) = k$ for some x, $1 \leqslant x \leqslant r$. Informally, this function gives the *position* of the holes having $b+1$ nodes in the sequence of the $q-1$ holes forming G.

The function *pos* we will use is based on the following observation. Let us consider a graph $G' = G(m_1, m_2, \ldots, m_\ell)$, where s of the ℓ holes contain $b+1$ nodes, while each of the remaining $\ell - s$ holes contains b nodes. From Fact 1 of Lemma 5.4, we get

$$s_{G'}(x_0, x_{\ell+1}) = \frac{(s(b+1) + (\ell - s)b) - 3\ell + 1}{\ell + 1} = \frac{b\ell + s - 3\ell + 1}{\ell + 1}.$$

If we now assume s fixed, we can use the latter equality to compute the minimum value for ℓ such that $s_{G'}(x_0, x_{\ell+1}) \leqslant \frac{p}{q}$. By imposing:

$$\frac{b\ell + s - 3\ell + 1}{\ell + 1} \leqslant \frac{p}{q}, \tag{5.2}$$

we deduce the following inequality:

$$\ell \cdot \big[(b-3)q - p\big] \leqslant p - q(s+1). \tag{5.3}$$

Notice that the multiplicative factor

$$D = (b-3)q - p$$

of ℓ in inequality (5.3) cannot be equal to zero. In fact, $(b-3)q - p = 0$ implies $\lfloor \frac{p-1}{q-1} \rfloor = \frac{p}{q}$, and this equality holds only if p/q is integer, a contradiction for the running hypothesis. To proceed further with this observation, we have to study when D is positive and negative. For each case, we will provide a different function *pos*.

Case $D > 0$: From inequality (5.3) we get

$$\ell \leqslant \frac{p - q(s+1)}{(b-3)q - p}. \tag{5.4}$$

Since we have already observed that $s_G(x_0, x_q) = p/q$, then inequality (5.4) holds when $\ell = q-1$ and $s = r$. In this case, it becomes equal to

$$q - 1 \leqslant \frac{p - q(r+1)}{(b-3)q - p}. \tag{5.5}$$

Now, fix the size of each hole according to any injective function $pos:\{1,2,\ldots,r\}\to\{1,2,\ldots,q-1\}$. According to Fact 3 of Lemma 5.4, we prove that the theorem holds by showing that $s_G(x_i,x_j)\leqslant p/q$, $0\leqslant i, j\leqslant q-1$ and $j-i\geqslant 2$.

By Fact 2 of Lemma 5.4, $p_G(x_i,x_j)\cup P_G(x_i,x_j)$ induces a subgraph G' isomorphic to $G(n_{i+1},n_{i+2},\ldots,n_{j-1})$. G' is made up of $\ell'=j-i-1$ holes, and, without loss of generality, we can assume that s' of such holes contain $b+1$ nodes. Notice that $\ell'\leqslant q-1$ and $s'\leqslant r$. It is easy to see that Eq. (5.4) holds also when $\ell=\ell'$ and $s=s'$. In fact, the right side of Eq. (5.4) depends only on the number of holes having $b+1$ nodes, while the left side depends on all the holes. This implies that if we replace $q-1$ by ℓ' and r by s' in the left and right side of inequality (5.5), respectively, then the value on the left side of inequality (5.5) decreases, while the value on the right side increases. Of course, the new relation we get is still valid, and this proves that $s_G(x_i,x_j)\leqslant p/q$. It is worth to note that the case $D>0$ occurs whenever p can be expressed as $p=kq+k'$, for two integers k and k' such that $k\geqslant q$ and $0\leqslant k'<q$.

Case $D<0$: In the analysis of this case, it is important to show that $r\geqslant 2$. To prove this property, it is convenient to express r as a function of p, q, and b as follows:

$$\begin{aligned} r &= (p-1) \bmod (q-1)\\ &= (p-1)-\left\lfloor\frac{p-1}{q-1}\right\rfloor(q-1)\\ &= (p-1)-(b-3)(q-1). \end{aligned}$$

Hence, $r\geqslant 2$ can be rewritten as $(p-1)-(b-3)(q-1)\geqslant 2$. This relation is equivalent to $p-(b-3)q+(b-3)\geqslant 3$. Since $D<0$, then $p-(b-3)q>0$, and hence $p-(b-3)q\geqslant 1$; moreover, $b-3$ is equal to $\lfloor\frac{p-1}{q-1}\rfloor$, and $p/q\geqslant 2$ implies $\lfloor\frac{p-1}{q-1}\rfloor\geqslant 2$. Hence, $p-(b-3)q+(b-3)\geqslant 3$ holds.

Since $D<0$, from Eq. (5.3) we get

$$\ell\geqslant\frac{p-q(s+1)}{(b-3)q-p}. \tag{5.6}$$

This equation induces the function *pos* as follows:

$$pos(x)=\begin{cases} x, & \text{if } \lceil\frac{p-q(x+1)}{(b-3)q-p}\rceil\leqslant x,\\ \lceil\frac{p-q(x+1)}{(b-3)q-p}\rceil, & \text{otherwise.}\end{cases}$$

Before completing the proof, we have to show that *pos* is a well-defined injective function from $\{1,2,\ldots,r\}$ to $\{1,2,\ldots,q-1\}$. To this aim, we prove the following three properties:

(1) $pos(1)=1$:
To prove that $pos(1)=1$, it is sufficient to show that $\frac{p-q(1+1)}{(b-3)q-p}\leqslant 1$, that is $(b-1)q-2p\leqslant 0$. This inequality is true since it follows from the hypotheses $(b-3)q-p<0$ and $p\geqslant 2q$.

(2) $pos(r)=q-1$:
To show that $pos(r)=q-1$, we show that $\frac{p-q(r+1)}{(b-3)q-p}=q-1$, that is

$$p-q(r+1)=(q-1)\big[(b-3)q-p\big]. \tag{5.7}$$

To prove that this equality is true, it is sufficient to plug the expression $r = (p-1) - (b-3)(q-1)$ into the left side of Eq. (5.7):

$$\begin{aligned} p - q(r+1) &= p - q\big[(p-1) - (b-3)(q-1) + 1\big] \\ &= p - q(p-1) + q(b-3)(q-1) - q \\ &= p - pq + q(b-3)(q-1) \\ &= -p(q-1) + q(b-3)(q-1) \\ &= (q-1)\big[(b-3)q - p\big]. \end{aligned}$$

(3) $pos(x+1) > pos(x)$, for each $1 \leqslant x \leqslant r-1$:
According to definition of *pos*, to prove this property it is sufficient to show that

$$\frac{p - q(x+2)}{(b-3)q - p} - \frac{p - q(x+1)}{(b-3)q - p} \geqslant 1,$$

that is

$$\frac{-q}{(b-3)q - p} \geqslant 1.$$

By using the hypothesis $(b-3)q - p < 0$ and the equality $b - 3 = \lfloor \frac{p-1}{q-1} \rfloor$, we can modify the last inequality to the equivalent equation:

$$\frac{p}{q} \leqslant \left\lfloor \frac{p-1}{q-1} \right\rfloor + 1. \tag{5.8}$$

Furthermore, since $\frac{p-1}{q-1} \leqslant \lfloor \frac{p-1}{q-1} \rfloor + 1$, to show that inequality (5.8) is true it is sufficient to shown that $\frac{p}{q} \leqslant \frac{p-1}{q-1}$. The last inequality corresponds to $p \geqslant q$, and it follows from the hypothesis $p \geqslant 2q$.

Since the function *pos* has been defined according to the observation based on inequality (5.2), then the following two properties holds:

P_1: For each i, $1 \leqslant i \leqslant q-1$, such that $n_i = b+1$, then $s_G(x_0, x_{i+1}) \leqslant p/q$.
P_2: If $n_i = b$ then $s_{\widehat{G}}(x_0, x_{i+1}) > p/q$, where $\widehat{G} = G(n_1, n_2, \ldots, n'_i)$ and $n'_i = b+1$ (i.e., $\widehat{G}$ is composed by using the first $i-1$ holes of G and one hole with size $n'_i = b+1$ instead of $n_i = b$). Hence, if $s_G(x_0, x_{i+1}) = p'/q'$ then $s_{\widehat{G}}(x_0, x_{i+1}) = (p'+1)/q' > p/q$.

In other words, the first property says that the stretch number of x_0 and every node x_{i+1} such that x_{i+1} belongs to a hole with $b+1$ nodes is at most p/q. The second property says that we cannot replace a hole with b nodes by a hole with $b+1$ nodes, otherwise the first property is no longer fulfilled.

Once we fixed the size of each hole by means of the function *pos*, according to Fact 3 of Lemma 5.4, we conclude the proof by showing that $s_G(x_i, x_j) \leqslant p/q$, $0 \leqslant i, j \leqslant q-1$ and $j - i \geqslant 2$. We distinguish two different cases:

Case $i = 0$: By Fact 2 of Lemma 5.4, $p_G(x_0, x_j) \cup P_G(x_0, x_j)$ induces a subgraph isomorphic to $G(n_1, n_2, \ldots, n_{j-1})$. We analyze two sub-cases: $n_{j-1} = b+1$ and $n_{j-1} = b$.

If $n_{j-1} = b + 1$, then $s_G(x_0, x_j) \leqslant p/q$ follows from property P_1 above.

If $n_{j-1} = b$, then there exist two holes $H_{n_{j'-1}}$, $j' < j$, and $H_{n_{j''-1}}$, $j'' > j$, both having $b + 1$ nodes, such that $n_k = b$, for each $j' \leqslant k \leqslant j'' - 2$. Such two holes having $b + 1$ nodes exist because: (i) $r \geqslant 2$, and (ii) $n_{j-1} = b$ implies $j < q$ because $n_{q-1} = b + 1$ by definition of function *pos*. As in the case above, $s_G(x_0, x_{j'}) \leqslant p/q$ and $s_G(x_0, x_{j''}) \leqslant p/q$ follow property P_1 above. Let us consider now the graph $\overline{G} = G(m_1, m_2, \ldots, m_{j''-1})$, where $m_k = n_k$, $1 \leqslant k \leqslant j'' - 2$, and $m_{j''-1} = b$. In other words, the first $j'' - 2$ holes used to build G and $\overline{G}$ coincide, whereas the last hole of $\overline{G}$ contains one node less than the last hole of G. This implies that $D_{\overline{G}}(x_0, x_{j''}) = D_G(x_0, x_{j''}) - 1$, and hence $s_{\overline{G}}(x_0, x_{j''}) \leqslant p/q$. Moreover, since $m_k = b$, $j' \leqslant k \leqslant j'' - 1$, we can apply Fact 6 of Lemma 5.4 to $\overline{G}$. By this fact, either

$$s_{\overline{G}}(x_0, x_{j'}) \leqslant s_{\overline{G}}(x_0, x_k) \leqslant s_{\overline{G}}(x_0, x_{j''}), \quad j' - 1 < k \leqslant j'' - 1$$

or

$$s_{\overline{G}}(x_0, x_{j'}) > s_{\overline{G}}(x_0, x_k) > s_{\overline{G}}(x_0, x_{j''}), \quad j' - 1 < k \leqslant j'' - 1$$

holds. In both cases, we deduce $s_{\overline{G}}(x_0, x_k) \leqslant p/q$, for each $j' - 1 < k \leqslant j'' - 1$, and hence $s_G(x_0, x_j) \leqslant p/q$.

Case $i > 0$: By contradiction, let us suppose that there exist two integers i and j such that $s_G(x_i, x_j) = p'/q' > p/q$; moreover, let us assume that $s_G(x_i, x_j) = s(G)$. By Fact 2 of Lemma 5.4, $p_G(x_i, x_j) \cup P_G(x_i, x_j)$ induces a subgraph isomorphic to $G(n_{i+1}, n_{i+2}, \ldots, n_{j-1})$. It follows that $n_i = b$, otherwise, by Fact 4 of Lemma 5.4, $s_G(x_{i-1}, x_j) > s(x_i, x_j)$, a contradiction for $s_G(x_i, x_j) = s(G)$.

Denote $s_G(x_0, x_{i+1}) = p''/q''$. Since $n_i = b$, if $\widehat{G} = G(n_1, n_2, \ldots, n'_i)$ and $n'_i = b + 1$ (i.e., $\widehat{G}$ is composed by using the first i holes of G, but the last one has size $b + 1$ instead of b), by property P_2 we get the following inequality:

$$s_{\widehat{G}}(x_0, x_{i+1}) = \frac{p'' + 1}{q''} > \frac{p}{q}.$$

Now, from definition of graph G, we can express the stretch of $s_G(x_0, x_j)$ by using $s_G(x_0, x_{i+1})$ and $s_G(x_i, x_j)$. In fact, $s_G(x_0, x_j) = \frac{p'+p''-1}{q'+q''-1}$. Then,

$$s_G(x_0, x_j) = \frac{p' + p'' - 1}{q' + q'' - 1} > \frac{\frac{p}{q}q' + (\frac{p}{q}q'' - 1) - 1}{q' + q'' - 1} = \frac{p}{q} + \left(\frac{p}{q} - 2\right)\frac{1}{q' + q'' - 1}.$$

Since $p/q \geqslant 2$, from the last expression we get $s_G(x_0, x_j) > p/q$, a contradiction for the case $i = 0$, where we shown that $s(x_0, x_j) \leqslant p/q$, $2 \leqslant j \leqslant q - 1$.

This concludes the proof. □

The results provided by Corollary 5.5, Theorems 5.3 and 5.6 can be summarized in the following two corollaries.

Corollary 5.7. *Let t be an admissible stretch number. Then, either $t \geqslant 2$ or $t = 2 - 1/i$ for some integer $i \geqslant 1$.*

Corollary 5.8. *For every admissible stretch number t, split composition can be used to generate a graph G with $s(G) = t$.*

Notice that, even if stretch numbers are rational numbers, we can also use every irrational number greater than 2 to define graph classes containing graphs with bounded induced distance. For instance, we can define $\mathrm{BID}(\pi)$, and $\mathrm{BID}(\pi) \neq \mathrm{BID}(k)$ for every rational number k. On the other hand, if we take an irrational number between 1 and 2 to define a class, then there exists a rational number to define the same class. For example, $\mathrm{BID}(\sqrt{3}) = \mathrm{BID}(5/3)$.

6. Recognition problem

The recognition problem for BID(1) can be solved in linear time [1,17]. In [7], this problem has been shown to be Co-NP-complete for the generic case (i.e., when k is not fixed), and the following question has been posed: What is the largest constant k such that the recognition problem for $\mathrm{BID}(k)$ can be solved in polynomial time?

In this section we show that Theorem 4.2 can be used to devise a polynomial algorithm to solve the recognition problem for the class $\mathrm{BID}(k)$, for every constant $k < 2$.

Lemma 6.1. *There exists a polynomial time algorithm to test whether a given graph G contains, as induced subgraph, a cycle C_n with $n \geqslant 6$ and $cd(C_n) \leqslant 1$.*

Proof. It is easy to see that a cycle C_n with $n \geqslant 6$ and $cd(C_n) \leqslant 1$ exists in G if and only if there are in G two nodes x and y such that all the following conditions hold:

(1) there exists a node u such that $p_G(x, y) = (x, u, y)$;
(2) there exists an induced path $p'_G(x, y)$ such that $|p'_G(x, y)| \geqslant 4$;
(3) every chord (if any) in the cycle C_n induced by $p_G(x, y) \cup p'_G(x, y)$ is incident to u.

Recalling that $I_G(x, y)$ denotes the set containing all the nodes (except x and y) that belong to a shortest path from x to y, let $M = I_G(x, y)$. Moreover, if $d_{G-M}(x, y) = 3$, let $X = I_{G-M}(x, y) \cap N(x)$, and $Y = I_{G-M}(x, y) \cap N(y)$. If path $p'_G(x, y)$ exists, then it must be one of the following paths:

P_1: an induced path from x to y not containing neither nodes of X nor nodes of Y;
P_2: an induced path from x to y not containing nodes of X, and containing one node of Y;
P_3: an induced path from x to y containing one node in X, and no node of Y;
P_4: an induced path from x to y containing one node of X and one node of Y.

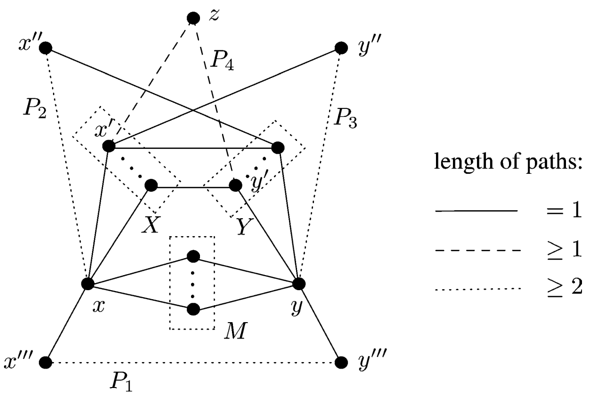

Fig. 4. P_1, P_2, P_3 and P_4 are the four kinds of induced paths from x to y that contribute to form a cycle C_n, $n \geqslant 6$, with $cd(C_n) \leqslant 1$.

In Fig. 4 these four paths correspond to the path through x''' and y'''; the path through x'' and one node in Y; the path through one node in X and y''; and the path through x', z, and y', respectively.

Procedure `Test` (Fig. 5) analyzes every pair of nodes $\{x, y\}$ having distance 2 in G, and test whether an induced path of type P_i, $1 \leqslant i \leqslant 4$, between x and y exists in G. It is easy to see that Procedure `Test` is correct. It remains to be shown that the procedure runs in time polynomial in the size of the input graph G.

All the pairs of nodes that are at distance 2 in G can be computed in $O(n^2)$ time at step 1. Each one of steps 3, 9, and 10 can be computed in $O(m)$ time, where m is the number of edges in G; each one of the steps 11, 14, and 18 can be computed in $O(m)$ time. Cycle at step 17 can be performed at most n^2 times. Hence, the total time to perform the Procedure `Test` is $O(n^4 m)$ time. □

Theorem 6.2. *For any fixed integer $i \geqslant 1$, the recognition problem for the class* BID$(2 - 1/i)$ *can be solved in polynomial time.*

Proof. For $i = 1$ the problem can be solved in linear time [1,17]. By Theorem 4.2, a brute-force, rather naive algorithm for solving the recognition problem for the class BID$(2 - 1/i)$, $i > 1$, is: test if G contains, as induced subgraph, a cycle C_n with $n \geqslant 6$ and $cd(C_n) \leqslant 1$, or a cycle C_{3i+2} with chord distance equal to i. According to Proof of Lemma 6.1, this means that a graph G belongs to BID$(2 - 1/i)$ if and only if Procedure `Test` returns false if applied to G, and does not exist a cycle C_{3i+2} in G such that $cd(C_{3i+2}) = i$.

To test the existence of a cycle C_{3i+2} in G such that $cd(C_{3i+2}) = i$ we can check whether any subset of $3i + 2$ nodes of G forms a cycle with chord distance equal to i. This test can be implemented in polynomial time since the number of subsets of nodes with $3i + 2$ elements is bounded by n^{3i+2}, where n is the number of nodes in G. □

In what follows we show that the strategy used to prove Theorem 6.2 cannot be applied to find polynomial solutions to the recognition problem for class BID(k), for each integer

procedure Test;
– **input**: a connected graph $G = (V, E)$.
– **output**: true if and only if there exists C_n, $n \geqslant 6$, in G such that $cd(C_n) \leqslant 1$.

for each $(x, y) \in G$ such that $d_G(x, y) = 2$
begin
  compute $M = I_G(x, y)$
  if x and y connected in $G - M$
  then begin
    if $d_{G-M}(x, y) > 3$
    then return(true) {there exists P_1; X and Y are empty}
    else begin {both X and Y are not empty}
      compute $X = I_{G-M}(x, y) \cap N(x)$
      compute $Y = I_{G-M}(x, y) \cap N(y)$
      if x, y are connected in $G - (M \cup X)$
      then return(true) {there exists P_1 or P_2}
      else {both P_1 and P_2 do not exist}
        if x, y are connected in $G - (M \cup Y)$
        then return(true) {there exists P_3}
        else {P_1, P_2, and P_3 do not exist}
          for each pair (x', y') such that $x' \in X$, $y' \in Y$, $(x', y') \notin E$
            if x', y' are connected in the subgraph
              $G - (M \cup (X \setminus \{x'\}) \cup (Y \setminus \{y'\}))$
            then return(true) {there exists P_4}
    end
  end
end
return(false)

Fig. 5. Testing the existence of a cycle C_n, $n \geqslant 6$, with $cd(C_n) \leqslant 1$.

$k \geqslant 2$. In particular, we show that it is not possible to characterize $\mathrm{BID}(k)$ by listing all its forbidden induced subgraphs, as in Theorem 4.2.

Theorem 6.3. *For each integers $k \geqslant 2$ and $i \geqslant 2$, there exists a minimal forbidden subgraph for the class* $\mathrm{BID}(k)$, *which is a cycle with chord distance equal to i.*

Proof. Let $k \geqslant 2$ and $i \geqslant 2$ be two integers. We define $G = G(n_1, n_2, \ldots, n_i)$ where $n_1 = 2k+2$, $n_i = 2k+2$, and $n_j = k+3$ for each $2 \leqslant j \leqslant i-1$. By definition of graph $G(n_1, n_2, \ldots, n_i)$, it easily follows that G is a cycle with chord distance equal to i. We now prove that $G \notin \mathrm{BID}(k)$, while each proper induced subgraph of G belongs to $\mathrm{BID}(k)$.

In what follows we use facts of Lemma 5.4. From Fact 1 it follows that

$$s_G(x_0, x_{i+1}) = \frac{2(2k+2) + (i-2)(k+3) - 3i + 1}{i+1} = k + \frac{k-1}{i+1}$$

and this implies that $G \notin \mathrm{BID}(k)$.

Let G' (G'', respectively) be the induced subgraph of G isomorphic to $G(n_2, n_3, \ldots, n_{i-1})$ ($G(n_2, \ldots, n_i)$, respectively). By Fact 5, each induced subgraph of G' has stretch number smaller or equal to $s(G')$, and by Fact 4, $s(G'') \geqslant s(G')$. Then, G'' is the proper induced subgraph of G having the maximum stretch number, and

$$s(G'') = s_G(x_1, x_{i+1}) = \frac{(2k+2) + (i-2)(k+3) - 3(i-1) + 1}{i} = k.$$

This proves that each proper induced subgraph of G belongs to $\mathrm{BID}(k)$. $\square$

7. Conclusions

In this paper we provide new results about graph classes that represent a parametric extension of the class of distance-hereditary graphs. In any graph G belonging to the generic new class $\mathrm{BID}(k)$, the distance between every two connected nodes in every induced subgraph of G is at most k times their distance in G. The smallest k such that $G \in \mathrm{BID}(k)$ is called stretch number of G.

The main results of the paper can be summarized as follows. Any rational number $k \geqslant 2$ is an admissible stretch number, that is, there exists a graph having stretch number k. Surprisingly, the only admissible stretch numbers smaller then 2 are $k = 2 - 1/i$, for every integer $i \geqslant 1$. In all cases, constructive proofs for the existence of a graph with an admissible stretch number are given. For each class $\mathrm{BID}(2 - 1/i)$, a characterization based on forbidden subgraphs is provided. Such a characterization eventually leads to a polynomially time recognition algorithm for the class $\mathrm{BID}(2 - 1/i)$, for every integer $i \geqslant 2$. The running time of the algorithm is bounded by $\mathrm{O}(n^{3i+2})$, when it is used for the class $\mathrm{BID}(2 - 1/i)$.

Many problems are left open. First of all, notice that the algorithm provided in the paper is only of theoretical value. Even for the class $\mathrm{BID}(3/2)$ (which is the closest one to the distance-hereditary graphs) the running time is already $\mathrm{O}(n^8)$. As a consequence, finding an efficient recognition algorithm for $\mathrm{BID}(3/2)$ is an interesting problem. A related natural question would be whether the recognition of $\mathrm{BID}(2 - 1/i)$ is fixed parameter tractable (taking i as parameter).

Moreover, several algorithmic problems are solvable in polynomial time for distance-hereditary graphs [11]. Can some of these results be extended to $\mathrm{BID}(k)$, $k > 1$?

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