AN ALGEBRAIC APPROACH TO ASSIGNMENT PROBLEMS

Rainer E. BURKARD

University of Cologne, Cologne, Federal Republic of Germany

Willi HAHN

Fachhochschule des Landes Rheinland Pfalz, Bingen, Federal Republic of Germany

Uwe ZIMMERMANN

University of Cologne, Cologne, Federal Republic of Germany

For assignment problems a class of objective functions is studied by algebraic methods and characterized in terms of an axiomatic system. It says essentially that the coefficients of the objective function can be chosen from a totally ordered commutative semigroup, which obeys a divisibility axiom. Special cases of the general model are the linear assignment problem, the linear bottleneck problem, lexicographic multicriteria problems, p-norm assignment problems and others. Further a polynomial bounded algorithm for solving this generalized assignment problem is stated. The algebraic approach can be extended to a broader class of combinatorial optimization problems.

Key words: Combinatorial Optimization, Assignment Problem, Ordered Semigroups, Bottleneck Objective, Lexicographical Objective, Labeling Method, Admissible Transformation.

1. Introduction

The starting point of this paper was the fact, that linear sum assignment problems

$$\min_{\varphi} \sum_{i=1}^{n} c_{i\varphi(i)} \tag{1}$$

and linear bottleneck assignment problems

$$\min_{\varphi} \max_{1 \leqslant i \leqslant n} c_{i\varphi(i)} \tag{2}$$

can be solved by a joint algorithm[1]. The relationship between the two problems can be expressed mathematically by defining a general linear assignment problem (GLAP) in an ordered semigroup. A similar formulation has been given by Gabovič [3]. Besides the above mentioned problems there is a broad class of problems of practical interest which can also be viewed as special cases of the

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[1] φ is a permutation of the set $N = \{1, \ldots, n\}$

GLAP. For instance this holds for the p-norm assignment problem

$$\min_{\varphi} \|(c_{i\varphi(i)})\|_p, \qquad 0 < p \leq \infty \tag{3}$$

and for the lexicographic multicriteria assignment problem

$$\operatorname*{lex\,min}_{\varphi} \begin{pmatrix} \sum\limits_{i=1}^{n} c_{i\varphi(i)}^{1} \\ \dots\dots \\ \sum\limits_{i=1}^{n} c_{i\varphi(i)}^{m} \end{pmatrix}.$$

   Further important examples are given in Section 5. In Section 2 the GLAP is defined and the underlying algebraic theory is developed in Section 3. The algorithm solving the GLAP is based on admissible transformations which are introduced in Section 4. Extensions of the algebraic approach to other combinatorial problems are mentioned in Section 6.

2. The general linear assignment problem (GLAP)

Let S be a nonempty set together with an internal composition $*$ and an order relation $\leq$. Then the following four axioms shall hold for the system $(S, *, \leq)$:

Axioms.   (I) *S is totally ordered with respect to "$\leq$",*
   (II) *$(S, *)$ is a commutative semigroup,*
   (III) *$a \leq b$ implies $a * c \leq b * c$ for all $a, b, c \in S$,*
   (IV) *for all $a, b \in S$ with $a < b$ there exists an element $x \in S$ such that*

   $a * x = b.$

Definition 1. *General linear assignment problem* (GLAP):
Let be given a system $(S, *, \leq)$ which obeys (I)–(IV) and elements $c_{ij} \in S$ for $i, j \in N$. Find a permutation $\tilde{\varphi}$ of the set N which minimizes $*_{i \in N} c_{i\varphi(i)}$, where

$$\operatorname*{*}_{i \in N} c_{i\varphi(i)} = c_{1\varphi(1)} * c_{2\varphi(2)} * \dots * c_{n\varphi(n)}.$$

Therefore the GLAP can be written as

$$\min_{\varphi} \operatorname*{*}_{i \in N} c_{i\varphi(i)}. \tag{3}$$

Remarks. (1) To define a GLAP it is sufficient to have an ordered set S which is a commutative semigroup. Under these conditions the problem (3) is properly posed. The minimal values can be obtained by complete enumeration of the $n!$ possible permutations. Axioms I, III and IV guarantee that the uniquely deter-