Structural induction and coinduction in a fibrational setting

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Abstract. *We present a categorical logic formulation of induction and coinduction principles for reasoning about inductively and coinductively defined types. Our main results provide sufficient criteria for the validity of such principles: in the presence of comprehension, the induction principle for initial algebras is admissible, and dually, in the presence of quotient types, the coinduction principle for terminal coalgebras is admissible. After giving an alternative formulation of induction in terms of binary relations, we combine both principles and obtain a mixed induction/coinduction principle which allows us to reason about minimal solutions* $X \cong \sigma(X)$ *where* X *may occur both positively and negatively in the type constructor* σ*. We further strengthen these logical principles to deal with contexts and prove that such strengthening is valid when the (abstract) logic we consider is contextually/functionally complete. All the main results follow from a basic result about adjunctions between 'categories of algebras' (inserters).*

Introduction

A well-stablished approach to the semantics of data types is to regard them as (Lambek) algebras for endofunctors $T\colon \mathbb{B} \to \mathbb{B}$ on a category $\mathbb{B}$ with suitable structure. Inductive data types correspond to initial algebras $(D, TD \overset{c}{\to} D)$, in which case T specifies the signature of *constructors* of the type and c gives the interpretation of such constructors in D. Dually, coinductive data types correspond to terminal coalgebras $(C, C \overset{d}{\to} TC)$, where T specifies the signature of *destructors* of the data type and d maps an element of the type C to its components. Of course, many other mathematical structures can be understood as initial algebras or terminal coalgebras, among the extensive relevant bibliography see e.g. [27, 43, 31, 45, 1, 19, 41, 15, 38, 16, 17, 21]. See also [9, 10] for an experimential programming language CHARITY, which essentially only contains algebras and coalgebras (and iterated combinations of these, which we do not consider here in detail, but see the comments before §§3.1).

Once the object of study has been singled out by a universal property (such as initiality/terminality of algebras/coalgebras), this property becomes the main tool to infer properties about the object. In fact, a main point of this paper is to formulate, in a canonical fashion, an *induction principle* for initial algebras and a *coinduction* principle for terminal coalgebras, considering *polynomial functors* $T\colon \mathbb{B} \to \mathbb{B}$, built up from the (universal) structure of the category $\mathbb{B}$. More specifically, we concentrate on bicartesian (closed) categories, together with the associated class of endofunctors determined by such structure.

We wish to consider logical propositions over (co-)algebras. Such propositions will be the formulas $\varphi(x)$ of a predicate logic, possibly containing free variables x ranging over types A. Categorically we capture such a logic by a *fibration* $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ (with suitable structure) over the category $\mathbb{B}$, where we think of the objects of the *total* category $\mathbb{P}$ as formulas (or propositions) in context and those of the *base* category $\mathbb{B}$ as types. The functor p sends a proposition to its underlying context, containing the types of its free variables. A typical case is the 'internal logic' fibration $\begin{smallmatrix}\mathrm{Sub}(\mathbb{B})\\ \downarrow\\ \mathbb{B}\end{smallmatrix}$, where $\mathrm{Sub}(\mathbb{B})$ is the category of subobjects: objects are subobjects $(X \rightarrowtail I)$, and morphisms $(X \rightarrowtail I) \to (Y \rightarrowtail J)$ are maps between the underlying objects $I \to J$ which commute with the given subobjects. The analysis we present will show that logical principles such as (co-)induction arise from the relationship between the (universally determined) structure of the total category $\mathbb{P}$ and the structure of the base category $\mathbb{B}$. The logical interpretation of this relationship hinges on the fact that the structure of $\mathbb{P}$ can be inferred from suitable structure of the fibration p. The structure of p that interests us here corresponds to (the interpretation) of connectives and quantifiers (among other logical operations) of the predicate logic which it represents.

Within this setting, we make a fundamental conceptual identification: an inductive predicate $P \in \mathbb{P}_D$ (in the total category $\mathbb{P}$) for an algebra $c\colon TD \to D$ (in the base category) amounts to an algebra $f\colon \mathrm{Pred}(T)P \to P$ in the total category, over the given algebra (D, c). The functor $\mathrm{Pred}(T)\colon \mathbb{P} \to \mathbb{P}$ is defined via the polynomial structure of T; that is, $\mathrm{Pred}(T)$ is built with the same type constructors (products, coproducts and exponentials) as T in the category $\mathbb{P}$. A dual observation applies to coinductive relations—sometimes called (strong) bisimulations [43]—and coalgebras.

Once such analysis is carried out, we will be in position to give sufficient criteria for the validity of the induction and coinduction principles, which constitute the main results of the paper. Given the nature of the conditions we impose, we can present these results (Theorem 5.1) as **admissibility properties of constructive predicate logic** (taking proofs into account):

- if the logic admits *comprehension*, it satisfies the induction principle for initial algebras of (polynomial) endofunctors;
- if the logic admits *quotients of relations*, it satisfies the coinduction principle for terminal coalgebras of (polynomial) endofunctors.

The second result is essentially the dual of the first. To make this duality explicit, we give a reformulation of induction, originally stated for predicates, in terms of binary relations. We further prove these two formulations of induction to be equivalent under mild exactness conditions (Theorem 3.4).

We are then able to combine induction and coinduction to give a reasoning principle for recursive data types, involving mixed variance functors, typically the exponential functor $\Rightarrow : \mathbb{B}^{\mathrm{op}} \times \mathbb{B} \to \mathbb{B}$, based on Freyd's analysis of such recursive types in terms of initial/terminal algebras on self-dual categories, *cf.* [16, 38]. The validity of this principle in the presence of comprehension and quotients seems to be the major novelty of this work from the point of view of (constructive) logic (see Theorem 6.4).

We finally analyse another intrinsic property of first-order predicate logic with respect to induction/coinduction, namely the 'stability' of such principles under the addition of indeterminates. Such stability property is necessary if we wish to use these principles in arbitrary contexts (of data and propositions). This is the case when we define functions of several arguments by induction on one of them (*e.g.* addition of natural numbers); the remaining arguments are considered fixed (but arbitrary) constants and play the role of a context. At the logical level we may have assumptions about such arguments, which form a propositional context. It is in the presence of these 'data with propositional hypotheses' context that we wish to apply the induction/coinduction principles. The logical properties involved to guarantee such stability are *contextual* and *functional completeness* as formulated in [22]. They amount to representability conditions with respect to the *addition of indeterminates*. Functional completeness guarantees the stability of initial algebras (and hence of their associated induction principle), while contextual completeness does the same with respect to terminal coalgebras (and coinduction), see Theorem 7.6.

Applications of coinduction principles occur prominently in [15] (internal full abstraction for the lazy lambda calculus) and in [38] (adequacy and strong extensionality for operational semantics). Both references are primarily concerned with (abstract) domain theory. Here we give an abstract analysis of induction and coinduction principles in the spirit of categorical logic, which achieves the right level of abstraction required to combine the salient features of the above approaches: we use an abstract notion of predicate (and of relation) as embodied by the notion of *fibration* similar to [38], but unlike this latter, we use the polynomial structure of the functor to define its 'relational lifting' (via logical predicates). Hence the functor defining the data type canonically determines its lifting, a desideratum of the approach in [15].

It is worthy to emphasise the conceptual simplicity and technical economy of the present work: all the admissibility and stability results are immediate consequences of a basic result about adjunctions between categories of algebras (Theorem A.5). Although the result could be proved by direct calculation for ordinary categories, the 2-categorical version is equally simple to prove via universality of inserters, and makes the result applicable to the stability of the induction/coinduction principles in §§7, where we work in the 2-category $\mathbf{Cat}^{\to}$. Since this purely 2-categorical excursion about the functoriality of inserters may be distracting in the main text, we relegate it to the Appendix.

The paper is organised as follows: in the next section we discuss some concrete examples of induction and coinduction principles to motivate their subsequent formal treatment. We continue with another preliminary section containing background material. In §§3 we start with the actual content of the paper: the formulation of the induction principle for T-algebras as an exactness condition. This is further elaborated in §§3.1, where the principle is reformulated for binary relations in terms of equality, in order to exhibit patently the duality of induction and coinduction for coalgebras. We briefly touch upon the relationship between induction principles for algebras of different functors in §§3.2. In §§4 we formulate the coinduction principle for coalgebras, while in §§5 we give sufficient criteria for the validity of the induction and coinduction principles; the criteria consist of effectively guaranteeing the relevant exactness conditions via the existence of adjoint functors. In §§6 we combine our previous formulations of induction and coinduction into a mixed principle suitable for minimal invariants of mixed variance polynomial functors $T\colon \mathbb{B}^{\mathrm{op}} \times \mathbb{B} \to \mathbb{B}$. We conclude in §§7 by strengthtening our formulations of logical principles to make them stable under weakening of context, so that the principles can be applied in arbitrary contexts, rather than the empty one (which was the case considered up to this point). We also extend our criteria of validity to incorporate this stable version, by recourse to contextual and functional completeness.

1 Examples of induction and coinduction

This section analyses examples of induction and coinduction, providing motivation for their formal treatment in §§3 and §§4. We consider both *definition* and *reasoning* by induction. For example, on an (initial) algebra of lists (of some fixed type A) with constructors $\mathsf{nil}\colon 1 \to \mathsf{list}(A)$ for the empty list and $\mathsf{cons}\colon A \times \mathsf{list}(A) \to \mathsf{list}(A)$ for adding an element to a list, we can inductively define a length map $\mathsf{len}\colon \mathsf{list}(A) \to \mathbb{N}$ by the two clauses

$$\mathsf{len}(\mathsf{nil}) = \mathsf{0}, \qquad \mathsf{len}(\mathsf{cons}(a, \ell)) = \mathsf{S}(\mathsf{len}(\ell)),$$

where $\mathsf{0}$ and S are the zero and successor constructors of the natural numbers $\mathbb{N}$. Formally we define $\mathsf{len}\colon \mathsf{list}(A) \to \mathbb{N}$ as the unique algebra map from the initial algebra of lists to the set of natural numbers, suitably equipped with a list-algebra structure $1 \to \mathbb{N}$, $A \times \mathbb{N} \to \mathbb{N}$ as in

$$\begin{tikzcd}
\mathsf{list}(A) \arrow[rr, dashed, "\mathsf{len}"] & & \mathbb{N} \\
1 + A \times \mathsf{list}(A) \arrow[u, "{[\mathsf{nil}, \mathsf{cons}]}"] \arrow[rr, dashed, "1 + A \times \mathsf{len}"'] & & 1 + A \times \mathbb{N} \arrow[u, "{[\mathsf{0}, \mathsf{S} \circ \pi']}"']
\end{tikzcd}$$

This is definition by induction. Reasoning by induction involves predicates (or relations): for a predicate $P \subseteq \mathsf{list}(A)$, assuming that $P(\mathsf{nil})$ and $P(\ell) \Rightarrow P(\mathsf{cons}(a, \ell))$ hold (for each $a \in A$ and $\ell \in \mathsf{list}(A)$), we conclude that P must be the whole of $\mathsf{list}(A)$. That is, every predicate (on the initial algebra) which is closed under the operations of the algebra must be the whole set (or must contain the truth predicate, as we shall say later). This requirement that P is closed under the operations of the algebra is expressed abstractly by the condition that P itself carries an algebra structure, in a category of predicates. In our analysis, validity of this induction principle follows from comprehension: the algebra structure on P in a category of predicates can be transferred to an algebra structure on the associated set $\{P\} = \{\ell \in \mathsf{list}(A) \mid P(\ell)\}$ in the category of sets. Initiality of $\mathsf{list}(A)$ then yields a unique algebra map $\mathsf{list}(A) \dashrightarrow \{P\}$. From this it follows that $P(\ell)$ holds for all $\ell \in \mathsf{list}(A)$, because comprehension $\{-\}$ is right adjoint to 'truth'.

Alternatively we may express induction in terms of (binary) relations: if $R \subseteq \mathsf{list}(A) \times \mathsf{list}(A)$ is a relation on lists satisfying $R(\mathsf{nil}, \mathsf{nil})$ and $R(\ell, \ell') \Rightarrow R(\mathsf{cons}(a, \ell), \mathsf{cons}(a, \ell'))$ for all $a \in A$, then R must be reflexive. That is, the induction principle for relations says that relations which are suitably closed under the operations (congruences) must contain the equality relation. Thus truth predicates and equality relations play a fundamental role in the formulation of induction (as a reasoning principle).

We turn to coinduction, which is a less familiar notion. Coinduction is associated with (terminal) coalgebras like induction is to (initial) algebras. Coalgebras $X \to TX$ of a functor T may be understood as abstract dynamical systems, consisting of a state space X together with a transition map, or 'dynamics', $X \to TX$, acting on X. We also consider both *definition* and *reasoning* by coinduction. For example, consider a (deterministic, partial) automaton consisting of a state space X, an attribute or output map $\mathsf{at}: X \to O$ and a procedure $\mathsf{pr}: X \times \Sigma \to 1 + X$. For every state $s \in X$ and every symbol a in the input alphabet Σ we get a result $\mathsf{pr}(s, a) \in 1 + X$. If $\mathsf{pr}(s, a) \in X$ the computation is succesful and yields a new state, but in case $\mathsf{pr}(s, a) = * \in 1$, the computation is unsuccesful (and the automaton halts). Such an automaton may be identified with a coalgebra $X \to O \times (1 + X)^{\Sigma}$. The *behaviour* of the automaton in a specific state $s \in X$ tells us what we can observe externally, by considering the possible output value in O resulting from a sequence of inputs in $\Sigma^* = \coprod_{n \geq 0} \Sigma^n$. Such observations form a set

$$\begin{aligned} C = \{\varphi: \Sigma^* \to 1 + O \mid \varphi(\langle\rangle) \neq * \text{ and } \forall \sigma \in \Sigma^*. \forall a \in \Sigma. \\ \varphi(\sigma) = * \;\Rightarrow\; \varphi(a \cdot \sigma) = *\}. \end{aligned}$$

For this set C, we have attribute and procedure operations

$$\begin{array}{ccc} C & \longrightarrow & O \\ \varphi & \longmapsto & \varphi(\langle\rangle) \end{array} \qquad \begin{array}{ccl} C \times \Sigma & \longrightarrow & 1 + C \\ (\varphi, a) & \longmapsto & \begin{cases} * & \text{if } \varphi(\langle a \rangle) = * \\ \lambda\sigma.\, \varphi(a \cdot \sigma) & \text{otherwise.} \end{cases} \end{array}$$

and thus forms the state space of an automaton. The induced map $C \to O \times (1 + C)^{\Sigma}$ is the terminal coalgebra: for an arbitrary automaton on X as above, we get a unique mediating map f in a situation:

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, "\langle \mathsf{at}, \mathsf{pr} \rangle"'] & C \arrow[d] \\ O \times (1 + X)^{\Sigma} \arrow[r, dashed, "O \times (1 + f)^{\Sigma}"'] & O \times (1 + C)^{\Sigma} \end{tikzcd}$$

This map f sends a state $s \in X$ to the function $f(s): \Sigma^* \to 1 + O$ in C given by

$$f(s)(\langle\rangle) = \mathsf{at}(s) \qquad f(s)(a \cdot \sigma) = \begin{cases} * & \text{if } \mathsf{pr}(s, a) = * \\ f(\mathsf{pr}(s, a))(\sigma) & \text{otherwise.} \end{cases}$$

An (applicative) bisimulation relation on such a coalgebra automaton of the form $\langle \mathsf{at}, \mathsf{pr}\rangle : X \to O \times (1+X)^{\Sigma}$ is a relation $R \subseteq X \times X$ on the states satisfying:

$$R(x,y) \Rightarrow \begin{cases} \mathsf{at}(x) = \mathsf{at}(y), \qquad \text{and} \\ \text{for each } a \in \Sigma\text{: } \mathsf{pr}(x,a) \in X \text{ iff } \mathsf{pr}(y,a) \in X, \\ \qquad \text{and in that case } R(\mathsf{pr}(x,a), \mathsf{pr}(y,a)). \end{cases}$$

We call two states $x, y \in X$ bisimilar, and write $x \underline{\leftrightarrow} y$, if there is an applicative bisimulation $R \subseteq X \times X$ with $R(x,y)$. This is equivalent to saying that bisimilarity is the union of all bisimulation relations.

The coinduction principle says that bisimilar elements x, y have the same behaviour: $f(x) = f(y)$ in C. More abstractly, it says that every bisimulation is contained in the kernel relation of the unique map to the terminal coalgebra. The task of showing that states have the same behaviour is thus reduced to showing that they are contained in some relation R which is suitably closed under the coalgebra operations. Such a relation R carries a coalgebra structure in a category of relations. In our analysis the coinduction principle holds in the presence of quotients: the coalgebra structure on R in the category of relations can be transferred to a coalgebra structure on the quotient set X/R in the category of sets. We get a unique coalgebra map $X/R \dashrightarrow C$, and thus a map of relations $R \to \mathrm{Eq}(C)$, since quotients are left adjoint to equality. Hence elements related by R are equal when mapped to C.

2 Preliminaries

In this section we explain the relevant technical notions that will be used in our abstract treatment of induction and coinduction from §§3 onwards. These are: algebras and coalgebras for polynomial endofunctors on bicartesian (closed) categories (2.1), logic interpreted in (bicartesian) fibrations (2.2), including comprehension and quotients (2.3), lifting of polynomial functors to fibred categories (2.4) and transfer of adjunctions to categories of (co)algebras (2.5).

2.1 Algebras and coalgebras of polynomial functors

Let $\mathbb{B}$ be a category and $T: \mathbb{B} \to \mathbb{B}$ an endofunctor on $\mathbb{B}$. An algebra (or, a T-algebra, to be explicit) is an object $X \in \mathbb{B}$ together with a morphism $a: TX \to X$. The object X is called the carrier, and the map a is the structure. As an example, the lists $\mathsf{list}(A)$ of type A in the previous section are algebras $1 + A \times \mathsf{list}(A) \to \mathsf{list}(A)$ of the functor $T(X) = 1 + A \times X$ on $\mathbb{B} = \mathbf{Sets}$. A morphism of algebras (or an algebra map, for short) from $(a: TX \to X)$ to $(b: TY \to Y)$ is a morphism $f: X \to Y$ in $\mathbb{B}$ between the carriers which commutes with the structures: $f \circ a = b \circ Tf$. We write $\mathrm{Alg}(T)$ for the category of algebras of the functor T. Initial algebras—*i.e.* initial objects in the category $\mathrm{Alg}(T)$—play a special role in data type theory, see *e.g.* [18, 49]. A standard result, due to Lambek, is that for an initial algebra $a: TX \to X$, a is an isomorphism.

Dually a T-coalgebra is a morphism of the form $c: X \to TX$. The object X is called the carrier or the state space, and the map b is called the structure, the transition map, or the dynamics of the coalgebra. A morphism of coalgebras from $(c: X \to TX)$ to $(d: Y \to TY)$ is a morphism $f: X \to Y$ in $\mathbb{B}$ with

$d \circ f = Tf \circ c$. We write $\mathrm{CoAlg}(T)$ for this category of coalgebras. Note that $\mathrm{CoAlg}(T)$ is $(\mathrm{Alg}(T^{\mathrm{op}}))^{\mathrm{op}}$, where T^{op} is the induced functor $\mathbb{B}^{\mathrm{op}} \to \mathbb{B}^{\mathrm{op}}$. Terminal coalgebras will be of most interest; their structure maps (or dynamics) are isomorphisms, dualizing the above observation for initial algebras. Both these categories of algebras and coalgebras can be characterised as inserters (see the Appendix for the relevant technical details).

We shall be especially interested in so-called *polynomial* functors T. They are built up from the identity, constants, and finite products and coproducts. Formally, call $\mathbb{B}$ a bicartesian category if it has finite products $(1, \times)$ and coproducts $(0, +)$. We do not require any distributivity at this stage. The class of polynomial functors $\mathbb{B} \to \mathbb{B}$ is inductively defined by the following clauses.

(i) The identity functor is polynomial, and for each object $A \in \mathbb{B}$, the constant functor $X \mapsto A$ is polynomial; this includes the special cases $A = 1$, $A = 0$.

(ii) If $T, S\colon \mathbb{B} \rightrightarrows \mathbb{B}$ are polynomial functors, then so are the product and coproduct (in the category $\mathbf{Cat}(\mathbb{B}, \mathbb{B})$ of endofunctors on $\mathbb{B}$):

$$X \mapsto T(X) \times S(X) \qquad \text{and} \qquad X \mapsto T(X) + S(X).$$

For example, the functor $X \mapsto 1 + A \times X$ used for lists in the previous section is polynomial. And the automaton functor $X \mapsto O \times (1 + X)^{\Sigma}$ is polynomial if the input alphabet Σ is finite: if it has n elements, then we can write this functor as

$$X \mapsto O \times \underbrace{(1 + X) \times \cdots \times (1 + X)}_{n \text{ times}}.$$

2.2 Bicartesian fibrations

In the previous subsection we have considered a functor T on a category $\mathbb{B}$, where we think of the objects of $\mathbb{B}$ as sets or types, and regard T as a signature of type constructors. In order to reason about such a situation we need a logic, consisting of a category $\mathbb{P}$ of predicates on types. This is formalised by requiring a functor $\mathbb{P} \to \mathbb{B}$, which is a fibration (see [32, 33, 35, 23] for an exposition of this point of view). For an object $A \in \mathbb{B}$, we write $\mathbb{P}_A$ for the subcategory of objects and maps of $\mathbb{P}$ that get sent to A. This is the category of predicates on A. The fibration gives us (using the Axiom of Choice) for every morphism $u\colon A \to B$ in $\mathbb{B}$ a substitution functor $u^*\colon \mathbb{P}_B \to \mathbb{P}_A$. A morphism from a fibration $\left(\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}\right)$ to another fibration $\left(\begin{smallmatrix}\mathbb{Q}\\ \downarrow q\\ \mathbb{A}\end{smallmatrix}\right)$ consists of a pair of functors $K\colon \mathbb{B} \to \mathbb{A}$ between the *base* categories and $H\colon \mathbb{P} \to \mathbb{Q}$ between the *total* categories, which commutes with the fibrations: $K \circ p = q \circ H$, and with substitution: $H(u^*(X)) \cong K(u)^*(HX)$, canonically. We then call H a fibred functor. A 2-cell $(K, H) \Rightarrow (K', H')$ between two such morphisms of fibrations consists of two natural transformations $\alpha\colon K \Rightarrow K'$, $\beta\colon H \Rightarrow H'$ with $q\beta = \alpha p$. This sets up the 2-category $\mathbf{Fib}$ of fibrations.

Just like we have used bicartesian categories above, we consider *bicartesian fibrations*, meaning that we have bicartesian structure both in the base category $\mathbb{B}$ and in the total category $\mathbb{P}$ of a fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ in such a way that the functor p (strictly)[1] preserves this structure. The following result shows how this (global) bicartesian structure can be obtained from fibrewise (local) bicartesian structure. The formulations of the induced global products and coproducts are sometimes referred to as the "logical predicate" formulas, *cf.* [20].

2.1. Lemma. (i) *Consider a fibration* $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *with fibred cartesian structure (*i.e. *cartesian structure in every fibre, which is preserved by substitution functors* u^**). Assume that the base category* $\mathbb{B}$ *also has bicartesian structure, the fibres have finite coproducts and the substitution functors* $\kappa^*\colon \mathbb{P}_{A+B} \to \mathbb{P}_A$ *(along coprojections) have left adjoints* $\coprod_\kappa$*. Then the total category* $\mathbb{P}$ *has bicartesian structure, which is strictly preserved by the functor* p*.*

(ii) *If additionally the substitution functors preserve finite coproducts and the following diagrams are pullback squares in* $\mathbb{B}$

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\kappa"'] & A' \arrow[d, "\kappa"] \\ A+B \arrow[r, "u+v"'] & A'+B' \end{tikzcd}$$

and the coproducts $\coprod_\kappa$ *satisfy the Beck-Chevalley condition with respect to these pullback squares, then the induced functor* $+\colon \mathbb{P}\times\mathbb{P} \to \mathbb{P}$ *from* (i) *is a fibred functor, so that* p *becomes a bicartesian object in the 2-category of fibrations.*

Proof. The terminal object $1 \in \mathbb{P}_1$ in the fibre over $1 \in \mathbb{B}$ is terminal object in $\mathbb{P}$, and similarly the initial object $0 \in \mathbb{P}_0$ over $0 \in \mathbb{B}$ is initial in $\mathbb{P}$. The product and coproduct in $\mathbb{P}$ of $X \in \mathbb{P}_A$ and $Y \in \mathbb{P}_B$ are respectively

$$\pi^*(X) \times \pi'^*(Y) \in \mathbb{P}_{A\times B} \qquad \text{and} \qquad \textstyle\coprod_\kappa(X) + \coprod_{\kappa'}(Y) \in \mathbb{P}_{A+B}$$

where $\times, +$ refer to the product and coproduct in the fibre. The Beck-Chevalley condition in (ii) is used to show that the coproduct functor is fibred. □

The additional conditions in the second point of the lemma do not hold in all of our examples. They are not needed for the theory below.

2.2. Examples. (i) The (classical) logic of predicates over sets is captured by the fibration $\begin{smallmatrix}\mathrm{Sub}(\mathbf{Sets})\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ of subobjects of sets. It satisfies the conditions of the lemma, and hence is a bicartesian fibration. For instance, the product of predicates $X \subseteq A$ and $Y \subseteq B$ is the predicate $\{(x,y) \mid x \in X \wedge y \in Y\} \subseteq A \times B$.

More generally, for every regular category $\mathbb{B}$, the associated subobject fibration $\begin{smallmatrix}\mathrm{Sub}(\mathbb{B})\\ \downarrow\\ \mathbb{B}\end{smallmatrix}$ is bicartesian. Furthermore, if $\mathbb{B}$ has disjoint and universal finite coproducts $(0,+)$, *e.g.* if $\mathbb{B}$ is a topos, the fibration satisfies the additional hypothesis of Lemma 2.1.(ii).

(ii) Let ω-$\mathbf{Cpo}_\perp$ be the category of pointed ω-cpos and strict continuous functions. Objects are posets with a bottom element — and least upper bounds (lub's) of ω-chains $(x_n)_{n\in\mathbb{N}}$ (where $x_n \leq x_{n+1}$). The morphisms are monotone functions which preserve bottoms and lubs of chains. Call a subset $X \subseteq A$ of $A \in \omega$-$\mathbf{Cpo}_\perp$ *admissible* if it contains $-$ and is closed under lubs of chains. Let $\begin{array}{c}\scriptstyle \mathrm{ASub}(\omega\text{-}\mathbf{Cpo}_\perp)\\ \downarrow\\ \scriptstyle \omega\text{-}\mathbf{Cpo}_\perp\end{array}$ be the fibration of these admissible subsets over ω-cpos. This fibration is bicartesian by Lemma 2.1.(i), since the coprojections $\kappa\colon A \hookrightarrow A + B$ are themselves admissible, so that we have coproducts $\coprod_\kappa$ by composition and Beck-Chevalley holds.

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[1] It is an easy coherence result that we can assume such structure to be preserved on-the-nose.

The category ω-$\mathbf{Cpo}_\perp$ has finite products in the usual way. However, it is not cartesian closed, but monoidal closed. The relevant tensor $\otimes$ is the "smash product" (or "wedge product" as it is called for pointed topological spaces) in which elements of the form $(x, -)$ and $(-, y)$ are identified with $(-, -)$. This tensor classifies bi-strict morphisms, that is, morphisms strict in each argument separately. The associated internal hom is the ω-cpo of strict continuous functions (with pointwise order).

(iii) We consider metric spaces (M, d) where the distance function d is restricted to take values in the unit interval $[0, 1]$. (This can always be enforced without changing the topology.) An ultrametric space is one in which the triangular inequality is strengthened to: $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ (with 'max' instead of '+' as for ordinary metric spaces). As morphisms between (ultra)metric spaces we take the non-expansive functions: those f with $d(f(x), f(y)) \leq d(x, y)$. An (ultra)metric space is complete if every Cauchy sequence has a limit. We write **Cms** and **Cums** for the categories of complete (ultra)metric spaces. We consider these with the fibrations $\begin{array}{c}\scriptstyle \mathrm{ClSub}(\mathbf{Cms})\\ \downarrow\\ \scriptstyle \mathbf{Cms}\end{array}$ and $\begin{array}{c}\scriptstyle \mathrm{ClSub}(\mathbf{Cums})\\ \downarrow\\ \scriptstyle \mathbf{Cums}\end{array}$ of closed subsets (*i.e.* those subsets which are closed under limits of Cauchy sequences). These satisfy the hypothesis of Lemma 2.1.(i)—and of (ii) as well—and hence are bicartesian fibrations. For more background information see [4, 44], and [6] for applications to the semantics of programming languages.

For completeness we recall that the cartesian product of two metric spaces (M_1, d_1) and (M_2, d_2) has the product $M_1 \times M_2$ as underlying set, with distance

$$d_\times(\langle x_1, x_2\rangle, \langle y_1, y_2\rangle) = \max\bigl\{d_1(x_1, y_1), d_2(x_2, y_2)\bigr\}.$$

We will additionally consider a tensor product $\otimes$ of metric spaces, which also has the cartesian product as underlying set. Its distance is given by '+' instead of 'max', whereby we take care to stay within the $[0, 1]$ interval:

$$d_\otimes(\langle x_1, x_2\rangle, \langle y_1, y_2\rangle) = \min\bigl\{1,\ d_1(x_1, y_1) + d_2(x_2, y_2)\bigr\}.$$

This tensor classifies bi-non-expansive maps. The category **Cms** is monoidal closed[2] and the category **Cums** is cartesian closed. In both cases the exponential (or internal hom) $M_1 \Rightarrow M_2$ is the set of non-expansive functions $M_1 \to M_2$ with distance between $f, g\colon M_1 \rightrightarrows M_2$ given by supremum:

$$d_\Rightarrow(f, g) = \sup_{x \in M_1} d(f(x), g(x)).$$

(iv) Finally we sketch a syntactic example. Assume we have a predicate logic over some (simple) type theory. This involves a category $\mathcal{T}$ of types. Objects are types A, and morphisms $A \to B$ are equivalence classes (with respect to conversion) of terms $x{:}\, A \vdash M{:}\, B$. We shall assume finite product types $(1, \times)$ and coproduct types $(0, +)$ in this calculus (see e.g. [26] for details).

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[2] Interestingly, the monoidal structure on ω-$\mathbf{Cpo}_\perp$ admits contraction (via diagonals) and the monoidal structure on **Cms** admits weakening (via projections), see [24].

On top of this category $\mathcal{T}$ of types there is a category $\mathcal{L}$ of predicates on types, which gives us the logic. Objects of $\mathcal{L}$ are propositions $(x{:}\, A \vdash \varphi{:}\, \mathsf{Prop})$ in context (or predicates); and morphisms $(x{:}\, A \vdash \varphi{:}\, \mathsf{Prop}) \to (y{:}\, B \vdash \psi{:}\, \mathsf{Prop})$ in $\mathcal{L}$ are morphisms $A \to B$ in $\mathcal{T}$, say given by a term $x{:}\, A \vdash M{:}\, B$, together with (a derivation of) the entailment $x{:}\, A \mid \varphi \vdash \psi[M/y]$. There is an obvious projection functor $\begin{smallmatrix}\mathcal{L}\\ \downarrow\\ \mathcal{T}\end{smallmatrix}$ which sends a predicate $(x{:}\, A \vdash \varphi{:}\, \mathsf{Prop})$ on A to its underlying type A. It is a (split) fibration, with substitution functors given by syntactic substitution of terms in predicate formulas.

Let us assume that we have "coherent" logic, with propositional connectives $\top, \wedge, -, \vee$ for finite conjunctions and disjunctions, existential quantifiers $\exists x{:}\, A.\, -$ and equality predicate $=_A$ for each type A. Then the category $\mathcal{L}$ of predicates has finite products and coproducts. For example, the coproduct of predicates $(x{:}\, A \vdash \varphi{:}\, \mathsf{Prop})$ and $(y{:}\, B \vdash \psi{:}\, \mathsf{Prop})$ is the predicate $\varphi + \psi$ on $z{:}\, A + B$ given by

$$(\varphi + \psi)(z) = \big(\exists x{:}\, A.\, z =_{A+B} \kappa x \wedge \varphi(x)\big) \vee \big(\exists y{:}\, B.\, z =_{A+B} \kappa' y \wedge \psi(y)\big).$$

Some additional logical assumptions are needed to make this $+$ a fibred functor $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$.

In all of these examples the fibre categories are pre-orders. This means that the fibrations model provability (that is, they account only for the existence of proofs or derivations). The theory that we develop applies to the more general situation with proper fibre categories, and hence to a logic with explict proofs. Universality takes care of the commutativity conditions inductive proofs must satisfy.

Besides categories of predicates we shall be using categories of (binary) relations. They can be obtained as follows. For a fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ with cartesian products $\times$ in its base category $\mathbb{B}$, form the fibration $\begin{smallmatrix}\mathrm{Rel}(\mathbb{P})\\ \downarrow\\ \mathbb{B}\end{smallmatrix}$ of relations by change-of-base (pullback):

$$\begin{tikzcd}
\mathrm{Rel}(\mathbb{P}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P} \arrow[d, "p"] \\
\mathbb{B} \arrow[r, "A \mapsto A \times A"'] & \mathbb{B}
\end{tikzcd}$$

Thus the fibre category $\mathrm{Rel}(\mathbb{P})_A$ over $A \in \mathbb{B}$ is the fibre category $\mathbb{P}_{A \times A}$ of binary relations on A. We have the following elementary result.

2.3. Lemma. *Let* $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *be a fibration as in Lemma 2.1.*(i) *with distributive coproducts in its base category. Then the associated category of relations* $\mathrm{Rel}(\mathbb{P})$ *is also bicartesian, and the functor* $\mathrm{Rel}(\mathbb{P}) \to \mathbb{B}$ *strictly preserves this structure. If the fibration* p *additionally satisfies the assumptions in* (ii)*, then the induced coproduct functor* $+{:}\, \mathrm{Rel}(\mathbb{P}) \times \mathrm{Rel}(\mathbb{P}) \to \mathrm{Rel}(\mathbb{P})$ *is a fibred functor.*

Proof. If coproducts $+$ in $\mathbb{B}$ are distributive, then we have left adjoints to substitution functors $(\kappa \times \kappa)^*\colon \mathbb{P}_{(A+B)\times(A'+B')} \to \mathbb{P}_{A\times A'}$, namely via adjoints of the composite

$$\begin{tikzcd}
A \times A' \arrow[r, "\kappa"] \arrow[drr, bend right=10, "\kappa \times \kappa"'] & A \times A' + A \times B \arrow[r, "\kappa"] & (A \times A' + A \times B') + (B \times A' + B \times B') \arrow[d, equal, "\wr"] \\
& & (A + B) \times (A' + B')
\end{tikzcd} \qquad \square$$

In all of the examples listed above the base category has distributive coproducts, so the result applies.

2.3 Comprehension and quotients

Assume every fibre category $\mathbb{P}_A$ of a fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ has a terminal object, call it $\mathbf{1}(A)$, or $\mathbf{1}_A$, and such objects are stable under substitution, *i.e.* they are preserved by substitution functors: for any $u\colon A \to B$, $u^*\mathbf{1}_B \cong \mathbf{1}_A$. Such (fibred) terminal objects amount to a functor $\mathbf{1}\colon \mathbb{B} \to \mathbb{P}$, (fibred) right adjoint to the functor p. In the logical view of (bicartesian) fibrations, these fibred terminal objects correspond to the (constantly) truth predicates over types.

2.4. Definition (After [30, 13]). A fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ with terminal object functor $\mathbf{1}\colon \mathbb{B} \to \mathbb{P}$ is said to admit *comprehension* if this functor $\mathbf{1}$ has a right adjoint. We usually write it as $\{-\}\colon \mathbb{P} \to \mathbb{B}$.

As for the examples in §§2.2, subobject fibrations always admit comprehension, by choosing for a subobject $(X \rightarrowtail I)$ a domain object X in the base category. The same applies to the fibrations of admissible subsets over ω-cpos, and of closed subsets over (ultra-)metric spaces. For the syntactic example $\begin{smallmatrix}\mathcal{L}\\ \downarrow\\ \mathcal{T}\end{smallmatrix}$ comprehension amounts to forming the extent of a predicate, that is, the type of all values where it (provably) holds:

$$\big(x\colon \sigma \vdash \varphi\colon \mathsf{Prop}\big) \mapsto \big(\vdash \{x\colon \sigma \mid \varphi\}\colon \mathsf{Type}\big).$$

The adjunction $\mathbf{1} \dashv \{-\}$ gives us appropriate introduction and elimination rules for such 'comprehension types'.

We turn to quotients. Here the situation is that quotients are left adjoints to equality. So we first have to say what it means for a fibration to have equality. This in turn involves left adjoints to contraction functors.

2.5. Definition (From [30]). Let $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ be a fibration with cartesian products $\times$ in $\mathbb{B}$. The fibration p is said to admit equality if for each object $A \in \mathbb{B}$ the "contraction" functor $\delta_A^*\colon \mathbb{P}_{A\times A} \to \mathbb{P}_A$, induced by the diagonal $\delta_A = \langle id, id\rangle\colon A \rightarrowtail A \times A$ on A, has a left adjoint $\coprod_{\delta_A}$.

In case the fibration has fibred terminal objects $\mathbf{1}(A) \in \mathbb{P}_A$ for $A \in \mathbb{B}$, we write

$$\mathrm{Eq}(A) \stackrel{\mathrm{def}}{=} \textstyle\coprod_{\delta_A}(\mathbf{1}(A)) \in \mathbb{P}_{A\times A} = \mathrm{Rel}(\mathbb{P})_A$$

for the equality relation on A. The assignment $A \mapsto \mathrm{Eq}(A)$ extends to a functor $\mathrm{Eq}\colon \mathbb{B} \to \mathrm{Rel}(\mathbb{P})$; the morphism part of this functor expresses that morphisms (in the base $\mathbb{B}$) map 'equal arguments to equal results'.

2.6. Remarks. (i) The substitution functors $\delta_A^*\colon \mathbb{P}_{A\times A} \to \mathbb{P}_A$ give rise to a global functor $\delta^*\colon \mathbb{P} \to \mathrm{Rel}(\mathbb{P})$. The adjunctions $\coprod_{\delta_A} \dashv \delta_A^*$ between the fibres induce a global left adjoint $\coprod_\delta\colon \mathrm{Rel}(\mathbb{P}) \to \mathbb{P}$ to δ^*. The equality functor is then the composite

$$\big(\mathbb{B} \xrightarrow{\mathrm{Eq}} \mathrm{Rel}(\mathbb{P})\big) = \big(\mathbb{B} \xrightarrow{\mathbf{1}} \mathbb{P} \xrightarrow{\coprod_\delta} \mathrm{Rel}(\mathbb{P})\big)$$

Hence, if a fibration has comprehension, the equality functor has a right adjoint $\{-\} \circ \delta^*$ by composition. The converse also holds: if $K\colon \mathrm{Rel}(\mathbb{P}) \to \mathbb{B}$ is right adjoint to Eq, then $K\pi^*$ is right adjoint to $\mathbf{1}$.

(ii) The functor $\mathrm{Eq}\colon \mathbb{B} \to \mathrm{Rel}(\mathbb{P})$ is characterised by the following universal property: regarded as a morphism of fibrations, $\mathrm{Eq}\colon \left(\begin{smallmatrix}\mathbb{B}\\ \downarrow\mathrm{id}\\ \mathbb{B}\end{smallmatrix}\right) \to \left(\begin{smallmatrix}\mathrm{Rel}(\mathbb{P})\\ \downarrow\\ \mathbb{B}\end{smallmatrix}\right)$ it is the absolute lifting (in the sense of [48]) of the terminal object functor $\mathbf{1}\colon \left(\begin{smallmatrix}\mathbb{B}\\ \downarrow\mathrm{id}\\ \mathbb{B}\end{smallmatrix}\right) \to \left(\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}\right)$ along $\delta^*\colon \left(\begin{smallmatrix}\mathrm{Rel}(\mathbb{P})\\ \downarrow\\ \mathbb{B}\end{smallmatrix}\right) \to \left(\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}\right)$. The 2-cell $\rho\colon \mathbf{1} \Rightarrow \delta^*\mathrm{Eq}$ is (pointwise) the proof of reflexivity; universality renders $\mathrm{Eq}(A)$ as the *least reflexive relation* on A.

2.7. Definition (From [25]). A fibration $\left(\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}\right)$ with equality as above admits quotients if the equality functor $\mathrm{Eq}\colon \mathbb{B} \to \mathrm{Rel}(\mathbb{P})$ has a left adjoint.

In all of the examples of 2.2 we have equality. It is usually given by the diagonal relation $A \hookrightarrow A \times A$ (which is admissible over ω-cpos, and closed over metric spaces).

It is not hard to show that the subobject fibration $\left(\begin{smallmatrix}\mathrm{Sub}(\mathbb{B})\\ \downarrow\\ \mathbb{B}\end{smallmatrix}\right)$ of a regular category $\mathbb{B}$ has quotients if and only if $\mathbb{B}$ has coequalisers. Similarly for admissible subsets over ω-cpos and closed subsets of metric spaces quotients are given by coequalisers. (For coequalisers in ω-cpos, see [14, 34] and in metric spaces, see [4, 44].) In the predicate logic example quotients are an extra feature of the logic, given by a mapping

$$\big(x\colon\sigma, x'\colon\sigma \vdash R(x,x')\colon \mathsf{Prop}\big) \mapsto \big(\vdash \sigma/R\colon \mathsf{Type}\big)$$

with suitable introduction and elimination rules provided by this adjunction (see [25]).

2.8. Remark. What we have defined above is the quotient of an arbitrary relation. Set theoretically, it is the quotient by the least equivalence relation generated by the given relation. In a diagram:

$$\begin{tikzcd}
 & \mathrm{ER}(\mathbb{P}) \arrow[ld, bend right, "\mathcal{Q}'"'] \arrow[dd, hook] \\
\mathbb{B} \arrow[ru, "\mathrm{Eq}"'] \arrow[rd, "\mathrm{Eq}"] \arrow[ru, phantom, bend right=20, "\dashv"] \arrow[rd, phantom, bend left=20, "\dashv"'] & \\
 & \mathrm{Rel}(\mathbb{P}) \arrow[lu, bend left, "\mathcal{Q}"]
\end{tikzcd}$$

where $\mathrm{ER}(\mathbb{P})$ is the full subcategory of $\mathrm{Rel}(\mathbb{P})$ of equivalence relations, $\mathcal{Q}$ is the quotient functor as defined above, and $\mathcal{Q}'$ is the quotient on equivalence relations only (as they are usually presented). Suppose that we can freely form the equivalence relation generated by a relation, *i.e.* that the inclusion $\mathrm{ER}(\mathbb{P}) \hookrightarrow \mathrm{Rel}(\mathbb{P})$ has a (fibred) left adjoint F, then having an adjoint $\mathcal{Q}$ is the same as having an adjoint $\mathcal{Q}'$.

(For this observation it is simpler to consider a pre-ordered fibration, so that it is unambiguous what the category $\mathrm{ER}(\mathbb{P})$ of equivalence relations is. See [36, 37] for further details about categories of relations on non-pre-ordered fibrations.)

Below we are interested in situations where the truth and equality functors preserve finite products and coproducts. We list a few easy observations.

2.9. Lemma. (i) *A terminal object functor* $\mathbf{1}$ *always preserves products, since it is a right adjoint.*

(ii) *If a fibration admits comprehension,* $\mathbf{1}$ *preserves coproducts.*

(iii) *If* $\mathbf{1}$ *preserves coproducts, then so does the equality functor* Eq*, by Remark 2.6.(i).*

(iv) *If a fibration has quotients, then* Eq *preserves products.* □

Having a terminal object functor $\mathbf{1}\colon \mathbb{B} \to \mathbb{P}$ of a fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ preserve finite coproducts $(0, +)$ means that:

- the terminal object $\mathbf{1}(0) \in \mathbb{P}_0$ in the fibre over the initial object $0 \in \mathbb{B}$ is initial in $\mathbb{P}$. Equivalently, $\mathbf{1}(0)$ is initial in $\mathbb{P}_0$, see Lemma 3.5 below. But this means that the initial and terminal object in the fibre $\mathbb{P}_0$ over 0 coincide, and thus that $\mathbb{P}_0$ is (equivalent to) the terminal category (one object, one arrow).

- for each pair of objects $A, B \in \mathbb{B}$ there is a canonical isomorphism

$$\textstyle\coprod_{\kappa}(\mathbf{1}A) +_{A+B} \coprod_{\kappa'}(\mathbf{1}B) \xrightarrow{\cong} \mathbf{1}(A+B).$$

This last condition essentially means that the union of the images of the coproduct coprojections κ, κ' cover the coproduct object $A + B \in \mathbb{B}$, in the sense that every element of $A + B$ must come from either A or B. We note that these conditions are satisfied for instance, when we consider *internal logic* fibrations, *i.e.* fibrations in which the predicates are the subobjects of the base category, provided coproduct coprojections κ, κ' are monic, or more generally, when we consider fibrations with comprehension, as in (ii) of the above lemma.

The requirement that the equality functor Eq preserve products and coproducts amounts to an extensionality condition, expressing that equality is structurally determined. This means that equality of elements on a product object is given componentwise, while equality on a coproduct object holds if the elements are both in the same component and equal therein.

2.4 Lifting functors to predicates and relations

In this subsection we show how a polynomial functor T acting on a category of types can be lifted (by induction on the structure) to a functor $\mathrm{Pred}(T)$ acting on predicates, and to a functor $\mathrm{Rel}(T)$ acting on relations. We shall use such lifted functors $\mathrm{Pred}(T)$ to capture inductive predicates on algebras in §§3, and $\mathrm{Rel}(T)$ for congruences in §§3.1 and coinductive relations on coalgebras in §§4.

2.10. Definition. Let $\begin{smallmatrix}\mathbb{P}\\ \downarrow{\scriptstyle p}\\ \mathbb{B}\end{smallmatrix}$ be a fibration where $\mathbb{P}$ is bicartesian over $\mathbb{B}$, with terminal object functor $\mathbf{1}\colon \mathbb{B} \to \mathbb{P}$.

(i) A polynomial functor $T\colon \mathbb{B} \to \mathbb{B}$ on the base category is lifted to a polynomial functor $\mathrm{Pred}(T)\colon \mathbb{P} \to \mathbb{P}$, called the *logical predicate lifting* of T, by induction on the structure of T. The bicartesian structure of $\mathbb{B}$ used in T is replaced by the bicartesian structure in $\mathbb{P}$. Every constant $A \in \mathbb{B}$ occurring in T is replaced by the "truth" constant $\mathbf{1}(A) \in \mathbb{P}$ in $\mathrm{Pred}(T)$.

(ii) Similarly, if the fibration has an equality functor $\mathrm{Eq}\colon \mathbb{B} \to \mathrm{Rel}(\mathbb{P})$, then such a polynomial functor T can be lifted to a functor $\mathrm{Rel}(T)\colon \mathrm{Rel}(\mathbb{P}) \to \mathrm{Rel}(\mathbb{P})$, called the *logical relation lifting* of T, by induction on the structure of T. Now we replace a constant $A \in \mathbb{B}$ occurring in T by the "equality" contant $\mathrm{Eq}(A) \in \mathrm{Rel}(\mathbb{P})$ in $\mathrm{Rel}(T)$.

For the functor $T = 1 + A \times (\_)$, whose initial algebra is the type of lists of elements of A, the logical predicate lifting is $\mathrm{Pred}(T) = 1 + \mathbf{1}(A) \times (\_)$, as an endofunctor on $\mathbb{P}$. And for the functor $T = A \times (\_)$, whose terminal coalgebra is the type of streams (or infinite lists) of elements of A, the logical relation lifting is $\mathrm{Rel}(T) = \mathrm{Eq}(A) \times (\_)$.

Notice that because functors $\begin{smallmatrix}\mathbb{P}\\ \downarrow\\ \mathbb{B}\end{smallmatrix}$ and $\begin{smallmatrix}\mathrm{Rel}(\mathbb{P})\\ \downarrow\\ \mathbb{B}\end{smallmatrix}$ strictly preserve the bicartesian structure we have by construction commuting diagrams

$$\begin{tikzcd}
\mathbb{P} \arrow[r, "\mathrm{Pred}(T)"] \arrow[d] & \mathbb{P} \arrow[d] \\
\mathbb{B} \arrow[r, "T"'] & \mathbb{B}
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
\mathrm{Rel}(\mathbb{P}) \arrow[r, "\mathrm{Rel}(T)"] \arrow[d] & \mathrm{Rel}(\mathbb{P}) \arrow[d] \\
\mathbb{B} \arrow[r, "T"'] & \mathbb{B}
\end{tikzcd}$$

The following will be used later.

2.11. Lemma. *Consider* $\begin{smallmatrix}\mathbb{P}\\ \downarrow{\scriptstyle p}\\ \mathbb{B}\end{smallmatrix}$ *and* $T\colon \mathbb{B} \to \mathbb{B}$ *as above.*

(i) *If* $\mathbf{1}\colon \mathbb{B} \to \mathbb{P}$ *preserves finite coproducts, then predicate lifting commutes with truth: there is a (canonical) natural isomorphism* $\mathrm{Pred}(T) \circ \mathbf{1} \overset{\cong}{\Longrightarrow} \mathbf{1} \circ T$ *as on the left diagram below.*

(ii) *If* $\mathrm{Eq}\colon \mathbb{B} \to \mathrm{Rel}(\mathbb{P})$ *preserves both products and coproducts, then relation lifting commutes with equality:* $\mathrm{Eq} \circ T \overset{\cong}{\Longrightarrow} \mathrm{Rel}(T) \circ \mathrm{Eq}$, *canonically, as in the right diagram:*

$$\begin{tikzcd}
\mathbb{P} \arrow[r, "\mathrm{Pred}(T)"] \arrow[dr, phantom, "\overset{\cong}{\Searrow}"] & \mathbb{P} \\
\mathbb{B} \arrow[u, "\mathbf{1}"] \arrow[r, "T"'] & \mathbb{B} \arrow[u, "\mathbf{1}"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathrm{Rel}(\mathbb{P}) \arrow[r, "\mathrm{Rel}(T)"] \arrow[dr, phantom, "\overset{\cong}{\Searrow}"] & \mathrm{Rel}(\mathbb{P}) \\
\mathbb{B} \arrow[u, "\mathrm{Eq}"] \arrow[r, "T"'] & \mathbb{B} \arrow[u, "\mathrm{Eq}"']
\end{tikzcd}$$

Proof. By induction on the structure of T. □

2.5 Transfer of adjunctions

In this section we mention the main technical result about transfer of adjunctions to categories of algebras and coalgebras. Part of this result occurs (independently) in [5]. An abstract proof using the characterization of categories of (co)algebras as interters is given in the Appendix.

2.12. Theorem. *Consider a natural transformation* $\alpha\colon SF \Rightarrow FT$ *in a situation*

$$\begin{tikzcd} \mathbb{A} \arrow[r, "S"] \arrow[dr, phantom, "\Searrow^{\alpha}"] & \mathbb{A} & & \mathrm{Alg}(S) \\ \mathbb{B} \arrow[u, "F"] \arrow[r, "T"'] & \mathbb{B} \arrow[u, "F"'] & \text{then it induces a functor} & \mathrm{Alg}(T) \arrow[u, "\mathrm{Alg}(F)"'] \end{tikzcd}$$

given by $(TX \xrightarrow{f} X) \mapsto (SFX \xrightarrow{\alpha_X} FTX \xrightarrow{Ff} FX)$.

And if α *is an isomorphism, then a right adjoint* G *to* F

$$\begin{tikzcd} \mathbb{A} \arrow[d, bend left, "G"] \arrow[d, phantom, "\dashv"] & & \mathrm{Alg}(S) \arrow[d, bend left, "\mathrm{Alg}(G)"] \arrow[d, phantom, "\dashv"] \\ \mathbb{B} \arrow[u, bend left, "F"] & \text{induces a right adjoint} & \mathrm{Alg}(T) \arrow[u, bend left, "\mathrm{Alg}(F)"] \end{tikzcd}$$

where the functor $\mathrm{Alg}(G)$ *arises from* $\beta\colon TG \Rightarrow GS$*, the adjoint transpose of* $FTG \cong SFG \overset{S\varepsilon}{\Longrightarrow} S$. □

We shall not prove this result here, because it is an instance of Theorem A.5 in the Appendix, which describes the construction $T \mapsto \mathrm{Alg}(T)$ as a special case of a 2-functorial inserter construction $\mathrm{Ins}(\_,\_)$, namely as $T \mapsto \mathrm{Ins}(T, \mathrm{id})$.

There is a dual version of the previous theorem, by applying the above in $\mathbf{Cat}^{\mathrm{co}}$.

2.13. Corollary. *For functors* $\mathbb{B} \xrightarrow{T} \mathbb{B}$, $\mathbb{B} \xrightarrow{G} \mathbb{A}$, $\mathbb{A} \xrightarrow{S} \mathbb{A}$, *a natural transformation* $\alpha\colon GS \Rightarrow TG$ *induces a functor* $\mathrm{CoAlg}(G)\colon \mathrm{CoAlg}(S) \to \mathrm{CoAlg}(T)$. *Furthermore, if* α *is an isomorphism, then, a left adjoint* $F \dashv G$ *induces a left adjoint* $\mathrm{CoAlg}(F) \dashv \mathrm{CoAlg}(G)$. □

3 Induction principle for T-algebras

Having laid down the technical prerequisites in the previous section, we are now ready to tackle the main topic of the paper, namely the formalisation of induction (and later on, coinduction) principles for T-algebras ($T\colon \mathbb{B} \to \mathbb{B}$) relative to a logic embodied by a bicartesian fibration over the base category $\mathbb{B}$.

Consider a bicartesian fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ and a polynomial endofunctor $T\colon \mathbb{B} \to \mathbb{B}$, toghether with its logical predicate lifting $\mathrm{Pred}(T)\colon \mathbb{P} \to \mathbb{P}$. The crucial observation for the formulation of the induction principle for T-algebras, motivated by the analysis of the examples in §§1, is that inductive predicates (those predicates which are closed under the operations of the underlying T-algebra) correspond precisely to Pred(T)-algebras (over the given T-algebra). That is, inductive predicates are the objects of the total category of the fibration

$$\begin{tikzcd}
\mathrm{Alg}(\mathrm{Pred}(T)) \arrow[d, "\mathrm{Alg}(p)"'] & (\mathrm{Pred}(T)P \xrightarrow{f} P) \arrow[d, mapsto] \\
\mathrm{Alg}(T) & (p(\mathrm{Pred}(T)P) = T(pP) \xrightarrow[pf]{} pP)
\end{tikzcd}$$

obtained by Theorem 2.12 (or by the functoriality of inserters, examined in the Appendix). The induction principle asserts that an inductive predicate is provably true in the image of the unique map from the initial algebra to its underlying T-algebra. In other words, the predicate holds on the "reachable part" (see [49]) of the underlying algebra. In categorical terms this can be expressed as follows: given an initial T-algebra $a: TX \to X$, the Pred(T)-algebra $\mathrm{Pred}(T)(\mathbf{1}_X) \to \mathbf{1}_{TX} \xrightarrow{\mathbf{1}(a)} \mathbf{1}_X$ is an initial Pred(T)-algebra; this guarantees the existence of an appropriate morphism into a given inductive predicate, which corresponds to the proof of the property abovementioned. Finally notice that this initial Pred(T)-algebra is the image of the given initial T-algebra under the functor $\mathrm{Alg}(\mathbf{1})\colon \mathrm{Alg}(T) \to \mathrm{Alg}(\mathrm{Pred}(T))$, induced by the adjunction $p \dashv \mathbf{1}$ (Theorem 2.12):

$$\begin{tikzcd}
\mathbb{P} \arrow[loop left, "\mathrm{Pred}(T)"] \arrow[d, "p"', bend right=0] \arrow[d, phantom, shift left=2, "\dashv"] & & \mathrm{Alg}(\mathrm{Pred}(T)) \arrow[d, "\mathrm{Alg}(p)"'] \arrow[d, phantom, shift left=2, "\dashv"] \\
\mathbb{B} \arrow[loop left, "T"] \arrow[u, bend right, "\mathbf{1}"'] & \text{yielding} & \mathrm{Alg}(T) \arrow[u, bend right, "\mathrm{Alg}(\mathbf{1})"']
\end{tikzcd}$$

We thus arrive at the following formulation.

3.1. Definition (*Induction principle in a fibration).* Let $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{array}$ be a bicartesian fibration, and let $T: \mathbb{B} \to \mathbb{B}$ be a polynomial functor. The fibration p satisfies the *induction principle w.r.t.* T if the induced functor $\mathrm{Alg}(\mathbf{1})\colon \mathrm{Alg}(T) \to \mathrm{Alg}(\mathrm{Pred}(T))$ preserves initial objects.

Logically, the principle can be formulated as follows. Let $!: D \to X$ be the unique algebra map from the initial T-algebra $c: TD \xrightarrow{\cong} D$ to the T-algebra $s: TX \to X$, and let $P \in \mathbb{P}_X$ be a predicate on X. We then have the following inference rule.

$$\frac{x: X \mid \mathrm{Pred}(T)(P)(x) \vdash P(sx)}{d: D \mid \emptyset \vdash P(!d)}$$

(where we have written the empty proposition context $\emptyset$ for the truth predicate $\mathbf{1}_D$ on D.) The antecedent of the rule says that P has a Pred(T)-algebra structure over $s: TX \to X$, while the conclusion says that P holds in the image of the algebra map $!: D \to X$ (*i.e.* in the 'reachable part' of X).

In the example involving lists in §§1 we have $D = A^* = \mathsf{list}(A)$ as initial algebra. For an arbitrary algebra $s = [u, h]: 1 + A \times X \to X$ and predicate $P \subseteq X$, the premise $\mathrm{Pred}(T)(P)(x) \vdash P(sx)$ of the rule amounts to $\vdash P(u)$ and

$P(x) \vdash P(h(a,x))$. The conclusion is that $P(!\alpha)$ holds for all $\alpha \in A^*$, where $!: A^* \to X$ is the unique map of lists given by $!\langle\rangle = u$ and $!(a \cdot \beta) = h(a, !\beta)$. In particular, for $P \subseteq D$, we get the standard 'list induction' principle: if $P(\mathsf{nil})$ and $P(\ell) \vdash P(\mathsf{cons}(a,\ell))$ (for arbitrary $a{:}\,A$) then $P(\ell)$ holds (for arbitrary $\ell{:}\,D$).

Notice that for the functor $T(X) = 1 + X$ an initial algebra is a natural numbers object (NNO) $1 + N \xrightarrow{\cong} N$. If induction holds, then the truth predicate $\mathbf{1}(N)$ on N is initial algebra of the lifted functor $\mathrm{Pred}(T)(P) = 1 + P$. Hence it is a NNO in the category of predicates. In this way, our formulation of the induction principle admits iterated use of initial algebra definitions, meaning that we can use initial algebras of polynomial endofunctors as constant objects involved in the definition of other such functors consistently. Similar considerations apply to our treatment of coinduction for coalgebras in §§4 below.

3.1 Induction and equality

In order to make explicit the duality between the induction principle for algebras above and the coinduction principle for coalgebras in §§4 below (whose formulation involves equality relations), we establish an equivalent formulation of induction in terms of binary relations and equality. This reformulation involves additional exactness and completeness conditions in a bicartesian fibration.

When a bicartesian fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ admits equality, in the sense of Definition 2.5 we can perform the logical relation lifting of a polynomial functor $T{:}\,\mathbb{B} \to \mathbb{B}$ to the functor $\mathrm{Rel}(T){:}\,\mathrm{Rel}(\mathbb{P}) \to \mathrm{Rel}(\mathbb{P})$ on the total category of binary relations (by choosing $\mathrm{Eq}(A)$ as a constant relation for an object $A \in \mathbb{B}$ occuring in T), see Definition 2.10.(ii). A $\mathrm{Rel}(T)$-algebra is a binary relation R on a given T-algebra, closed under the algebra operations, that is R is a *congruence*.

The following lemma summarises the conditions we need to relate $\mathrm{Pred}(T)$-algebras and $\mathrm{Rel}(T)$-algebras.

3.2. Lemma. *Assume $\mathbb{B}$ is a distributive category, $T{:}\,\mathbb{B} \to \mathbb{B}$ is a polynomial functor and $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ is a bicartesian bifibration, with direct images ($\coprod_u \dashv u^*$), satisfying Beck-Chevalley and Frobenius conditions.*

(i) *There is a (canonical) natural isomorphism*

$$\coprod_\delta \circ \mathrm{Pred}(T) \underset{\cong}{\overset{\alpha}{\Longrightarrow}} \mathrm{Rel}(T) \circ \coprod_\delta$$

(ii) *If furthermore substitution functors preserve finite coproducts, then*

$$\mathrm{Pred}(T) \circ \delta^* \underset{\cong}{\overset{\beta}{\Longrightarrow}} \delta^* \circ \mathrm{Rel}(T)$$

canonically.

Proof. By induction on the structure of the polynomial functor T. We indicate the argument for the two non-trivial cases:

(i) The only case that requires proof is that of products: for $P, Q \in \mathbb{P}$ over $A, B \in \mathbb{B}$ respectively, we must show

$$\coprod_{\delta_{A\times B}}(P \times Q) \cong \coprod_{\delta_A}(P) \times \coprod_{\delta_B}(Q)$$

Consider the following diagram

$$\begin{tikzcd}
A \arrow[d, swap, "\delta_A"] & A \times B \arrow[l, "\pi_{A,B}"] \arrow[d, "\delta_A \times \mathrm{id}"] \arrow[dr, bend left, "\pi'_{A,B}"] \arrow[dl, phantom, very near start, "\llcorner"] & \\
A \times A & (A \times A) \times B \arrow[l, swap, "\pi_{A\times A,B}"] \arrow[r, "\pi'_{A\times A,B}"] \arrow[d, swap, "\mathrm{id} \times \delta_B"] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "\delta_B"] \\
& (A \times A) \times (B \times B) \arrow[ul, bend left, "\pi_{A\times A,B\times B}"] \arrow[r, swap, "\pi'_{A\times A,B\times B}"] & B \times B
\end{tikzcd}$$

the above isomorphism is obtained by applying Beck-Chevalley to both (pullback) squares and using the Frobenius condition on the middle vertical arrows (recall the description of products in a bicartesian fibration from the proof of Lemma 2.1.(i)).

(ii) The only case requiring proof is that of coproducts. Given relations $R \in \mathbb{P}_{A\times A}$ and $S \in \mathbb{P}_{B\times B}$, we must show

$$\delta^*_{A+B}(R+S) \cong \delta^*_A(R) + \delta^*_B(S)$$

for which we use the fact that coproduct injections are monics in $\mathbb{B}$, which is a distributive category, as shown in [8, Lemma 2.1]. Hence the following diagram is a pullback:

$$\begin{tikzcd}
A \arrow[r, "\kappa"] \arrow[d, swap, "\delta_A"] & A + B \arrow[d, "\delta_{A+B}"] \\
A \times A \arrow[r, swap, "\kappa \times \kappa"] & (A+B) \times (A+B)
\end{tikzcd}$$

so that we can apply Beck-Chevalley to it. The remaining details are routine (using the description of coproducts in a bicartesian fibration in the proof of Lemma 2.1.(i)). □

From a logical point of view, the first item in the above lemma means that an inductive predicate can be extended to a congruence relation by diagonalisation: if P carries a $\mathrm{Pred}(T)$-algebra structure, then the relation $R(x,y) \equiv (x = y) \wedge P(x)$ has a $\mathrm{Rel}(T)$-algebra structure. The second item expresses the fact that the reflexive part of a congruence is an inductive predicate: if R has a $\mathrm{Rel}(T)$-algebra structure, the predicate $P(x) \equiv R(x,x)$ has a $\mathrm{Pred}(T)$-algebra structure.

3.3. Remark. We should point out that the condition that direct images satisfy Beck-Chevalley may fail. It fails for example in the fibration of admissible subsets in ω-cpos (see Pitts' counter example in [11, Chapter 1, Exercise (7)]) and similarly in the fibration of closed subsets of metric spaces. But it does hold in subobject fibrations of regular categories. Nevertheless, all the examples in 2.2 validate the isomorphisms stated in the above lemma. For the fibration of admissible subsets $\begin{array}{c}\mathrm{ASub}(\omega\text{-}\mathbf{Cpo}_\perp)\\ \downarrow \\ \omega\text{-}\mathbf{Cpo}_\perp\end{array}$, the main technical point in this respect is that there is a reflection $\mathrm{ASub} \leftrightarrows \mathrm{Sub}'(\omega\text{-}\mathbf{Cpo}_\perp)$, where $\begin{array}{c}\mathrm{Sub}'(\omega\text{-}\mathbf{Cpo}_\perp)\\ \downarrow \\ \omega\text{-}\mathbf{Cpo}_\perp\end{array}$ is the fibration obtained from the classical logic fibration $\begin{array}{c}\scriptstyle \mathrm{Sub}(\mathbf{Sets})\\ \downarrow\\ \scriptstyle \mathbf{Sets}\end{array}$ by change of base along the forgetful functor $\omega\text{-}\mathbf{Cpo}_\perp \to \mathbf{Sets}$ (see [20, §4.3.2] for related considerations and details). The fibration $\begin{array}{c}\scriptstyle \mathrm{Sub}'(\omega\text{-}\mathbf{Cpo}_\perp)\\ \downarrow\\ \scriptstyle \omega\text{-}\mathbf{Cpo}_\perp\end{array}$ clearly satisfies the hypothesis of the above lemma. It is then routine to verify that the reflection preserves the relevant constructions as to transfer the isomorphisms to $\begin{array}{c}\scriptstyle \mathrm{ASub}(\omega\text{-}\mathbf{Cpo}_\perp)\\ \downarrow\\ \scriptstyle \omega\text{-}\mathbf{Cpo}_\perp\end{array}$. An entirely analogous argument applies to the fibration $\begin{array}{c}\scriptstyle \mathrm{ClSub}(\mathbf{Cms})\\ \downarrow\\ \scriptstyle \mathbf{Cms}\end{array}$.

If the equality functor $\mathrm{Eq}\colon \mathbb{B} \to \mathrm{Rel}(\mathbb{P})$ commutes with lifting, then we get by Theorem 2.12 a functor $\mathrm{Alg}(\mathrm{Eq})$ in a situation:

$$\begin{tikzcd}
\mathrm{Alg}(\mathrm{Rel}(T)) & & \big(\mathrm{Rel}(T)\mathrm{Eq}(X) \overset{\cong}{\to} \mathrm{Eq}(TX) \xrightarrow{\mathrm{Eq}(a)} \mathrm{Eq}(X)\big) \\
\mathrm{Alg}(T) \arrow[u, "\mathrm{Alg}(\mathrm{Eq})"] & & (TX \underset{a}{\to} X) \arrow[u, mapsto]
\end{tikzcd}$$

We can now express the induction principle for algebras in terms of equality, as follows.

3.4. Theorem. *Let* $\begin{array}{c}\scriptstyle \mathbb{P}\\ \downarrow p\\ \scriptstyle \mathbb{B}\end{array}$ *be a fibration as in Lemma 3.2.(i). Then: the functor* $\mathrm{Alg}(\mathbf{1})\colon \mathrm{Alg}(T) \to \mathrm{Alg}(\mathrm{Pred}(T))$ *preserves initial objects if and only if the functor* $\mathrm{Alg}(\mathrm{Eq})\colon \mathrm{Alg}(T) \to \mathrm{Alg}(\mathrm{Rel}(T))$ *does.*

Informally: the induction principle holds in unary form for predicates if and only if it holds in binary form for relations.

Proof. In one direction, if the functor $\mathrm{Alg}(\mathbf{1})$ preserves initial objects, then so does $\mathrm{Alg}(\mathrm{Eq}) = \mathrm{Alg}(\coprod_\delta) \circ \mathrm{Alg}(\mathbf{1})$, since $\mathrm{Alg}(\coprod_\delta)$ is a left adjoint, namely to $\mathrm{Alg}(\delta^*)$. Notice that $\mathrm{Alg}(\coprod_\delta)$ exists and has an adjoint because the natural transformation α in the mentioned lemma is invertible.

In the other direction, assume $a\colon TD \overset{\cong}{\to} D$ is an initial T-algebra in $\mathbb{B}$. By Beck-Chevalley we get an isomorphism $\delta^* \circ \coprod_\delta \cong \coprod_{\mathrm{id}} \circ \mathrm{id}^* \cong \mathrm{id}$, so that $\mathrm{Alg}(\delta^*) \circ \mathrm{Alg}(\coprod_\delta) \cong \mathrm{id}$. For an arbitrary $\mathrm{Pred}(T)$-algebra $g\colon \mathrm{Pred}(T)(P) \to P$ in $\mathbb{P}$ we get adjoint correspondences

$$\begin{array}{c}
\mathrm{Alg}(\mathbf{1})(a) \longrightarrow g \cong \mathrm{Alg}(\delta^*)\big(\mathrm{Alg}(\coprod_\delta)(g)\big) \\
\hline\hline
\mathrm{Alg}(\coprod_\delta)\big(\mathrm{Alg}(\mathbf{1})(a)\big) = \mathrm{Alg}(\mathrm{Eq})(a) \longrightarrow \mathrm{Alg}(\coprod_\delta)(g)
\end{array}$$

By assumption, $\mathrm{Alg}(\mathrm{Eq})(a)$ is initial object in the category $\mathrm{Alg}(\mathrm{Rel}(T))$, and so we may conclude that $\mathrm{Alg}(\mathbf{1})(a)$ is initial object in $\mathrm{Alg}(\mathrm{Pred}(T))$. □

The equivalence in the above theorem means that a fibration satisfies the induction principle if and only if the canonical congruence $\mathrm{Rel}(T)(\mathrm{Eq}(D)) \overset{\cong}{\to} \mathrm{Eq}(T(D)) \xrightarrow{\mathrm{Eq}(a)} \mathrm{Eq}(D)$ over an initial algebra $a\colon TD \overset{\cong}{\to} D$ is initial in the category $\mathrm{Alg}(\mathrm{Rel}(T))$ of congruences. On the logical side, this amounts to saying that every congruence $f\colon \mathrm{Rel}(T)R \to R$ over a T-algebra $b\colon TY \to Y$ is reflexive when restricted along the unique morphism $!\colon D \to Y$ (induced by initiality), *i.e.* that the relation $!^*(R)$ is provably reflexive. In particular, every congruence over an initial algebra is reflexive. This alternative formulation of the induction principle for T-algebras appears in [43] for the case of natural numbers. It also shows up in the derivations of induction and coinduction principles in [40] in the context of a formal logic for parametric polymorphism.

Our formulation of the induction principle is such that it can be used to prove certain properties about any T-algebra, and not just above the initial one. A more standard formulation would require that the canononically induced $\mathrm{Pred}(T)$-algebra over an initial T-algebra be initial among $\mathrm{Pred}(T)$-algebras over the same initial T-algebra. That is, for an initial algebra $a\colon TD \xrightarrow{\cong} D$, the algebra $\mathrm{Pred}(T)(\mathbf{1}D) \xrightarrow{\cong} \mathbf{1}(TD) \xrightarrow{\mathbf{1}(a)} \mathbf{1}D$ should be initial in the fibre category $\mathrm{Alg}(\mathrm{Pred}(T))_{a:TD\to D}$ of inductive predicates over the initial algebra $a\colon TD \xrightarrow{\cong} D$. Since the functor $\mathrm{Alg}(p)\colon \mathrm{Alg}(\mathrm{Pred}(T)) \to \mathrm{Alg}(T)$ is a fibration, both formulations are equivalent, as the following result shows.

3.5. Lemma. *Given a fibration* $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{array}$ *with* $0 \in \mathbb{B}$ *an initial object, an object* $X \in \mathbb{P}_0$ *is initial in the fibre* $\mathbb{P}_0$ *over* 0 *if and only if it is (globally) initial in the total category* $\mathbb{P}$. □

3.2 Relating induction principles of different data types

It is well-known that for many familiar inductive data types such as lists and trees, we can carry out inductive proofs about their elements by associating some 'measure' of them into the natural numbers $\mathbb{N}$ and using induction over $\mathbb{N}$. For instance, binary trees with leaves of (some fixed) type A are the elements of the initial algebra $A+\mathsf{Tree}(A)\times\mathsf{Tree}(A) \xrightarrow{\cong} \mathsf{Tree}(A)$ of the functor $TX = A+X\times X$. The height of a tree is given by the tree-morphism $h\colon \mathsf{Tree}(A) \dashrightarrow \mathbb{N}$, induced by the following T-algebra on $\mathbb{N}$.

$$A + \mathbb{N} \times \mathbb{N} \xrightarrow{[0 \circ !, \max]} \mathbb{N}$$

More generally, in any category $\mathbb{B}$ which has colimits of ω-chains, that is colimits of diagrams of the form $\mathbf{Cat}(\omega, \mathbb{B})$ (ω being the preordered category of natural numbers) and an initial object 0, every endofunctor $T\colon \mathbb{B} \to \mathbb{B}$ which preserves such ω-colimits has an initial algebra obtained as the colimit D of the following ω-chain

$$\begin{tikzcd}
0 \arrow[r, "\iota"] & T0 \arrow[r, "T\iota"] & T^2 0 \arrow[r, "T^2\iota"] & \cdots \arrow[r] & D
\end{tikzcd}$$

where $\iota\colon 0 \to T0$ is the unique morphism out of the initial object, *cf.* [31, 45]. If $\mathbb{B}$ has a natural numbers object $\mathbb{N}$, we have the following cocone over the above diagram:

$$\begin{tikzcd}
0 \arrow[r, "\iota"] \arrow[drr, "h_0"'] & T0 \arrow[r, "T\iota"] \arrow[dr, "h_1"] & T^2 0 \arrow[r, "T^2\iota"] \arrow[d, "h_2"] & \cdots \arrow[r] & D \arrow[dll, dashed, "h"] \\
& & \mathbb{N} & &
\end{tikzcd}$$

where the h_n's are defined as the composites

$$h_n \overset{\text{def}}{=} \Big(T^n 0 \xrightarrow{!} 1 \xrightarrow{\mathsf{0}} \underbrace{\mathbb{N} \xrightarrow{\mathsf{S}} \mathbb{N} \xrightarrow{\mathsf{S}} \cdots \xrightarrow{\mathsf{S}} \mathbb{N}}_{n \text{ times}}\Big)$$

This definition yields a cocone with vertex $\mathbb{N}$, which induces a 'height' map $h{:}\, D \dashrightarrow \mathbb{N}$, from the carrier D of the initial algebra $TD \overset{\cong}{\Rightarrow} D$ to $\mathbb{N}$. Via this map we can reason by induction on the height (or 'depth') of the 'elements' of the initial algebra, as we now explain.

Given a predicate on the type $\mathsf{Tree}(A)$, say $t{:}\, \mathsf{Tree}(A) \vdash P(t){:}\, \mathsf{Prop}$, we know we can assert $t{:}\, \mathsf{Tree}(A) \mid \emptyset \vdash P(t)$ if we can show for every $n \in \mathbb{N}$

$$Q(n) = \forall t \in \mathsf{Tree}(A).\, h(t) = n \Rightarrow P(t).$$

And of course, this proposition can be established by ordinary induction on $\mathbb{N}$. The formula Q is the expression in the internal language of the fibration of the predicate $\prod_h(P)$, where $\prod_h$ is right adjoint to h^* in:

$$\begin{tikzcd}
\mathbb{P}_{\mathsf{Tree}(A)} \arrow[r, bend left, "\prod_h"] \arrow[r, phantom, "\perp"] & \mathbb{P}_{\mathbb{N}} \arrow[l, bend left, "h^*"] \\
\mathsf{Tree}(A) \arrow[r, "h"'] & \mathbb{N}
\end{tikzcd}$$

Logically, $\prod_h$ 'universal quantification along h'. Then, the adjunction laws set up a bijective correspondence

$$\frac{t{:}\, \mathsf{Tree}(A) \mid \emptyset \vdash P(t)}{n{:}\, \mathbb{N} \mid \emptyset \vdash \prod_h .(P(t)) = Q}$$

which gives the formal counterpart to the abovementioned reduction of induction on trees to induction on natural numbers via their associated 'height'.

4 Coinduction principle for T-coalgebras

We now turn to consider a logical principle for terminal coalgebras. Unlike the situation with algebras, for which the induction principle gives a method to prove any proposition over them, the coinduction principle gives only a way of proving equality of elements of the coalgebra. In the context of data types, this principle is still quite useful, since elements of terminal coalgebras are generally infinite objects, and a method to show two of them equal is therefore necessary.

The formulation of coinduction is entirely dual to that of induction (in terms of binary relations as in §§3.1). Given a polynomial endofunctor $T{:}\, \mathbb{B} \to \mathbb{B}$ and a bicartesian fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ admitting equality, if the equality functor $\mathrm{Eq}{:}\, \mathbb{B} \to \mathrm{Rel}(\mathbb{P})$ preserves products and coproducts, then we obtain a functor $\mathrm{CoAlg}(\mathrm{Eq}){:}\, \mathrm{CoAlg}(T) \to \mathrm{CoAlg}(\mathrm{Rel}(T))$, as in the algebraic case. That is, given a coalgebra $d{:}\, C \to TC$, the equality relation $\mathrm{Eq}(C)$ has a canonical $\mathrm{Rel}(T)$-coalgebra structure $\mathrm{Eq}(C) \to \mathrm{Eq}(TC) \overset{\cong}{\Rightarrow} \mathrm{Rel}(T)(\mathrm{Eq}(C))$ over d.

It follows from the analysis in §§1 that $\mathrm{Rel}(T)$-coalgebras can be regarded as (applicative) bisimulations or coinductive relations. This means that a given $\mathrm{Rel}(T)$-coalgebra $f\colon R \to \mathrm{Rel}(T)(R)$ over a coalgebra $c\colon X \to TX$ is a relation R on X which is preserved by the destructor operation c. The coinduction principle asserts that elements x, y of X which are R-related are equal when mapped to the terminal coalgebra.

Logically, if $!\colon (X, c) \dashrightarrow (D, d)$ denotes the unique coalgebra map to the terminal coalgebra $d\colon D \xrightarrow{\cong} TD$, we have the following rule

$$\frac{x, y\colon X \mid R(x, y) \vdash \mathrm{Rel}(T)(R)(cx, cy)}{x, y\colon X \mid R(x, y) \vdash\ !x =_D\ !y}$$

where the premise expresses that R has a $\mathrm{Rel}(T)$-coalgebra structure over $c\colon X \to TX$. Furthermore, from a constructive point of view, it is natural to require that different proofs of the entailment in the premise of the rule yield different proofs of the entailment in the conclusion. These considerations lead us to require that the canonical $\mathrm{Rel}(T)$-coalgebra on $\mathrm{Eq}(D)$ be terminal.

This principle is formally captured by the following definition.

4.1. Definition (*Coinduction principle in a fibration).* Let $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{array}$ be a bicartesian fibration with equality (preserving bicartesian structure), and let $T\colon \mathbb{B} \to \mathbb{B}$ be a polynomial functor. The fibration p satisfies the *coinduction principle* with respect to T if the induced functor $\mathrm{CoAlg}(\mathrm{Eq})\colon \mathrm{CoAlg}(T) \to \mathrm{CoAlg}(\mathrm{Rel}(T))$ preserves terminal objects.

4.2. Example. For a given set A, consider the polynomial functor $T(X) = A \times X$. Its terminal coalgebra in **Sets** has as carrier the set $L = A^{\mathbb{N}}$ of infinite sequences of A's. The structure $\langle h, t\rangle\colon L \xrightarrow{\cong} A \times L$ consists of the head h and tail t functions. The fibration $\begin{array}{c}\mathrm{Sub}(\mathbf{Sets})\\ \downarrow\\ \mathbf{Sets}\end{array}$ of subsets satisfies the coinduction principle with respect to this functor T, *i.e.* that the equality relation $\mathrm{Eq}(L) = \{(\ell, \ell) \mid \ell \in L\} \subseteq L \times L$ is the terminal coalgebra of the induced lifted functor $\mathrm{Rel}(T)(R) = \mathrm{Eq}(A) \times R$ on the total category $\mathrm{Rel}(\mathrm{Sub}(\mathbf{Sets}))$ of relations. Indeed, a relation $R \subseteq X \times X$ carrying a $\mathrm{Rel}(T)$-coalgebra structure consists of a T-coalgebra structure $\langle f, g\rangle\colon X \to A \times X$ on its underlying set, such that

$$R(x, y) \;\Rightarrow\; \left\{\begin{array}{l} f(x) = f(y) \wedge \\ R(g(x), g(y)) \end{array}\right.$$

for all $x, y \in X$. The induced T-coalgebra map $k\colon X \dashrightarrow L = A^{\mathbb{N}}$ given by $k(x) = \lambda n \in \mathbb{N}.\, f(g^{(n)}(x))$ is the unique map of $\mathrm{Rel}(T)$-coalgebras $k\colon R \dashrightarrow \mathrm{Eq}(L)$, since $R(x, y) \Rightarrow k(x) = k(y)$.

We illustrate the use of the coinduction principle for such infinite lists. We can define maps $\mathsf{even}, \mathsf{odd}\colon L \rightrightarrows L$, which take out the evenly and oddly listed elements. These are obtained by terminality of $\langle h, t\rangle\colon L \xrightarrow{\cong} A \times L$ in

$$\begin{tikzcd}
L \arrow[r, dashed, "\mathsf{even}"] \arrow[d, "\langle h, t \circ t\rangle"'] & L \arrow[d, "\cong"', "\langle h, t\rangle"] & L \arrow[l, dashed, "\mathsf{odd}"'] \arrow[d, "\langle h, t\rangle \circ t"] \\
A \times L \arrow[r, dashed] & A \times L & A \times L \arrow[l, dashed]
\end{tikzcd}$$

Also we can define a merge operation in

$$\begin{tikzcd}
L \times L \arrow[r, dashed, "\mathsf{merge}"] \arrow[d, "{\langle h \circ \pi, \langle \pi', t \circ \pi \rangle \rangle}"'] & L \arrow[d, "\cong"', "{\langle h, t \rangle}"] \\
A \times (L \times L) \arrow[r, dashed] & A \times L
\end{tikzcd}$$

that is, $\mathsf{merge}(a \cdot \alpha, \beta) = a \cdot \mathsf{merge}(\beta, \alpha)$. Showing that $\mathsf{even}(\mathsf{merge}(\alpha, \beta)) = \alpha$ and $\mathsf{odd}(\mathsf{merge}(\alpha, \beta)) = \beta$ (in **Sets**) amounts to showing (by coinduction) that the relations on L

$$\begin{aligned} R &= \{\langle \alpha, \mathsf{even}(\mathsf{merge}(\alpha, \beta)) \rangle \mid \alpha, \beta \in L\} \\ S &= \{\langle \beta, \mathsf{odd}(\mathsf{merge}(\alpha, \beta)) \rangle \mid \alpha, \beta \in L\}. \end{aligned}$$

are bisimulations (*i.e.* carry $\mathrm{Rel}(T)$-coalgebra structures).

We can further show that $\mathsf{merge}(\mathsf{even}(\alpha), \mathsf{odd}(\alpha)) = \alpha$, by first showing that $\mathsf{odd}(\alpha) = \mathsf{even}(t(\alpha))$. We thus get an isomorphism $L \cong L \times L$.

The same argument may be carried out in the fibration of admissible subsets over ω-cpo's, since the relations involved are admissible.

5 Validity of the induction and coinduction principles

Having formalised induction and coinduction principles in a fibration, we proceed to give sufficient criteria for their validity. We will show that, like in ordinary set theory, if the logic admits comprehension, the induction principle for algebras is valid in it. And dually, if the logic admits quotients of relations, it satisfies the coinduction principle for coalgebras.

The validity of induction and coinduction principles—which have been formalised as exactness conditions for certain functors—will be guaranteed to hold in presence of suitable adjoints. The existence of these latter is inferred from comprehension and quotients, as appropriate, as the following theorem shows.

5.1. Theorem. *Consider a polynomial functor* $T \colon \mathbb{B} \to \mathbb{B}$ *and a bicartesian fibration* $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{array}$.

(i) *If the fibration admits comprehension, it satisfies the induction principle with respect to* T.

(ii) *If the fibration admits finite-product-preserving equality and quotients, it satisfies the coinduction principle with respect to* T.

Proof. Both statements are consequences of Theorem 2.12 and Corollary 2.13 respectively: the comprehension and quotient adjunctions

$$\begin{tikzcd}
\mathbb{P} \arrow[loop left, "\mathrm{Pred}(T)"] \arrow[d, bend left, "\{-\}"] \arrow[d, phantom, "\dashv"] \\
\mathbb{B} \arrow[u, "\mathbf{1}"] \arrow[loop left, "T"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{Rel}(\mathbb{P}) \arrow[loop right, "\mathrm{Rel}(T)"] \arrow[d, bend right, "\mathcal{Q}"'] \arrow[d, phantom, "\dashv"] \\
\mathbb{B} \arrow[u, "\mathrm{Eq}"'] \arrow[loop right, "T"]
\end{tikzcd}$$

induce adjunctions between associated categories of algebras and coalgebras:

$$\begin{tikzcd}
\mathrm{Alg}(\mathrm{Pred}(T)) \arrow[d, bend left, "\mathrm{Alg}(\{-\})"] & & \mathrm{CoAlg}(\mathrm{Rel}(T)) \arrow[d, bend right, "\mathrm{CoAlg}(\mathcal{Q})"'] \\
\mathrm{Alg}(T) \arrow[u, "\mathrm{Alg}(\mathbf{1})"] \arrow[u, phantom, shift left=3, "\dashv"] & \text{and} & \mathrm{CoAlg}(T) \arrow[u, "\mathrm{CoAlg}(\mathrm{Eq})"'] \arrow[u, phantom, shift right=3, "\dashv"]
\end{tikzcd}$$

Hence the functor $\mathrm{Alg}(\mathbf{1})$ preserves initial objects, and the functor $\mathrm{CoAlg}(\mathrm{Eq})$ preserves terminal objects. □

The above formal argument is an abstract counterpart of the concrete set theoretic arguments presented in §§1: the functor $\mathrm{Alg}(\{\_\})\colon \mathrm{Alg}(\mathrm{Pred}(T)) \to \mathrm{Alg}(T)$ extracts the subalgebra[3] of a given algebra determined by an inductive predicate on it. And the functor $\mathrm{CoAlg}(\mathcal{Q})\colon \mathrm{CoAlg}(\mathrm{Rel}(T)) \to \mathrm{CoAlg}(T)$ takes a T-coalgebra with an applicative bisimulation on it and produces a T-coalgebra, by quotienting the given one by the bisimulation.

5.2. Example. We shall illustrate the details of the argument in the coinductive case for the terminal coalgebra $\langle h, t\rangle\colon L \xrightarrow{\cong} A \times L$ of infinite lists of the functor $T(X) = A \times X$ from Example 4.2. There we already saw that the coinduction principle holds with respect to T via a direct argument. Here we spell out the argument used in Theorem 5.1 above.

Assume therefore that we have an arbitrary T-coalgebra $\langle f, g\rangle\colon X \to A \times X$, with a relation $R \subseteq X \times X$ on X carrying a $\mathrm{Rel}(T)$-coalgebra structure over $\langle f, g\rangle$. We write X/R for the result of quotienting X by the equivalence relation generated by $R \subseteq X \times X$. This quotient X/R may be described as the coequaliser in **Sets**:

$$\begin{tikzcd}
R \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] & X \arrow[r, two heads, "c"] & X/R
\end{tikzcd}$$

where $r_0, r_1\colon R \rightrightarrows X$ are the composites of $R \hookrightarrow X \times X$ with the projections $X \times X \rightrightarrows X$. Consider the diagram

$$\begin{tikzcd}
R \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] & X \arrow[r, two heads, "c"] \arrow[dr, "{\langle f, c \circ g\rangle}"'] & X/R \arrow[d, dashed, "\ell"] \\
& & A \times (X/R)
\end{tikzcd}$$

By the coinduction assumption, $R(x, y)$ implies $f(x) = f(y)$ and $R(g(x), g(y))$. This means that

$$\langle f, c \circ g\rangle \circ r_0 = \langle f, c \circ g\rangle \circ r_1$$

as in the diagram above. The resulting mediating map $\ell\colon X/R \dashrightarrow A \times (X/R)$ is then a T-coalgebra in **Sets** on the quotient X/R. Hence there is a unique coalgebra map $\overline{\ell}: X/R \dashrightarrow L = A^{\mathbb{N}}$. We then get a diagram

---

[3] Notice that our formalisation gives a precise description of the notion of subalgebra as an algebra in the category of subobjects (here generalised to a fibration), rather than a mere subobject in the category of algebras.

$$\begin{tikzcd}
X \arrow[r, "c"'] \arrow[d, "\langle f, g\rangle"'] \arrow[rr, bend left, "k"] & X/R \arrow[r, "\overline{\ell}"'] \arrow[d, "\ell"] & L = A^{\mathbb{N}} \arrow[d, "\cong"', "\langle h, t\rangle"] \\
A \times X \arrow[r, "\mathrm{id} \times c"] \arrow[rr, bend right, "\mathrm{id} \times k"'] & A \times (X/R) \arrow[r, "\mathrm{id} \times \overline{\ell}"] & A \times L
\end{tikzcd}$$

that shows that $\overline{\ell} \circ c$ is the unique coalgebra map $k: X \dashrightarrow A^{\mathbb{N}}$. This yields $k \circ r_0 = k \circ r_1: R \to A^{\mathbb{N}}$, so that k becomes a map of relations $R \to \mathrm{Eq}(L)$. This is the conclusion we seek: $R(x, y) \Rightarrow k(x) = k(y)$.

6 Mixed induction/coinduction for recursive types

In this section we show how to combine the induction and coinduction principles of §§3 and §§4 in order to extend our reasoning principles to structures involving the mixed variance exponent functor $\Rightarrow: \mathbb{B}^{\mathrm{op}} \times \mathbb{B} \to \mathbb{B}$ on a bicartesian closed category $\mathbb{B}$.

We thus adopt the following setting: let $\mathbb{B}$ be a bicartesian closed category, *i.e.* a bicartesian category in which the functor $(\_) \times A: \mathbb{B} \to \mathbb{B}$ has a right adjoint, for every object $A \in \mathbb{B}$, and let $T: \mathbb{B}^{\mathrm{op}} \times \mathbb{B} \to \mathbb{B}$ be a polynomial functor, *i.e.* a functor in the smallest class of functors $\mathbf{Cat}(\mathbb{B}^{\mathrm{op}} \times \mathbb{B}, \mathbb{B})$ containing the second projection and constant functors, which is closed under products and coproducts (given pointwise) and exponentials, in the sense that if $F, G: \mathbb{B}^{\mathrm{op}} \times \mathbb{B} \rightrightarrows \mathbb{B}$ are in the class, so is the functor

$$\mathbb{B}^{\mathrm{op}} \times \mathbb{B} \xrightarrow{(\gamma \times \mathrm{id}) \circ \Delta} (\mathbb{B} \times \mathbb{B}^{\mathrm{op}}) \times (\mathbb{B}^{\mathrm{op}} \times \mathbb{B}) \xrightarrow{F^{\mathrm{op}} \times G} \mathbb{B}^{\mathrm{op}} \times \mathbb{B} \xrightarrow{\Rightarrow} \mathbb{B}$$

where $\gamma: \mathbb{B}^{\mathrm{op}} \times \mathbb{B} \xrightarrow{\cong} \mathbb{B} \times \mathbb{B}^{\mathrm{op}}$ is the evident 'twist' isomorphism, which makes $\mathbb{B}^{\mathrm{op}} \times \mathbb{B}$ a self-dual category.

We are interested in minimal invariants of such a polynomial functor T, *i.e.* in objects D such that $D \cong T(D, D)$, with a universal property. After Freyd's work on algebraically compact categories [16, 17], it is standard to reduce the analysis of such objects D to the case of initial algebras/terminal coalgebras of an associated functor $\widehat{T}$. This is possible because any functor $T: \mathbb{B}^{\mathrm{op}} \times \mathbb{B} \to \mathbb{B}$ uniquely determines a 'symmetric' functor $\widehat{T}: \widehat{\mathbb{B}} \to \widehat{\mathbb{B}}$, where $\widehat{\mathbb{B}} = \mathbb{B}^{\mathrm{op}} \times \mathbb{B}$ is the cofree self-dual category on $\mathbb{B}$, and $\widehat{T}(X, Y) = (T(Y, X), T(X, Y))$. It follows from *ibid.* that

$$\text{minimal invariant of } T \equiv \text{ initial } \widehat{T}\text{-algebra } \equiv \text{ terminal } \widehat{T}\text{-coalgebra.}$$

Explicitly, this goes as follows. A category $\mathbb{B}$ is called **algebraically compact**, according to Freyd, if every functor $T: \mathbb{B} \to \mathbb{B}$ of a suitable kind has an initial algebra $a: T(D) \xrightarrow{\cong} D$ such that its inverse $a^{\perp 1}: D \xrightarrow{\cong} T(D)$ is a terminal coalgebra. As examples, the category ω-$\mathbf{Cpo}_{\perp}$ is algebraically compact when we consider locally continuous functors, see [45, 2]. And the category $\mathbf{Cms}$ of complete metric spaces is algebraically compact for locally contractive functors, see [3, 42]. (The identity functor is not locally contractive, but the functor $id_{\frac{1}{2}}$ is; it maps a metric space (X,d) to the space $(X, \frac{1}{2}d)$, with (pointwise) half the original distance. Neither ω-$\mathbf{Cpo}_{\perp}$ nor **Cms** is cartesian closed, but these categories are monoidal closed. The category **Cums** of complete *ultra* metric spaces is cartesian closed, and algebraically compact for locally contractive functors. (In [7] these functors (instead of the category) would be called algebraically compact.)

A basic result of [17] is that if $\mathbb{B}$ is algebraically compact, then so is $\widehat{\mathbb{B}} = \mathbb{B}^{\mathrm{op}} \times \mathbb{B}$, the cofree self-dual category on $\mathbb{B}$. This will be useful in the following way. If we have a functor $T\colon \mathbb{B}^{\mathrm{op}} \times \mathbb{B} \to \mathbb{B}$, it induces a functor $\widehat{T}\colon \widehat{\mathbb{B}} \to \widehat{\mathbb{B}}$ with

$$\begin{cases} \langle X, Y\rangle & \mapsto & \langle T(Y,X), T(X,Y)\rangle \\ \langle f, g\rangle & \mapsto & \langle T(g,f), T(f,g)\rangle. \end{cases}$$

Then we can determine by algebraic compactness an initial algebra in $\mathbb{B}^{\mathrm{op}} \times \mathbb{B}$,

$$\widehat{T}(D_1, D_2) \xrightarrow[\cong]{\langle a_1, a_2\rangle} \langle D_1, D_2\rangle$$

such that the inverse $\langle a_1^{\perp 1}, a_2^{\perp 1}\rangle$ is a terminal colgebra[4]. It is not hard to verify that swapping components yields a new map

$$\widehat{T}(D_2, D_1) \xrightarrow[\cong]{\langle a_2^{\perp 1}, a_1^{\perp 1}\rangle} \langle D_2, D_1\rangle$$

which is also an initial algebra, and the inverse of which is also a terminal coalgebra. This yields a unique mediating isomorphism $\langle D1, D_2\rangle \cong \langle D_2, D_1\rangle$ between these algebras and coalgebras. We then get $D_1 \cong D_2$. Rephrasing things with this new insight, we have a single isomorphism $a\colon T(D,D) \overset{\cong}{\Rightarrow} D$ with the following universal property: for each pair of maps $c\colon X \to T(Y,X)$, $d\colon T(X,Y) \to Y$ there is a unique pair of maps $f\colon X \dashrightarrow D$, $g\colon D \dashrightarrow Y$ making the following diagram commute.

$$\begin{tikzcd}
T(Y,X) \arrow[r, "T(g,f)"] & T(D,D) \arrow[r, "T(f,g)"] \arrow[d, "a"', "\cong"] & T(X,Y) \arrow[d, "d"] \\
X \arrow[u, "c"] \arrow[r, "f"'] & D \arrow[r, "g"'] & Y
\end{tikzcd}$$

In order to get a suitable induction/coinduction principle for such invariant objects $T(D,D) \overset{\cong}{\Rightarrow} D$, we must extend our logical relation lifting of polynomial functors to encompass the exponential functor. In order to do so, we assume a fibration $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{array}$ such that the total category $\mathbb{P}$ is bicartesian closed and p (strictly) preserves such structure. One way to guarantee cartesian closure of $\mathbb{P}$ out of logical operations is given in the following proposition (for a complete proof see [20, Corollary 3.3.11]).

---

[4] This procedure can also be followed for locally contractive functors on complete (ultra) metric spaces, even though the identity functor is not locally contractive.

6.1. Proposition. *Let* $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{array}$ *be a fibration satisfying the following three conditions.*

(i) $\mathbb{B}$ *is cartesian closed;*

(ii) p *is a fibred-ccc,* i.e. *every fibre category is cartesian closed and reindexing functors preserve such structure;*

(iii) p *admits 'simple' products,* i.e. *for every cartesian projection* $\pi' \colon A \times B \to B$*, the 'weakening' functor* $(\pi')^* \colon \mathbb{P}_B \to \mathbb{P}_{A\times B}$ *has a right adjoint* $\prod_A$*, and these right adjoints satisfy the Beck-Chevalley condition.*

The total category $\mathbb{P}$ *is then cartesian closed and* p *(strictly) preserves this structure.*

Proof. Finite products of $\mathbb{P}$ have been spelled out in Lemma 2.1. As for exponentials, given objects $P \in \mathbb{P}_A$ and $Q \in \mathbb{P}_B$, their exponential $P \Rightarrow Q$ in $\mathbb{P}$ (over $A \Rightarrow B$ in $\mathbb{B}$) is given by the formula

$$P \Rightarrow Q = \textstyle\prod_A \Big(\pi^*(X) \Rightarrow \mathrm{ev}^*(Y)\Big)$$

where $\Rightarrow$ on the right-hand-side is the exponential in the fibre over $A \times (A \Rightarrow B)$, $A \xleftarrow{\pi} A \times (A \Rightarrow B) \xrightarrow{\pi'} A \Rightarrow B$ is a product diagram in $\mathbb{B}$ and $\mathrm{ev}\colon A \times (A \Rightarrow B) \to B$ is the evaluation morphism (an instance of the counit of the exponential adjunction $(\_) \times A \dashv A \Rightarrow (\_)$). $\square$

The above expression for the exponential object in $\mathbb{P}$ is the traditional 'logical predicate' formula for higher types in lambda calculus [20].

Assume a fibration $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{array}$ is given with $\mathbb{P}$ bicartesian closed, and with an equality functor $\mathrm{Eq}\colon \mathbb{B} \to \mathrm{Rel}(\mathbb{P})$ preserving cartesian closed structure. This means in particular that equality is given extensionally (pointwise) for elements of the internal hom (or 'function space'): for $f, g\colon A \Rightarrow B$

$$\begin{aligned} f =_{A \Rightarrow B} g \quad &\equiv \quad \forall x, y\colon A.\, x =_A y \;\Rightarrow\; fx =_B gy \\ &\equiv \quad \forall x\colon A.\, fx =_B gx. \end{aligned}$$

We can define for such p a logical relation lifting of any polynomial functor as in Definition 2.10.(ii) obtaining the following diagram

$$\begin{tikzcd}
\mathrm{Rel}(\mathbb{P})^{\mathrm{op}} \times \mathrm{Rel}(\mathbb{P}) \arrow[r, "\mathrm{Rel}(T)"] & \mathrm{Rel}(\mathbb{P}) \\
\mathbb{B}^{\mathrm{op}} \times \mathbb{B} \arrow[u, "\mathrm{Eq}^{\mathrm{op}} \times \mathrm{Eq}"] \arrow[r, "T"'] \arrow[ur, phantom, "\overset{\cong}{\Nwarrow}"] & \mathbb{B} \arrow[u, "\mathrm{Eq}"']
\end{tikzcd}$$

and hence a functor $\mathrm{Alg}(\mathrm{Eq}^{\mathrm{op}} \times \mathrm{Eq})\colon \mathrm{Alg}(\widehat{T}) \to \mathrm{Alg}(\widehat{\mathrm{Rel}(T)})$.

When the fibration admits quotients, there is an equivalent condition for pointwise equality involving a Frobenius condition for these quotients.

6.2. Proposition. *If a fibration* $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{array}$ *admits quotients via an adjunction* $\mathcal{Q} \dashv \mathrm{Eq}\colon \mathbb{B} \rightleftarrows \mathrm{Rel}(\mathbb{P})$*, then the functor* Eq *preserves exponentials if and only if the adjunction* $\mathcal{Q} \dashv \mathrm{Eq}$ *satisfies the Frobenius condition: the canonical 2-cell*

$$\mathcal{Q} \circ ((\_) \times \mathrm{Eq}(A)) \overset{\alpha}{\Longrightarrow} ((\_) \times A) \circ \mathcal{Q}$$

is an isomorphism (This α exists because Eq *preserves products, as it is a right adjoint.)*

Proof. Clearly, for any $R \in \mathrm{Rel}(\mathbb{P})$, we have canonical natural isomorphisms

$$\mathcal{Q}(R \times \mathrm{Eq}(A)) \xrightarrow{\cong} \mathcal{Q}(R) \times A \quad \text{if and only if} \quad \mathrm{Eq}(A \Rightarrow B) \xrightarrow{\cong} \mathrm{Eq}(A) \Rightarrow \mathrm{Eq}(B)$$

since we have (composite) adjunctions

$$\begin{array}{rcl} ((\_) \times A) \circ \mathcal{Q} & \dashv & \mathrm{Eq} \circ (A \Rightarrow (\_)) \\ \mathcal{Q} \circ ((\_) \times \mathrm{Eq}(A)) & \dashv & (\mathrm{Eq}(A) \Rightarrow (\_)) \circ \mathrm{Eq}. \end{array} \qquad \square$$

The following formulation of a logical principle for a mixed variance polynomial functor is the evident generalization of Definitions 3.1 and 4.1 for covariant polynomial functors.

6.3. Definition. Consider a bicartesian closed category $\mathbb{B}$, a polynomial functor $T\colon \mathbb{B}^{\mathrm{op}} \times \mathbb{B} \to \mathbb{B}$ and a fibration $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{array}$ with $\mathbb{P}$ bicartesian closed, with both p and Eq structure-preserving. The fibration p satisfies the *mixed induction / coinduction* principle with respect to T if the induced functor $\mathrm{Alg}(\mathrm{Eq}^{\mathrm{op}} \times \mathrm{Eq})\colon \mathrm{Alg}(\widehat{T}) \to \mathrm{Alg}(\widehat{\mathrm{Rel}(T)})$ preserves initial objects.

Logically, the above principle can be expressed as follows: let

- (X, Y) be a $\widehat{T}$-algebra, with structure $c\colon X \to T(Y, X)$ and $d\colon T(X, Y) \to Y$,
- $a\colon T(D, D) \xrightarrow{\cong} D$ be the minimal invariant, with unique $\widehat{T}$-algebra map $(!_X, !_Y)\colon (D, D) \dashrightarrow (X, Y)$ in $\widehat{\mathbb{B}} = \mathbb{B}^{\mathrm{op}} \times \mathbb{B}$,
- R and S be relations over X and Y respectively,

then we have the following rule.

$$\frac{\begin{array}{l} x, x'\colon X \mid R(x, x') \vdash \\ \quad \widehat{\mathrm{Rel}(T)}(S, R)(cx, cx') \end{array} \qquad \begin{array}{r} y, y'\colon Y \mid \widehat{\mathrm{Rel}(T)}(R, S)(y, y') \vdash \\ S(gy, gy') \end{array}}{x, x'\colon X \mid R(x, x') \vdash\, !_X x =_D\, !_X x' \qquad y, y'\colon D \mid y =_D y' \vdash S(!_Y y, !_Y y')}$$

The premise of the rule asserts that the pair of relations (R, S) carries a $\widehat{\mathrm{Rel}(T)}$-algebra structure over (X, Y). The conclusion tells that we have a coinduction principle on the contravariant side and an induction principle on the covariant one.

We can then combine our criteria of validity of induction and coinduction to give the following criterion of validity for the mixed principle.

6.4. Theorem. *If a fibration* $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{array}$ *satisfying the conditions of Definition 6.3, admits both comprehension and quotients then it satisfies the mixed induction/coinduction principle for any mixed variance polynomial functor on* $\mathbb{B}$.

Proof. Quotients and comprehension yield left and right adjoints to equality:

$$\begin{tikzcd}
\mathrm{Rel}(\mathbb{P}) \arrow[d, bend right=60, "\mathcal{Q}"'] \arrow[d, bend left=60, "\{-\} \circ \delta^*"] \\
\mathbb{B} \arrow[u, "\mathrm{Eq}" description] \arrow[u, phantom, shift right=5, "\dashv"] \arrow[u, phantom, shift left=5, "\dashv"]
\end{tikzcd}$$

By combining these adjoints we get a right adjoint $\mathcal{Q} \times (\{-\} \circ \delta^*) \colon \mathrm{Rel}(\mathbb{P})^{\mathrm{op}} \times \mathrm{Rel}(\mathbb{P}) \to \mathbb{B}^{\mathrm{op}} \times \mathbb{B}$ to the functor $\mathrm{Eq}^{\mathrm{op}} \times \mathrm{Eq}$ in

$$\begin{tikzcd}
\mathrm{Rel}(\mathbb{P})^{\mathrm{op}} \times \mathrm{Rel}(\mathbb{P}) \arrow[loop above, "\widehat{\mathrm{Rel}(T)}"] \arrow[d, bend left=40] \arrow[d, phantom, shift left=2, "\dashv"] \\
\mathbb{B}^{\mathrm{op}} \times \mathbb{B} \arrow[loop below, "\widehat{T}"] \arrow[u, "\mathrm{Eq}^{\mathrm{op}} \times \mathrm{Eq}"]
\end{tikzcd}
\qquad \text{yielding} \qquad
\begin{tikzcd}
\mathrm{Alg}(\widehat{\mathrm{Rel}(T)}) \arrow[d, bend left=40] \arrow[d, phantom, shift left=2, "\dashv"] \\
\mathrm{Alg}(\widehat{T}) \arrow[u, "\mathrm{Alg}(\mathrm{Eq}^{\mathrm{op}} \times \mathrm{Eq})"]
\end{tikzcd}$$

Therefore $\mathrm{Alg}(\mathrm{Eq}^{\mathrm{op}} \times \mathrm{Eq})$ preserves initial objects. □

The previous theorem describes validity for our mixed variance principle in pure (cartesian) form. As it stands, it does not apply to our main examples $\begin{array}{c}\scriptstyle\mathrm{ASub}(\omega\text{-}\mathbf{Cpo}_\perp)\\ \downarrow \\ \scriptstyle\omega\text{-}\mathbf{Cpo}_\perp\end{array}$, $\begin{array}{c}\scriptstyle\mathrm{ClSub}(\mathbf{Cms})\\ \downarrow \\ \scriptstyle\mathbf{Cms}\end{array}$ and $\begin{array}{c}\scriptstyle\mathrm{ClSub}(\mathbf{Cums})\\ \downarrow \\ \scriptstyle\mathbf{Cums}\end{array}$, see Examples 2.2.(ii) and (iii): the category **Cums** is cartesian closed, but the categories ω-$\mathbf{Cpo}_\perp$ and **Cms** are only monoidal closed. And the metric categories **Cms** and **Cums** are algebraically compact with respect to a class of functors (namely the locally contractive functors) which does not include the identity functor. This second problem is not so serious, as it only requires a minor adaptation of the main result, specifying the appropriate class of polynomial functors with $\mathrm{id}_{\frac{1}{2}}$ replacing[5] id.

The first problem involves replacing the cartesian closed structure $(1, \times, \Rightarrow)$ in mixed variance polynomial functors by monoidal closed structure $(I, \otimes, \multimap)$. There are canonical liftings of this structure $(I, \otimes, \multimap)$ on ω-$\mathbf{Cpo}_\perp$ to $(I, \otimes, \multimap)$ on $\mathrm{ASub}(\omega\text{-}\mathbf{Cpo}_\perp)$ and of $(I, \otimes, \multimap)$ on **Cms** to $(I, \otimes, \multimap)$ on $\mathrm{ClSub}(\mathbf{Cms})$, determined by the universal property of $\otimes$ in both these categories (bistrict morphism classifier in ω-$\mathbf{Cpo}_\perp$ and bi-non-expansive map classifier in **Cms**), namely:

- The tensor product $(P \subseteq A) \otimes (Q \subseteq B)$ of two admissible subsets $P \subseteq A, Q \subseteq B$ of ω-cpo's A, B is the subset $(P \otimes Q \subseteq A \otimes B)$. The associated unit is $\mathbf{1}(I) = (I \subseteq I)$, where $I = \{-, \top\}$ is the neutral element for $\otimes$ on ω-$\mathbf{Cpo}_\perp$.
- The associated internal hom $(P \subseteq A) \multimap (Q \subseteq B)$ on $\mathrm{ASub}(\omega\text{-}\mathbf{Cpo}_\perp)$ is the subset $(\{f \mid f(P) \subseteq Q\} \subseteq A \multimap B)$.

The lifting for metric spaces is similar:

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[5] With the further addition that the functor $\mathrm{id}_{\frac{1}{2}} \colon \mathbf{Cms} \to \mathbf{Cms}$ has a logical relation lifting $\mathrm{Pred}(\mathrm{id}_{\frac{1}{2}}) \colon \mathrm{ClSub}(\mathbf{Cms}) \to \mathrm{ClSub}(\mathbf{Cms})$ given by $(P \subseteq A) \mapsto (\mathrm{id}_{\frac{1}{2}}(P) \subseteq \mathrm{id}_{\frac{1}{2}}(A))$. And similarly for **Cums** instead of **Cms**.

- The tensor product $(P \subseteq A) \otimes (Q \subseteq B)$ of two closed subsets $P \subseteq A, Q \subseteq B$ of complete metric spaces A, B is $(P \otimes Q \subseteq A \otimes B)$, with neutral element $\mathbf{1}(1) = (1 \subseteq 1)$, where $1 = \{*\}$ is the neutral element for $\otimes$ on **Cms**.
- The internal hom on $\mathrm{ClSub}(\mathbf{Cms})$ is given by $(P \subseteq A) \multimap (Q \subseteq B) = (\{f \mid f(P) \subseteq Q\} \subseteq A \multimap B)$.

In the same way we have canonical liftings to total categories of relations (on ω-cpo's and on complete metric spaces). With these modifications, we can apply our previous setup (*i.e.* formulation of (co)induction principles and the criteria for their admissibility) to polynomial functors determined by the above monoidal closed structures.

7 Stability of initial algebras and terminal coalgebras under weakening of context

So far we have considered (co)inductive data types and their associated (co)induction principles in terms of initiality *in the empty context*. For instance, the initiality of $\mathbb{N}$ allows us to define functions out of it, *eg.* $h: \mathbb{N} \dashrightarrow A$, by endowing the set A with a $1 + (\_)$-algebra structure. But we also want to use this method when the inductive data type occurs in an arbitrary context, *eg.* to define addition $\mathsf{add}: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by induction on the second argument. This requires that the initiality of $\mathbb{N}$ be preserved when we move from the empty context to the context $n: \mathbb{N}$ (for the first argument of add). This operation is called *context weakening*. Technically, we say initiality is *stable under addition of indeterminates*, the indeterminate being $n: \mathbb{N}$. This is also called *initiality with parameters*, see [26, 9, 10].

A similar extension is needed then for the associated induction principle, since when we perform context weakening $\Gamma \mapsto (\Gamma, x: A)$, the element x may be subject to some (propositional) hypothesis. That is, we are generally interested in proving relative entailments $P \vdash Q$ rather than 'absolute' assertions $\emptyset \vdash Q$. For instance, we may want to prove $n: \mathbb{N}, m: \mathbb{N} \mid p: \mathsf{Even}(m) \vdash q: \mathsf{Even}(\mathsf{add}(2 * n, m))$ for some q, in which case we use induction on n with $m: \mathbb{N}$ and $p: \mathsf{Even}(m)$ as parameters. The stable version of the induction principle for $\mathbb{N}$ is formulated in [29, 12] in logical terms. We will give a categorical account in §§7.2 below, extended to the mixed variance case, as well as a criterion for its admissibility (Theorem 7.6).

Abstractly, both extensions (with type and proposition parameters) are instances of the same phenomenon: let $\mathcal{K}$ be a 2-category with finite products and inserters and let C be an object of $\mathcal{K}$ with a terminal object $\mathbf{1}$, given by an adjunction $! \dashv 1: \mathbf{1} \to C$ in $\mathcal{K}$. For a global element $A: \mathbf{1} \to C$ (or C-object), we can consider the 'object C with an indeterminate element $x: 1 \Rightarrow A$', written $C[x: A]$. This object is equipped with a 1-cell $\eta_A: C \to C[x: A]$ and a 2-cell $\alpha_x: \eta_A 1 \Rightarrow \eta_A A$, and is universal among objects with such data. Given an endomorphism (1-cell) $T: C \to C$ in $\mathcal{K}$, we can consider the object of T-algebras $\mathrm{Alg}(T)$, namely the inserter of T and the identity on C (in $\mathcal{K}$). Similarly, since any polynomial functor $T: C \to C$ induces[6] $T[x: A]: C[x: A] \to C[x: A]$ with $T[x: A]\eta_A = \eta_A T$, we can consider the object $\mathrm{Alg}(T[x: A])$ and the induced morphism $\mathrm{Alg}(\eta_A)\colon \mathrm{Alg}(T) \to \mathrm{Alg}(T[x\colon A])$. Stability means that $\mathrm{Alg}(\eta_A)$ preserves initial objects, for every C-object $A\colon \mathbf{1} \to C$.

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[6] In general, a functor T lifts to a functor $T[x: A]: C[x: A] \to C[x: A]$ if it admits a *strength*. Any polynomial functor admits a strength [26, 9].

With the above formalisation of stability, it follows from Theorem A.5 that stability is guaranteed whenever the object A is *functionally complete*, *i.e.* when η_A has a right adjoint. Similarly, stability of terminal coalgebras is guaranteed whenever $\mathbb{B}$ is *contextually complete*, *i.e.* when η_A has a left adjoint. We spell this out in more detail for categories and fibrations in the following subsections. Further details on indeterminates and on contextual and functional completeness can be found in [22]. We refer to [46] for the relevant definitions of comonads and their associated morphisms, as well as of Kleisli objects for them, in a 2-category. In any case, these concepts are not essential to understand what follows.

7.1 Stability of initial algebras and terminal coalgebras in a distributive bicategory

The material in this subsection is based on [26], although the formulations and proofs are different. It is a preliminary to the treatment of stability of (co)induction principles in §§7.2.

Given a bicartesian category $\mathbb{B}$ and an object $A \in \mathbb{B}$, we let $\mathbb{B}[x\colon A]$ denote the universal bicartesian category $\eta_A\colon \mathbb{B} \to \mathbb{B}[x\colon A]$ which has a global element of type A, *i.e.* a morphism $x\colon 1 \to \eta_A(A)$. Universality means (at the 1-dimensional level) that given a bicartesian category $\mathbb{C}$, a functor $F\colon \mathbb{B} \to \mathbb{C}$ preserving finite products and coproducts, and a morphism $a\colon F1 \to FA$ in $\mathbb{C}$, there is a unique functor $\overline{F}\colon \mathbb{B}[x\colon A] \dashrightarrow \mathbb{C}$ preserving finite products and coproducts such that

$$\overline{F}\eta_A = F \qquad \text{and} \qquad \overline{F}(x) = a.$$

The category $\mathbb{B}[x\colon A]$ can be characterised as the Kleisli category of the comonad $(\_) \times A$, written $\mathbb{B}/\!\!/A$, when $\mathbb{B}$ is *distributive*, *i.e.* when $(\_) \times A$ preserves finite coproducts.

From a logical point of view, we think of $\mathbb{B}[x\colon A]$ as the theory with the same types of $\mathbb{B}$, whose terms have a parameter of type A, *i.e.* whose terms are of the form $\Gamma, x\colon A \vdash t\colon B$ in $\mathbb{B}$. This interpretation is obtained by considering the internal language of the Kleisli category of the comonad $(\_) \times A$ on $\mathbb{B}$.

A functor $T\colon \mathbb{B} \to \mathbb{B}$ lifts to a functor $T/\!\!/A\colon \mathbb{B}/\!\!/A \to \mathbb{B}/\!\!/A$ such that $(T/\!\!/A) \circ \eta_A = \eta_A \circ T$, whenever it is endowed with appropriate additional structure. Technically, this structure is exactly what makes T a morphism of comonads; it is essentially the same as requiring T to have a strength. More specifically, we require a natural transformation $\theta\colon (\times \circ (T \times \mathrm{id})) \Rightarrow (T \circ \times)\colon \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ satisfying the following coherence conditions:

$$\begin{tikzcd}
TX \arrow[dd, equal] & TX \times A \arrow[l, "\pi"'] \arrow[r, "\delta"] \arrow[dd, "\theta"] & (TX \times A) \times A \arrow[d, "\theta \times \mathrm{id}"] \\
 & & T(X \times A) \times A \arrow[d, "\theta"] \\
TX & T(X \times A) \arrow[l, "T(\pi)"] \arrow[r, "T(\delta)"'] & T((X \times A) \times A)
\end{tikzcd}$$

Every polynomial functor T admits such structure in a canonical way and hence can be lifted to $\mathbb{B}/\!\!/A$.

7.1. Definition. Consider a bicartesian category $\mathbb{B}$ and a polynomial functor $T: \mathbb{B} \to \mathbb{B}$ on $\mathbb{B}$.

(i) We say that $\mathbb{B}$ admits *stable initial T-algebras* if it admits an initial T-algebra and for every object $A \in \mathbb{B}$, the induced functor

$$\mathrm{Alg}(\eta_A): \mathrm{Alg}(T) \longrightarrow \mathrm{Alg}(T/\!\!/A)$$

preserves initial objects.

(ii) Dually, $\mathbb{B}$ admits *stable terminal T-coalgebras* if it admits a terminal T-coalgebra and for every object A, the functor

$$\mathrm{CoAlg}(\eta_A): \mathrm{CoAlg}(T) \longrightarrow \mathrm{CoAlg}(T/\!\!/A)$$

preserves terminal objects.

We recall from [22] that a category $\mathbb{B}$ is *functionally complete* if for every object $A \in \mathbb{B}$, the induced functor $\eta_A: \mathbb{B} \to \mathbb{B}[x: A]$ has a right adjoint, and that it is *contextually complete* when every such η_A has a left adjoint. A bicartesian category $\mathbb{B}$ is contextually complete if it is distributive, and is functionally complete if it is additionally cartesian closed. As an immediate consequence of Theorem 2.12 we have the following result.

7.2. Proposition. *Let $\mathbb{B}$ be a bicartesian category and $T: \mathbb{B} \to \mathbb{B}$ a polynomial functor.*

(i) *If $\mathbb{B}$ is contextually complete, then terminal T-coalgebras are stable.*

(ii) *If $\mathbb{B}$ is functionally complete, then initial T-algebras are stable.* □

7.2 Stability of initial algebras and terminal coalgebras in a bicartesian fibration

Just as we require inductive data types to be stable under addition of indeterminates to use the initial algebra property in an arbitrary context, we must require an analogous stability of their associated induction principles. Similar considerations apply to coalgebras and coinduction. In order to express such stability, we consider, for a given fibration (logic), an associated fibration with indeterminates both on the base and total categories.

7.3. Remark. Although the treatment of indeterminates for fibrations to follow parallels that for categories in §§7.1, there is a subtle technical difference. All the concepts previously defined by universal properties in **Cat**, should be considered in their bicategorical variants in **Fib**, *i.e.* up-to-equivalence rather than up-to-isomorphism. This is because the pseudo-functorial nature of the cleavages of fibrations allows only the existence of the bicategorical cocompleteness properties required (Kleisli objects), rather than the 2-categorical versions previously mentioned. The strict 2-categorical version does apply if we restrict attention to split fibrations and splitting-preserving morphisms.

Given a bicartesian fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ and an object P of $\mathbb{P}$, say over $A \in \mathbb{B}$, the fibration with an indeterminate of P, written $p[\langle x, h\rangle\colon P]\colon \mathbb{P}/\!\!/(P) \to \mathbb{B}[x\colon A]$ is the universal fibration $(\eta_P, \eta_A)\colon p \to p[\langle x, h\rangle\colon P]$ with a global element $x\colon 1 \to \eta_A(A)$ in $\mathbb{B}[x\colon A]$ and a global element $h\colon \mathbf{1}_1 \to x^*(\eta_P(P))$ in $(\mathbb{P}/\!\!/(P))_1$. Universality means that given a bicartesian fibration $\begin{smallmatrix}\mathbb{Q}\\ \downarrow q\\ \mathbb{C}\end{smallmatrix}$, a morphism of fibrations $(H, K)\colon p \to q$ preserving finite products and coproducts, and global elements $a\colon K1 \to K(A)$ and $b\colon H\mathbf{1}_1 \to a^*(HP)$, there is a unique (up to isomorphism) morphism $(H', K')\colon p[\langle x, h\rangle\colon P] \dashrightarrow q$ preserving finite products and coproducts such that

$$(H', K') \circ (\eta_P, \eta_A) \cong (H, K), \qquad K'x = a, \qquad \phi \circ H'h = b.$$

where $\phi\colon H'(x^*(\eta_P(P))) \overset{\cong}{\Rightarrow} (K'x)^*(HP)$ is the canonical comparison isomorphism in the fibration q.

It is easy to extend Lemma 2.1 to make the total category $\mathbb{P}$ a distributive category when the base and the fibres are so and when the coreindexing functors $\coprod$ satisfy Beck-Chevalley and Frobenius conditions [20, Prop. 4.5.8]. We call such a bicartesian p, with the base and total categories distributive, a *distributive fibration*. In this case, we can characterise $p[\langle x, h\rangle\colon P]$ as a Kleisli fibration $p/\!\!/(P)$ for the comonad $((\_) \times P, (\_) \times pP)$ on p (in **Fib**), as explained in [22]. See also [32] for a concrete description and a different application of this construction.

From a logical perspective, we think of the fibration $p[\langle x, h\rangle\colon P]$ as a logic with the same types and propositions as those of p, but whose terms have a parameter of type $A = pP$, *i.e.* whose terms are of the form $\Gamma, x\colon A \vdash t\colon B$, and whose entailment relation allows an additional hypothesis $P(x)$, *i.e.* the entailments have the form

$$\Gamma, x\colon A \mid \Theta, h\colon P(x) \vdash q\colon Q$$

That is, we are assuming the presence of an additional element x of type A, and a predicate P on that type whose instance at x is provably true. Both these elements represent the additional data with their associated properties forming the context in which we are working, for instance when carrying out an inductive proof. Semantically, such interpretation of $p[\langle x, h\rangle\colon P]$ can be obtained via the internal language of the Kleisli fibration $p/\!\!/(P)$.

A polynomial morphism of fibrations $(\mathrm{Pred}(T), T)\colon p \to p$ as considered in §§3, induces an endomorphism of fibrations $p[\langle x, h\rangle\colon P] \to p[\langle x, h\rangle\colon P]$ in a situation:

$$\begin{tikzcd}
\mathbb{P} \arrow[rr, "\mathrm{Pred}(T)"] \arrow[dr, "\eta_P"] \arrow[dd, "p"'] & & \mathbb{P} \arrow[dr, "\eta_P"] \arrow[dd, "p" near end] & \\
& \mathbb{P}/\!\!/(P) \arrow[rr, dashed, "\mathrm{Pred}(T)[h\colon P]" near start] \arrow[dd, "{p[\langle x, h\rangle\colon P]}" near start] & & \mathbb{P}/\!\!/(P) \arrow[dd, "{p[\langle x, h\rangle\colon P]}"] \\
\mathbb{B} \arrow[rr, "T"' near end] \arrow[dr, "\eta_A"'] & & \mathbb{B} \arrow[dr, "\eta_A"] & \\
& \mathbb{B}/\!\!/A \arrow[rr, dashed, "{T[x\colon A]}"'] & & \mathbb{B}/\!\!/A
\end{tikzcd}$$

Hence we have an induced morphism of fibrations

$$\mathrm{Alg}(\eta_P, \eta_A) \colon \mathrm{Alg}(\mathrm{Pred}(T), T) \longrightarrow \mathrm{Alg}(\mathrm{Pred}(T)[h{:}\,P], T[x{:}\,A])$$

where for an endomorphism $(H, K) \colon p \to p$ in $\mathbf{Cat}^{\to}$, with p a fibration, the fibration $\mathrm{Alg}(H, K)$ is obtained as the inserter of (H, K) and the identity on p; its base category is $\mathrm{Alg}(K)$ and its total category is $\mathrm{Alg}(H)$ (see the Appendix). Now we can formalise the stability of the (co)induction principle for (co)algebras.

7.4. Definition. Consider a polynomial functor $T \colon \mathbb{B} \to \mathbb{B}$ and a bicartesian fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$.

(i) The fibration p satisfies the *stable* induction principle with respect to T if the functor $\mathrm{Alg}(\mathbf{1}) \colon \mathrm{Alg}(T) \to \mathrm{Alg}(\mathrm{Pred}(T))$ preserves initial objects, and moreover, for every object $P \in \mathbb{P}$ over $A \in \mathbb{B}$, the morphism

$$\mathrm{Alg}(\eta_P, \eta_A) \colon \mathrm{Alg}(\mathrm{Pred}(T), T) \longrightarrow \mathrm{Alg}(\mathrm{Pred}(T)[h{:}\,P], T[x{:}\,A])$$

preserves initial objects (both on the base and the total categories).

(ii) The fibration p, additionally admitting equality, satisfies the *stable* coinduction principle with respect to T if the functor $\mathrm{CoAlg}(\mathrm{Eq}) \colon \mathrm{CoAlg}(T) \to \mathrm{CoAlg}(\mathrm{Rel}(T))$ preserves terminal objects and moreover, for every $P \in \mathbb{P}_A$, the morphism

$$\mathrm{CoAlg}(\eta_P, \eta_A) \colon \mathrm{CoAlg}(\mathrm{Rel}(T), T) \longrightarrow \mathrm{CoAlg}(\mathrm{Rel}(T)[h{:}\,P], T[x{:}\,A])$$

preserves terminal objects.

(iii) Assuming $\mathbb{B}$ and $\mathbb{P}$ bicartesian closed, with both functors p and $\mathrm{Eq} \colon \mathbb{B} \to \mathrm{Rel}(\mathbb{P})$ structure preserving, p satisfies the *stable* mixed induction/coinduction principle with respect to T if the induced functor $\mathrm{Alg}(\mathrm{Eq}^{\mathrm{op}} \times \mathrm{Eq}) \colon \mathrm{Alg}(\widehat{T}) \to \mathrm{Alg}(\widehat{\mathrm{Rel}(T)})$ preserves initial objects and moreover, for every $P \in \mathbb{P}_A$, the morphism of fibrations

$$\mathrm{Alg}((\eta_P^{\mathrm{op}} \times \eta_P), (\eta_A^{\mathrm{op}} \times \eta_A)) \colon \mathrm{Alg}(\widehat{\mathrm{Rel}(T)}, \widehat{T}) \longrightarrow \mathrm{Alg}(\mathrm{Rel}\widehat{(T)[h{:}\,P]}, \widehat{T[x{:}\,A]})$$

preserves initial objects.

7.5. Remark. The above definition could equivalently be expressed by requiring that every fibration $p[\langle x, h\rangle {:}\, P]$ with an indeterminate object P, satisfy the induction principle with respect to the induced morphism of fibrations $(\mathrm{Pred}(T)[h{:}\,P], T[x{:}\,pP]) \colon p[\langle x, h\rangle {:}\, P] \to p[\langle x, h\rangle {:}\, P]$, provided the base category $\mathbb{B}$ admits stable initial algebras. This makes logical sense, as we want to reason by induction in the fibration $p[\langle x, h\rangle {:}\, P]$, which has an indeterminate of type pP, satisfying the hypothesis P; this is exactly what the above formulation means. Similar considerations apply to coalgebras and coinduction.

In analogy with ordinary categories, we say that the fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ is *functionally complete* when, for every object $P \in \mathbb{P}_A$, the morphism $(\eta_P, \eta_A) \colon p \to p[\langle x, h\rangle {:}\, P]$ has a right adjoint (in $\mathbf{Fib}$). This holds for instance when p admits (or models) universal quantifiers $\forall$ and implication $\Rightarrow$ (as a model of first-order logic). And we call p *contextually complete* when the above morphisms (η_P, η_A)

admit left adjoints. Contextual completeness holds for distributive fibrations because the corresponding fibration with an indeterminate $p[\langle x, h\rangle : P]$ is a Kleisli object; the left adjoint is part of the resolution of the comonad $((\_) \times P, (\_) \times A)$ (again we refer to [22] for details). Then, we can apply Theorem A.5 (in the 2-category $\mathbf{Cat}^{\rightarrow}$) to show the following.

7.6. Theorem. *Let* $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *be a distributive fibration.*

(i) *If p satisfies the coinduction principle with respect to a polynomial functor T, then it also satisfies the corresponding stable coinduction principle.*

(ii) *If p is functionally complete and satisfies the induction principle with respect to to a polynomial endofunctor T, then it also satisfies the stable induction principle with respect to T.*

7.7. Corollary. *If the fibration* $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *is contextually and functionally complete, and satifies the mixed induction/coinduction principle for* $T : \mathbb{B}^{\mathrm{op}} \times \mathbb{B} \to \mathbb{B}$, *then it also satisfies the stable induction/coinduction principle for T.*

The fibrations of Example 2.2 are functionally complete: $\begin{smallmatrix}\mathrm{Sub}(\mathbf{Sets})\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ is so because it models $\forall$ and $\Rightarrow$, while $\begin{smallmatrix}\mathrm{ASub}(\omega\text{-}\mathbf{Cpo}_{\perp})\\ \downarrow\\ \omega\text{-}\mathbf{Cpo}_{\perp}\end{smallmatrix}$ is functionally complete although it does not model implication ($\Rightarrow$); functional completeness holds essentially because of the reflection mentioned in Remark 3.3. The same considerations apply to (ultra) metric spaces and closed subsets. Thus, the above abstract formulation seems to capture better this kind of example than a purely syntactic approach would. As for the syntactic example, we must assume our logic has implication and universal quantification $\forall x : A.(\_)$, as explained in [22]. Functional completeness (in this syntactic setting) is implicitly used in [29, § II.4] to show validity of the stable induction principle over the natural numbers object in a topos.

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References

1. P. Aczel and N. Mendler, A final coalgebra theorem, In: D.H. Pitt, A. Poigné, D.E. Rydeheard (eds.), *Category Theory and Computer Science* (Springer LNCS 389, Berlin 1989) p. 357–365.
2. S. Abramsky and A. Jung, Domain Theory, In: S. Abramsky, Dov M. Gabbai and T.S.E. Maibaum (eds.), *Handbook of Logic in Computer Science 3* (Oxford Univ. Press, 1994) p. 1–168.
3. P. America and J. Rutten, Solving reflexive domain equations in a category of complete metric spaces *Journ. Comp. & Syst. Sci.* **39(3)** (1989) p. 343–375.
4. M.A. Arbib and E.G. Manes, *Arrows, Structures and Functors. The Categorical Imperative* (Academic Press, New York 1975).
5. R. Backhouse and M. Bijsterveld, The “Beautiful Theorem”, manuscript, Univ. Eindhoven. Presented at *IFIP Working Group 2.3*, June 1994.
6. J. de Bakker and E. Vink, *Control Flow Semantics* (The MIT Press, Boston 1996).
7. M. Barr, Algebraically compact functors, *Journ. of Pure & Appl. Alg.* **82** (1992) p. 211–231.

8. J.R.B. Cockett, Introduction to Distributive Categories, *Math. Struct. in Comp. Science* **1** (1991), p. 1–20.
9. J.R.B. Cockett and D. Spencer, Strong Categorical Datatypes I. In: R.A.G. Seely (ed.), *Category Theory 1991* (CMS Conference Proceedings 13, 1992) p. 141–169.
10. J.R.B. Cockett and D. Spencer, Strong Categorical Datatypes II: A term logic for categorical programming, *Theor. Comp. Science* **139** (1995) p. 69–113.
11. R.L. Crole, *Categories for Types* (Camb. Univ. Press, Math. Textbooks, 1993).
12. R.L. Crole and A.M. Pitts, New Foundations for Fixpoint Computations: FIX-hyperdoctrines and the FIX-logic, *Inform. & Comput.* **98**(**2**) (1992) p. 171–210.
13. Th. Ehrhard, A Categorical Semantics of Constructions. In: *Logic in Comp. Sci. 3* (IEEE Comp. Sci. Press, 1988) p. 264–273.
14. A. Fiech, Colimits in the category Cpo, Techn. Rep., Kansas State University, 1992.
15. M. Fiore, A coinduction principle for recursive data types based on bisimulation. In: *Logic in Comp. Sci. 8* (IEEE Comp. Sci. Press, 1993) p. 110–119.
16. P.J. Freyd, Recursive types reduced to inductive types. In: *Logic in Comp. Sci. 5* (IEEE Comp. Sci. Press, 1990) p. 498–507.
17. P.J. Freyd, Remarks on algebraically compact categories. In: M.P. Fourman and P.T. Johnstone and A.M. Pitts (eds.) *Applications of Categories in Computer Science* (Camb. Univ. Press, LMS 177, 1992) p. 95–106.
18. J.A. Goguen, J. Thatcher and E. Wagner, An initial algebra approach to the specification, correctness and implementatiton of abstract data types. In: R. Yeh (ed.), *Current Trends in Programming Methodology* (Prentice Hall, 1978) p. 80–149.
19. T. Hagino,A Typed Lambda Calculus with Categorical Type Constructors. In: D.H. Pitt and A Poigné and D.E. Rydeheard (eds.), *Category and Computer Science* (Springer LNCS 283, 1987) p. 140–157.
20. C. Hermida, Fibrations, logical predicates and indeterminates, PhD. Thesis, Univ. of Edinburgh, 1993. Techn. rep. LFCS-93-277. Also available as Aarhus Univ. DAIMI Techn. rep. PB-462.
21. C. Hermida and B. Jacobs, An algebraic view of structural induction, In: L. Pacholski and J. Tiuryn (eds.), *Computer Science Logic 1994* (Springer LNCS 933, Berlin 1995) p. 412–426.
22. C. Hermida and B. Jacobs, Fibrations with Indeterminates: Contextual and Functional Completeness for Polymorphic Lambda Calculi, *Math. Struct. in Comp. Sci.* **5** (1995), p. 501–531.
23. B. Jacobs, *Categorical Type Theory*, PhD. thesis, Univ. Nijmegen, 1991.
24. B. Jacobs, Semantics of Weakening and Contraction, *Annals of Pure & Appl. Logic* **69**(**1**) (1994) p. 73–106.
25. B. Jacobs, Quotients in simple type theory, manuscript, Univ. of Utrecht, June 1994.
26. B. Jacobs, Parameters and Parametrization in Specification using distributive categories, *Fund. Informaticae* **24**(**3**) (1995), p. 209–250.
27. A. Joyal and I. Moerdijk, *Algebraic Set Theory* (Cambridge Univ. Press, 1995).
28. G.M. Kelly, Elementary observations on 2-categorical limits, *Bull. Austr. Math. Soc.* **39** (1989) p. 301–317.
29. J. Lambek and P.J. Scott, *Introduction to Higher-Order Categorical Logic* (Camb. Univ. Press, Studies in Adv. Math. 7, 1986).
30. F.W. Lawvere, Equality in hyperdoctrines and comprehension scheme as an adjoint functor. In: A. Heller (ed.), *Applications of Categorical Algebra* (AMS Providence, 1970) p. 1–14.
31. D. Lehmann and M. Smyth, Algebraic specification of datatypes: a synthetic approach *Math. Systems Theory* **14** (1981) p. 97 – 139.

32. M. Makkai, The fibrational formulation of intuitionistic predicate logic I: completeness according to Gödel, Kripke, and Läuchli. Part 1, *Notre Dame Journ. Formal Logic* **34(3)** (1993), p. 334–377.
33. M. Makkai, The fibrational formulation of intuitionistic predicate logic I: completeness according to Gödel, Kripke, and Läuchli. Part 2, *Notre Dame Journ. Formal Logic* **34(4)** (1993), p. 471–499.
34. G. Markowsky, Categories of chain-complete posets, *Theor. Comp. Science* **4** (1977) p. 125–135.
35. D. Pavlović, *Predicates and Fibrations*, PhD. thesis, Univ. Utrecht, 1990.
36. D. Pavlović, Maps I: relative to a factorization system, *Journ. Pure & Appl. Algebra* **60(2)** (1995), p. 9–34.
37. D. Pavlović, Maps II: Chasing diagrams in categorical proof theory, *Journ. of the IGPL* **4(2)** (1996), p. 1–36.
(Available from URL `http://www.mpi-sb.mpg.de/igpl/Journal`)
38. A.M. Pitts, Relational properties of recursively defined domains. In: *Logic in Comp. Sci. 8* (IEEE Comp. Sci. Press, 1993) p. 86–97.
39. A.M. Pitts, A co-induction principle for recursively defined domains, *Theor. Comp. Sci.* **124(2)** (1994) p. 195–219.
40. G. Plotkin and M. Abadi, A Logic for Parametric Polymorphism. in: M. Bezem and J.F. Groote (eds.) *Typed Lambda Calculi and Applications* (Springer LNCS 664, Berlin 1993) p. 361–375.
41. H. Reichel, An approach to object semantics based on terminal co-algebras *Math. Struct. in Comp. Sci.* **5** (1995), p. 129–152.
42. J. Rutten and D. Turi, On the foundations of final semantics: non-standard sets, metric spaces and partial orders. In: J.W. de Bakker, W.P. de Roever, and G. Rozenberg (eds.), *Semantics: Foundations and Applications* (Springer LNCS 666, Berlin 1993) p. 477–530.
43. J. Rutten and D. Turi, Initial Algebra and Final Coalgebra Semantics for Concurrency. In: J.W. de Bakker, W.P. de Roever, and G. Rozenberg (eds.), *A Decade of Concurrency* (Springer LNCS 803, Berlin 1994) p. 530–582.
44. M.B. Smyth, Topology. In: S. Abramsky, Dov M. Gabbai and T.S.E. Maibaum (eds.), *Handbook of Logic in Computer Science*, Volume 1, *Background: Mathematical Structures*, Oxford Univ. Press, 1992, p. 641–761.
45. M.B. Smyth and G.D. Plotkin, The category theoretic solution of recursive domain equations, *SIAM Journ. Comput.*, **11** (1982) p. 761 – 783.
46. R. Street, The formal theory of monads, *Journ. Pure & Appl. Algebra* **2** (1972) p. 149–168.
47. R. Street, Fibrations and Yoneda's Lemma in a 2-category. In: G.M. Kelly (ed.), *Proc. Sydney Category Theory Seminar 1972/1973* (Springer LNM 420, Berlin 1974) p. 104–133.
48. R. Street and R.F.C. Walters, Yoneda Structures on 2-Categories, *Journ. of Algebra* **50** (1978) p. 350–379.
49. M. Wirsing, Algebraic Specification, in: J. van Leeuwen (ed.), *Handbook of Theoretical Computer Science* (Elsevier/MIT Press 1990) Volume B, p. 673–788.

A 2-functoriality of inserters

The notion of *inserter* in a 2-category is taken from [47, 28].

A.1. Definition (Inserter). Given parallel morphisms $f, g\colon A \rightrightarrows C$ in a 2-category $\mathcal{K}$, their **inserter** consists of a morphism $p\colon \mathrm{Ins}(f,g) \to A$ together with a 2-cell $\lambda\colon fp \Rightarrow gp$ which is universal among such data: given a 1-cell $h\colon X \to A$

and a 2-cell $\sigma\colon fh \Rightarrow gh$, there is a unique morphism $h'\colon X \dashrightarrow \mathrm{Ins}(f,g)$ such that

$$ph' = h \qquad \text{and} \qquad \lambda h' = \sigma$$

and, furthermore, given a pair of such data $(h\colon X \to A,\ \sigma\colon fh \Rightarrow gh)$ and $(k\colon X \to A,\ \mu\colon fk \Rightarrow gk)$, and a 2-cell $\alpha\colon h \Rightarrow k$, such that

$$g\alpha \circ \sigma = \mu \circ f\alpha$$

there is a unique 2-cell $\alpha'\colon h' \Rightarrow k'$ such that $p\alpha' = \alpha$.

In **Cat**, the inserter of a pair of parallel functors $F, G\colon \mathbb{A} \rightrightarrows \mathbb{B}$ is given by the category $\mathrm{Ins}(F,G)$ with

objects	pairs $(A, a\colon FA \to GA)$, where A is an object in $\mathbb{A}$ and a is a morphism $FA \to GA$ in $\mathbb{B}$.
morphisms	$f\colon (A,a) \to (B,b)$ are morphisms $f\colon A \to B$ in $\mathbb{A}$ such that $Gf \circ a = b \circ Ff$.

In order to exhibit the 2-functoriality of the assignment $(f, g\colon A \rightrightarrows B) \mapsto \mathrm{Ins}(f,g)$, we need appropriate notions of morphisms and 2-cells between parallel morphisms.

A.2. Definition. Given a 2-category $\mathcal{K}$, the 2-category $\mathcal{K}^{\rightrightarrows}$ has

objects	pairs of parallel morphisms $(f, g\colon A \rightrightarrows B)$.
morphisms	$(f, g\colon A \rightrightarrows B) \to (f', g'\colon A' \rightrightarrows B')$ are 4-tuples (a, ρ, b, δ) of 1-cells $a\colon A \to A'$, $b\colon B \to B'$ and 2-cells $\rho\colon f'a \Rightarrow bf$ and $\delta\colon bg \Rightarrow g'a$ in $\mathcal{K}$, as displayed in:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"'] \arrow[dr, phantom, "\overset{\delta}{\Searrow}"] & A \arrow[l, "g"'] \arrow[d, "a"] \\
A' \arrow[r, "f'"'] \arrow[ur, phantom, "\overset{\rho}{\Nearrow}"] & B' & A' \arrow[l, "g'"]
\end{tikzcd}$$

2-cells	$(a, \rho, b, \delta) \Rightarrow (a', \rho', b', \delta')$ are given by two 2-cells $\alpha\colon a \Rightarrow a'$ and $\beta\colon b \Rightarrow b'$ in $\mathcal{K}$ satisfying $\beta f \circ \rho = \rho' \circ f'\alpha \quad \text{and} \quad g'\alpha \circ \delta = \delta' \circ \beta g.$

Identities and composition in $\mathcal{K}^{\rightrightarrows}$ are inherited from $\mathcal{K}$. Horizontal composition of 2-cells is well-defined by the interchange law in $\mathcal{K}$.

Now we can state the desired 2-functoriality of inserters.

A.3. Proposition. *The assignment* $(f, g\colon A \rightrightarrows B) \mapsto \mathrm{Ins}(f,g)$ *extends to a 2-functor*

$$\mathrm{Ins}(\_,\_)\colon \mathcal{K}^{\rightrightarrows} \longrightarrow \mathcal{K}.$$

Proof. We define $\mathrm{Ins}(\_,\_)$ on 1-cells and on 2-cells in $\mathcal{K}^{\rightrightarrows}$.

Given a morphism $(a,\rho,b,\delta)\colon (f,g\colon A \rightrightarrows B) \to (f',g'\colon A' \rightrightarrows B')$, the universality of $\mathrm{Ins}(f',g')$ gives us a morphism $h\colon \mathrm{Ins}(f,g) \to \mathrm{Ins}(f',g')$ induced by the 1-cell $ap\colon \mathrm{Ins}(f,g) \to A'$ together with the 2-cell $\delta p \circ b\lambda \circ \rho p\colon f'ap \Rightarrow g'cp$ as shown below.

$$\begin{tikzcd}
 & & \mathrm{Ins}(f,g) \arrow[dl, "p"'] \arrow[dr, "p"] & & \\
 & A \arrow[dl, "a"'] \arrow[dr, "f"] & \overset{\lambda}{\Longrightarrow} & A \arrow[dl, "g"'] \arrow[dr, "b"] & \\
A' \arrow[drr, "f'"'] & \overset{\rho}{\Longrightarrow} & B \arrow[d, "b"'] & \overset{\delta}{\Longrightarrow} & A' \arrow[dll, "g'"] \\
 & & B' & &
\end{tikzcd}$$

On 2-cells, given $(\alpha,\beta)\colon (a,\rho,b,\delta) \Rightarrow (a',\rho',b',\delta')$ in $\mathcal{K}^{\rightrightarrows}$, the universality of $\mathrm{Ins}(f',g')$ also gives us a 2-cell $\sigma\colon h \Rightarrow h'$ induced by the 2-cell $\alpha p\colon ap \Rightarrow a'p$, since it satisfies

$$g'\alpha b \circ (\delta p \circ b\lambda \circ \rho p) = (\delta' p \circ b'\lambda \circ \rho' p) \circ f'\alpha p$$

by definition of 2-cells in $\mathcal{K}^{\rightrightarrows}$ and the interchange law. ☐

Recall that an adjunction in a 2-category $\mathcal{K}$ is given by the following data: two 1-cells $f\colon A \to B$ and $g\colon B \to A$ and two 2-celss $\eta\colon \mathrm{id}_A \Rightarrow gf$ and $\varepsilon\colon fg \Rightarrow \mathrm{id}_B$, satisfying the triangular laws

$$\varepsilon f \circ f\eta = \mathrm{id}_f \qquad \text{and} \qquad g\varepsilon \circ \eta g = \mathrm{id}_g .$$

We write this data as $\eta,\varepsilon\colon f \dashv g\colon A \rightleftarrows B$ and say that g is right adjoint to f. The equational nature of adjunctions in a 2-category implies that adjunctions are preserved by 2-functors (just like ordinary functors preserve isomorphisms). Thus, we have the following easy corollary about $\mathrm{Ins}(\_,\_)$.

A.4. Corollary. *An adjunction in $\mathcal{K}^{\rightrightarrows}$ induces an adjunction between the corresponding inserters in $\mathcal{K}$.* ☐

Notice that an adjunction in $\mathcal{K}^{\rightrightarrows}$

$$(\eta_a,\eta_b),\, (\varepsilon_a,\varepsilon_b)\colon\, (a,\rho,b,\delta) \vdash (a',\rho',b',\delta')\colon \Big(A \overset{f}{\underset{g}{\rightrightarrows}} B\Big) \rightleftarrows \Big(A' \overset{f'}{\underset{g'}{\rightrightarrows}} B'\Big)$$

consists of adjunctions $\eta_a,\varepsilon_a\colon a \vdash a'\colon A \rightleftarrows A'$ and $\eta_b,\varepsilon_b\colon b \vdash b'\colon B \rightleftarrows B'$. By the definition of 2-cells in $\mathcal{K}^{\rightrightarrows}$, it follows that the adjoint mate of ρ', *i.e.* $\varepsilon_b f'a \circ b\rho' a \circ bf\eta_a$ is inverse to ρ, and similarly that the adjoint mate of δ is inverse to δ'. Hence, in such an adjoint situation, both ρ and δ' must be isomorphisms.

A.5. Theorem. *Consider a diagram in $\mathcal{K}$*

$$\begin{tikzcd}
A \arrow[r, "t"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\alpha}{\Nearrow}\cong"] & A \arrow[d, "f"] \\
B \arrow[r, "s"'] & B
\end{tikzcd}$$

in which α is an isomorphism and f has a right adjoint, $\eta, \varepsilon \colon f \dashv g \colon A \rightleftarrows B$. The adjoint mate of $\alpha^{\perp 1}$, namely

$$\beta = gs\varepsilon \circ g'\alpha^{\perp 1}g \circ \eta tg \colon tg \Longrightarrow gs$$

induces a morphism $\bar{g} \colon \mathrm{Ins}(s, \mathrm{id}_B) \to \mathrm{Ins}(t, \mathrm{id}_A)$ which is right adjoint to the morphism $\bar{f} \colon \mathrm{Ins}(t, \mathrm{id}_A) \to \mathrm{Ins}(s, \mathrm{id}_B)$ induced by the above diagram.

Proof. The morphisms $\bar{f}, \bar{g}$ arise by applying the 2-functor $\mathrm{Ins}(\_,\_) \colon \mathcal{K}^{\rightrightarrows} \to \mathcal{K}$ of Proposition A.3 to the given data, construed as morphisms in $\mathcal{K}^{\rightrightarrows}$. As such, these morphisms are adjoints in $\mathcal{K}^{\rightrightarrows}$, and so the adjunction $(\bar{f} \dashv \bar{g})$ follows by Corollary A.4. □

In **Cat**, the inserter $\mathrm{Ins}(T, \mathrm{id}_{\mathbb{A}})$ of a functor $T \colon \mathbb{A} \to \mathbb{A}$ is the category $\mathrm{Alg}(T)$ of T-algebras. The morphism $\bar{F} \colon \mathrm{Ins}(T, \mathrm{id}_{\mathbb{A}}) \to \mathrm{Ins}(S, \mathrm{id}_{\mathbb{B}})$ from the above corollary has action

$$\big(X, TX \xrightarrow{x} X\big) \longmapsto \big(FX, S(FX) \xrightarrow{\alpha_X} FTX \xrightarrow{Fx} FX\big)$$

and simiarly, its right adjoint $\bar{G} \colon \mathrm{Ins}(S, \mathrm{id}_{\mathbb{B}}) \to \mathrm{Ins}(T, \mathrm{id}_{\mathbb{A}})$ has action

$$\big(Y, SY \xrightarrow{y} Y\big) \longmapsto \big(GY, T(GY) \xrightarrow{\beta_Y} GTY \xrightarrow{Gy} GY\big)$$

as used in Theorem 2.12, namely as $\bar{F} = \mathrm{Alg}(F)$ and $\bar{G} = \mathrm{Alg}(G)$.