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On the Decidability of Certain Semi-Lattice Based Modal Logics

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Abstract. Sequent calculi are proof systems that are exceptionally suitable for proving the decidability of a logic. Several *relevance logics* were proved decidable using a technique attributable to Curry and Kripke. Further enhancements led to a proof of the decidability of *implicational ticket entailment* by Bimbó and Dunn in [12, 13]. This paper uses a different adaptation of the same core proof technique to prove a group of *positive modal logics* (with disjunction but no conjunction) decidable.

Keywords: Sequent calculi · Modal logic · Decidability · Relevance logic · Heap number · Semi-lattice based logic

1 Modal Logics

The well-known modal logic **S4** is arguably one of the most successful modal systems ever invented. It is a system that grew out of Lewis's original system of strict implication defined in [29] by the addition of the axiom $\neg\Diamond\neg p \rightarrow \neg\Diamond\neg\neg\Diamond\neg p$, where \rightarrow is strict implication (see [15]). **S4** was given the nowadays standard formulation of a normal modal logic as an explicit extension of classical propositional logic by Gödel in [24]. **S4** has a close connection to intuitionistic logic and topology, and it has a straightforward relational semantics over pre-ordered (or partially ordered) frames. The list of remarkable features goes on and on.

S4 can be formulated by adding two rules, namely, $(\Box\Box)$ and $(\Box\Diamond)$ to the propositional part of **LK** from [23].

$$\frac{I^\Box \Vdash \varphi}{I^\Box \Vdash \Box\varphi} \quad (\Box\Box) \qquad \frac{\varphi, \Gamma \Vdash \Delta}{\Box\varphi, \Gamma \Vdash \Delta} \quad (\Box\Diamond) \qquad (1)$$

This formulation assumes that the other modality, which is often denoted by \Diamond is defined (i.e., $\Diamond\varphi$ is simply an abbreviation for $\neg\Box\neg\varphi$). This is unproblematic in the case of classical logic, however, we do not always want to have a negation in a logic or we simply want to have both these modalities as primitives.¹

The sequent calculus formulation of **S4** with both modalities amends $(\Box\Box)$ to permit multiple formulas in the succedent. The new $(\Box\Box)$ rule and the rules for \Diamond were introduced in Kripke [27], and they are as follows.

¹ See for example Dunn [20] and Kripke [27].

$$\frac{\Gamma^\square \Vdash \Delta^\diamond, \varphi}{\Gamma^\square \Vdash \Delta^\diamond, \square\varphi} \quad (\Vdash\square) \qquad \frac{\Gamma \Vdash \Delta, \varphi}{\Gamma \Vdash \Delta, \diamond\varphi} \quad (\Vdash\diamond) \qquad \frac{\varphi, \Gamma^\square \Vdash \Delta^\diamond}{\diamond\varphi, \Gamma^\square \Vdash \Delta^\diamond} \quad (\diamond\Vdash)$$

Our goal in this paper is to investigate the problem of decidability for logics that contain a pair of modalities that have introduction rules analogous to the ones above, but they lack much of what an underlying 2-valued calculus gives. We are not concerned with interpretations here, however, we note that it is clear that once we start to drop rules from **LK**, the “meanings” of the connectives change. In order to preclude confusions stemming from connotations, we will use a pair of neutral symbols— \triangleright and \triangleleft —for the two unary connectives we take to be modalities. Another effect of omitting rules from **LK** is that space opens up for new versions of connectives—even without the introduction of multiple structural connectives. We will take advantage of this opportunity by including both \vee and $+$ in all our logics.

Our strategy is to fix a common set of connective rules for a group of logics. The choice of the connectives and of the rules for them is motivated by relevance logic (see, for example, [1, 2]). We will vary the structural rules and we will select 9 logics to scrutinize. We will refer to the whole group of these logics or to an arbitrary element of the group as $L\mathfrak{X}^*$.

Definition 1. The *signature* for $L\mathfrak{X}^*$ is $\langle \circ^2, \rightarrow^2, +^2, \vee^2, \triangleright^1, \triangleleft^1 \rangle$ (with the arities indicated in the superscripts). The *set of formulas* is generated by the following context-free grammar (CFG) in Backus–Naur form (BNF).

$$\varphi := \text{Prop} \mid (\varphi \circ \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi + \varphi) \mid (\varphi \vee \varphi) \mid \triangleright\varphi \mid \triangleleft\varphi,$$

where **Prop** is a non-terminal symbol that can be rewritten as any of the denumerably many propositional letters.²

REMARK 1. Occasionally, it is convenient to be able to refer to the connectives by names, which are somewhat mnemonic. We call \circ *fusion*, \rightarrow *implication*, $+$ *fission*, \vee *disjunction*, \triangleright *solid modality* and \triangleleft *fluid modality*. The latter two terms are chosen to keep the usual modal connotations at bay.

In the $L\mathfrak{X}^*$ logics, we want \circ and $+$ to be connectives that are commutative and associative. Then, it is felicitous to formulate the notion of sequents using multisets. In order to make this paper more or less self-contained (and to minimize the chance of terminological confusions), we include the definition of a multiset as well as an illustration of the concept.

Definition 2. A *multiset* is the set of finite sequences comprising the same elements that is closed under permutation.³

² We may use other letters than φ , from the latter part of the Greek alphabet, as variables for formulas.

³ In this paper, we only have use for *finite* multisets; thus, we use the term in a narrower sense than it is used elsewhere in the literature.

An essentially equivalent definition of multisets can be given as certain functions—see, for example, the definition of multisets in [10]. We are not interested in the reconstruction of sequences or multisets as sets here, and we take sets, multisets and sequences to be different ways of collecting objects together. Thus, informally speaking, a multiset is a finite set, in which the elements may appear more than once, hence, the number of listings matters.

EXAMPLE 2. If the multiset \mathbb{A} has two a 's and b 's as its elements, then we could list the elements of \mathbb{A} as a, a, b, b , or equivalently, as b, a, a, b , etc. Of course, $\langle a, a, b, b \rangle$ may be a different 4-tuple than $\langle b, a, a, b \rangle$ is, but a permutation transforms one into the other. On the other hand, $\{a, b\} = \{a, a, b, b\}$. The latter specification of a set is not only informal, but unnecessarily repetitive.

NOTATION 3. Obviously, we can describe a multiset by listing its elements. To distinguish an array from a multiset, we may enclose the latter into $[]$, and we use $;$ as the separator, because our multisets of formulas are associated to fusions or fissions of formulas. The letters $\alpha, \beta, \gamma, \dots$ range over multisets of formulas of the $L\mathfrak{X}^*$ logics. If φ is an element of α thrice, then we may say that the *type* φ is in α , and the *tokens* φ, φ and φ are in α .

Definition 3. If α and β are multisets, then $\alpha \cap \beta$ (the *intersection* of α and β) and $\alpha \uplus \beta$ (the *union* of α and β) are multisets. $\alpha \cap \beta$ has all the types that are in both α and β , and the number of tokens for each type is the lesser number of tokens of that type in the two. $\alpha \uplus \beta$ has all the types that are either in α or in β , and the number of tokens for a type is the sum of the number of tokens of that type in α and that in β .

We defined both \cap and \uplus to stress the lack of (informal) duality between them. \cap is min on the number of tokens, whereas \uplus is *not* max, rather $+$. For our purposes, \uplus is the important operation.

Definition 4. A *sequent* is an ordered pair of multisets of formulas. We write $\alpha \Vdash \beta$ instead of $\langle \alpha, \beta \rangle$. α is the *antecedent* and β is the *succedent* of the sequent.

NOTATION 4. The empty set is unique and so is the empty multiset, which we denote by \emptyset . However, when \emptyset appears in a sequent, we replace it with space. To formulate the rules of our calculi, we will use $\alpha; \varphi$ (or $\varphi; \alpha$) instead of $\alpha \uplus [\varphi]$. Similarly, $\alpha; \beta$ is a shorthand for $\alpha \uplus \beta$.

Definition 5. The $L\mathfrak{X}^*$ logics comprise axiom (I) and rules from among the following.

$$\begin{array}{c}
 \varphi \Vdash \varphi \quad (\text{I}) \\
 \\
 \frac{\alpha \Vdash \varphi; \beta}{\alpha \Vdash \varphi \vee \psi; \beta} \quad (\Vdash \vee_1) \qquad \frac{\alpha \Vdash \psi; \beta}{\alpha \Vdash \varphi \vee \psi; \beta} \quad (\Vdash \vee_2) \qquad \frac{\alpha; \psi \Vdash \beta \quad \alpha; \varphi \Vdash \beta}{\alpha; \varphi \vee \psi \Vdash \beta} \quad (\vee \Vdash) \\
 \\
 \frac{\alpha \Vdash \varphi; \beta \quad \gamma \Vdash \psi; \delta}{\alpha; \gamma \Vdash \psi \circ \varphi; \beta; \delta} \quad (\Vdash \circ) \qquad \frac{\alpha; \psi; \varphi \Vdash \beta}{\alpha; \varphi \circ \psi \Vdash \beta} \quad (\circ \Vdash)
 \end{array}$$

$$\begin{array}{c}
\frac{\alpha; \psi \Vdash \varphi; \beta}{\alpha \Vdash \psi \rightarrow \varphi; \beta} \quad (\Vdash \rightarrow) \qquad \frac{\alpha \Vdash \psi; \beta \quad \gamma; \varphi \Vdash \delta}{\alpha; \gamma; \psi \rightarrow \varphi \Vdash \beta; \delta} \quad (\rightarrow \Vdash) \\
\\
\frac{\alpha \Vdash \psi; \varphi; \beta}{\alpha \Vdash \varphi + \psi; \beta} \quad (\Vdash +) \qquad \frac{\alpha; \psi \Vdash \beta \quad \gamma; \varphi \Vdash \delta}{\alpha; \gamma; \varphi + \psi \Vdash \beta; \delta} \quad (+ \Vdash) \\
\\
\frac{\alpha^\triangleright \Vdash \varphi; \beta^\triangleleft}{\alpha^\triangleright \Vdash \triangleright \varphi; \beta^\triangleleft} \quad (\Vdash \triangleright) \qquad \frac{\alpha; \varphi \Vdash \beta}{\alpha; \triangleright \varphi \Vdash \beta} \quad (\triangleright \Vdash) \\
\\
\frac{\alpha \Vdash \varphi; \beta}{\alpha \Vdash \triangleleft \varphi; \beta} \quad (\Vdash \triangleleft) \qquad \frac{\alpha^\triangleright; \varphi \Vdash \beta^\triangleleft}{\alpha^\triangleright; \triangleleft \varphi \Vdash \beta^\triangleleft} \quad (\triangleleft \Vdash) \\
\\
\frac{\alpha \Vdash \psi; \psi; \beta}{\alpha \Vdash \psi; \beta} \quad (\Vdash W) \qquad \frac{\alpha; \varphi; \varphi \Vdash \beta}{\alpha; \varphi \Vdash \beta} \quad (W \Vdash) \\
\\
\frac{\alpha \Vdash \triangleleft \varphi; \triangleleft \varphi; \beta}{\alpha \Vdash \triangleleft \varphi; \beta} \quad (\Vdash \triangleleft W) \qquad \frac{\alpha; \triangleright \psi; \triangleright \psi \Vdash \beta}{\alpha; \triangleright \psi \Vdash \beta} \quad (\triangleright W \Vdash) \\
\\
\frac{\alpha \Vdash \beta}{\alpha \Vdash \triangleleft \varphi; \beta} \quad (\Vdash \triangleleft K) \qquad \frac{\alpha \Vdash \beta}{\alpha; \triangleright \psi \Vdash \beta} \quad (\triangleright K \Vdash) \\
\\
\frac{\alpha \Vdash \beta}{\alpha \Vdash \psi; \beta} \quad (\Vdash K) \qquad \frac{\alpha \Vdash \beta}{\alpha; \varphi \Vdash \beta} \quad (K \Vdash)
\end{array}$$

Superscript modalities such as α^\triangleleft and β^\triangleright indicate that for each token ψ in the multiset there is a formula φ such that ψ is $\triangleleft \varphi$ or $\triangleright \varphi$, respectively.

REMARK 5. The axiom is labeled with the identity combinator I . The contraction rules are labeled with w after the binary regular duplicator W , and the thinning rules are labeled with k after the binary regular cancellator K . Although these rules are not combinatory rules in the sense of [21], the analogy between structural rules and combinatory effects is profound. This correlation was observed and noted long ago (see, for example, Curry [17]).

If we keep all the operational rules fixed, then there are still plenty of logics that could be defined.⁴ However, the vast majority of those logics would be less than well motivated. We deem a handful of them worthy of interest.

Definition 6. The $L\mathcal{X}^*$ logics that we consider are defined by axiom (I) and the connective rules together with the structural rules with checkmarks as indicated in Table 1.⁵

⁴ A quick approximation suggests that there are 89 logics that can be expected to be distinct.

⁵ \times excludes a pair of rules; \star shows that the rules are easily derivable, hence, it is better to omit them—for the sake of economy in proofs.

Table 1. Structural rules in nine logics

| Rules | bci | bci [∇] | bci _Δ | bci _Δ [∇] | bciw | bck | bciw _Δ | bck [∇] | s4 |
|--------------|-----|------------------|------------------|-------------------------------|------|-----|-------------------|------------------|----|
| (W⊨), (⊨W) | × | × | × | × | ✓ | × | ✓ | × | ✓ |
| (▷W⊨), (⊨▷W) | × | ✓ | × | ✓ | ★ | × | ★ | ✓ | ★ |
| (▷K⊨), (⊨▷K) | × | × | ✓ | ✓ | × | ★ | ✓ | ★ | ★ |
| (K⊨), (⊨K) | × | × | × | × | × | ✓ | × | ✓ | ✓ |

NOTATION 6. The labels for the logics are intended to be somewhat reminiscent of but not identical to common abbreviations for certain logics. For example, the principal simple types of the combinators **B**, **C** and **I** are provable in **bci**. However, we included not only \rightarrow , but also \circ , $+$, \vee and the modalities \triangleright and \triangleleft (which are not in **BCI**). Likewise, **s4** differs from the logic **S4**.

Definition 7. A *proof* is a tree, in which the vertices are occurrences of sequents; the leaves are instances of (**I**), and a parent node is justified when that node and its children constitute an instance of a rule. The *root* of the proof tree is the sequent proved.

A formula φ is a *theorem* of an $L\mathfrak{X}^*$ logic iff $\Vdash \varphi$ has a proof.

Lemma 8. *The logic s4 is the negation-free fragment of the normal modal logic S4.*

Proof (sketch). From Sect. 2, we (will) know that the cut rule is admissible in **s4**, that is, **s4** is a well-formulated sequent calculus. We also assume that we know that **S4** can be formalized as an extension of the propositional part of **LK** from [23]. Namely, the two rules for \Box in (1) have to be added, and if \Diamond is a primitive too, then two more rules are included for \Diamond and the ($\Vdash\Box$) rule is modified by permitting a parametric set Δ^\Diamond on the right-hand side of the \Vdash .

The signature of **s4** differs from that of usual formulations of **S4**. In other words, we have to explain how to “translate” our formulas. In the presence of (**K⊨**), (**⊨K**), (**W⊨**) and (**⊨W**), $\Vdash (\varphi \vee \psi) \rightarrow (\varphi + \psi)$ and $\Vdash (\psi + \varphi) \rightarrow (\psi \vee \varphi)$ are provable. This means that $+$ is a notational variant of \vee . Also, \circ is idempotent and the following three sequents are provable: $\Vdash (\varphi \circ (\psi \vee \varphi)) \rightarrow \varphi$, $\Vdash \psi \rightarrow ((\psi \vee \varphi) \circ \psi)$, $\Vdash (\varphi \circ (\psi \vee \chi)) \rightarrow ((\varphi \circ \psi) \vee (\varphi \circ \chi))$. Implication is the residual of fusion, that is, \rightarrow behaves as \triangleright does in **LK**. This means that \rightarrow , \circ and $\vee/+$ are exactly like the positive fragment of classical propositional logic. Setting \triangleright to \Box and \triangleleft to \Diamond , the ($\triangleright\Vdash$), ($\Vdash\triangleright$), ($\triangleleft\Vdash$) and ($\Vdash\triangleleft$) rules are the rules for \Box and \Diamond . There are no other connectives unaccounted for in **s4**. \ddots

2 Cut Theorems

We formulated our nine $L\mathfrak{X}^*$ logics without the cut rule. However, this does not mean that we would want to neglect the cut rule, rather, the opposite. The

cut rule is extremely important for a proof that a sequent calculus defines an algebraizable logic, and that it is equivalent to an axiomatic system.

Definition 9. The *cut rule* is the following.

$$\frac{\alpha \Vdash \psi; \beta \quad \gamma; \psi \Vdash \delta}{\alpha; \gamma \Vdash \beta; \delta} \text{ (cut)}$$

Later we may refer to this cut rule as the *single cut*—to distinguish this rule from some other versions of cut. It is easy to see that the cut rule is not a derived rule in any $L\mathcal{X}^*$ logic. However, anything provable with cut is provable without the cut. This is the essence of Theorem 15.

Definition 10. The multiset of formulas in each rule is divided into three categories: principal, subaltern and parametric formulas. The *parametric formulas* are those in $\alpha, \alpha^\triangleright, \beta, \beta^\triangleright, \gamma$ and δ . In a proof (where the rules are instantiated with concrete sequents), any of these may be \emptyset . The *principal formulas* are the newly introduced formulas in the lower sequent of a rule, as well as, the displayed formulas in the lower sequent in the contraction rules. The *subalterns* are the formulas from which the principal formulas result—save in the thinning rules, where there are none. There is a 1–1 correspondence between the elements of multisets of parametric formulas bearing the same letter in a premise and in the conclusion, and we assume that a particular such bijection is fixed when needed.⁶

There is a range of terms and definitions used in the literature in proofs of cut theorems; hence, we briefly state the notions used in the proof of the next theorem.

Definition 11. A formula φ is an *ancestor* of ψ when it is in the transitive closure of the relation emerging from the above analysis through (i) and (ii).

- (i) A subaltern is an ancestor of the principal formula in a rule.
- (ii) A parametric formula in an upper sequent is an ancestor of its matching token in a lower sequent.⁷

For the next two definitions, we assume that we are given a proof, which may contain applications of the cut rule. We focus on a cut that has no cuts above it in that given proof.

Definition 12. The *left rank* of the cut is the maximal number of consecutive sequents above the left premise of the cut in which ancestors of the cut formula that are the same type as the cut formula occur in the succedent increased by 1. The *right rank* is the number calculated dually. The *rank of the cut* is the sum of the left and right ranks of the cut.

⁶ This analysis is fairly usual. For the ideas behind it and examples of it, we refer to [17] (and also to [9]).

⁷ This notion is an adaptation of a similar notion from Curry [17].

Definition 13. The *contraction measure* of the cut is the number of applications of contraction rules to ancestors of the cut formula that are the same type as the cut formula in the subproof rooted in the lower sequent of the application of the cut rule.

REMARK 7. The previous two definitions depend on the notion of ancestors, and they reflect Curry's insight that the subformula property allows tracking a formula to its origins within a proof. Then, the trace yields a tighter control over the proof itself.

Definition 14. The *degree* of a formula φ is denoted by $\mathfrak{d}(\varphi)$.

- (i) If $\varphi \in \text{Prop}$, then $\mathfrak{d}(\varphi) = 0$.
- (ii) If φ is $\odot\psi$ (where \odot is a unary connective), then $\mathfrak{d}(\varphi) = \mathfrak{d}(\psi) + 1$.
- (iii) If φ is $\psi\vee\varsigma$ (where \vee is a binary connective), then $\mathfrak{d}(\varphi) = \mathfrak{d}(\psi) + \mathfrak{d}(\varsigma) + 1$.

Theorem 15 (Cut theorem). In any $L\mathcal{X}^*$ logic, the cut rule is admissible.

Proof. The structure of the proof is fairly usual. A proof contains finitely many applications of the cut rule. If there is an application of the cut rule, then there is one that is at the top, in the sense that the subtree of the proof tree rooted in the lower sequent of the cut contains no other applications of the cut rule. We show that this subtree can be transformed into a proof tree with the same root but with no applications of the cut rule. Then, finitely many iterations of the argument replace the original proof tree with finitely many cuts with a proof tree (of the same sequent) with no applications of the cut rule.

The main part of the proof is by *triple induction* on the degree of the cut formula, on the contraction measure of the cut and on the rank of the cut. We cannot provide an exhaustive list of cases here; rather, we include two sample steps, and omit the remaining details.⁸

1. If modalities are introduced in the premises of the cut, then one of the cases goes as follows (and it is justified by $\mathfrak{d}(\psi)+1 = \mathfrak{d}(\odot\psi)$). (We omit \vdash everywhere; that is, the top sequents are not assumed to be axioms. The symbol " \rightsquigarrow " indicates the transformation on the proof tree.)

$$\begin{array}{c} \text{(cut)} \quad \frac{\frac{\alpha \Vdash \psi; \beta}{\alpha \Vdash \odot\psi; \beta} \quad \frac{\gamma^\triangleright; \psi \Vdash \delta^\triangleleft}{\gamma^\triangleright; \odot\psi \Vdash \delta^\triangleleft} \quad (\triangleleft \Vdash)}{\alpha; \gamma^\triangleright \Vdash \beta; \delta^\triangleleft} \quad \rightsquigarrow \quad \frac{\alpha \Vdash \psi; \beta \quad \gamma^\triangleright; \psi \Vdash \delta^\triangleleft}{\alpha; \gamma^\triangleright \Vdash \beta; \delta^\triangleleft} \quad (\text{cut}) \end{array}$$

2. The next sample step illustrates a reduction in the rank of the cut.

$$\begin{array}{c} \text{(cut)} \quad \frac{\frac{\epsilon \Vdash \chi; \eta \quad \frac{\alpha \Vdash \varphi; \beta \quad \gamma; \chi; \psi \Vdash \delta}{\alpha; \gamma; \varphi \rightarrow \psi; \chi \Vdash \beta; \delta} \quad (\rightarrow \Vdash)}{\alpha; \gamma; \epsilon; \varphi \rightarrow \psi \Vdash \beta; \delta; \eta} \quad \rightsquigarrow \end{array}$$

⁸ More details of a triple-inductive proof of the admissibility of the cut rule for a logic with no lattice operators may be found in [8]. Various enhancements of a more usual double-inductive proof of the cut theorem were introduced in [6, 7], where a goal was to accommodate constants like \mathbf{Y} , \mathbf{y} and \mathbf{t} .

$$\begin{array}{c}
\frac{\alpha \Vdash \varphi; \beta \quad \frac{\epsilon \Vdash \chi; \eta \quad \gamma; \psi; \chi \Vdash \delta}{\gamma; \epsilon; \psi \Vdash \delta; \eta} \text{ (cut)}}{\alpha; \gamma; \epsilon; \varphi \rightarrow \psi \Vdash \beta; \delta; \eta} \text{ } (\rightarrow \Vdash) \\
\vdots
\end{array}$$

The upshot of the theorem is that the $L\mathfrak{X}^*$ logics are reasonable logics (i.e., they are structural, in algebraic terminology). Also, we may focus on cut-free proofs without a loss of provable sequents.

Lemma 16. *Cut-free proofs in the $L\mathfrak{X}^*$ logics possess the subformula property. That is, if φ occurs (as a type) anywhere in a proof of $\alpha \Vdash \beta$, then φ is a subformula of a formula in α or in β .*

Proof. The $L\mathfrak{X}^*$ logics have no special zeroary connectives, hence, the claim follows by a simple inspection of the rules. (Cf. LE_{\rightarrow}^t in [7] for a more complicated situation.) We note that the contraction rules may reduce the number of tokens, but they do not omit types. \vdots

3 Decidability

The decidability of a logic may be proved in various ways. This is especially true for propositional modal logics, for which semantic methods have been used widely. Probably, the best-known semantic technique is *filtration* that relies on the relational semantics of normal modal logics, but *algebraic methods* have been successfully applied in some cases. It is not completely straightforward (or easy) to define set-theoretic semantics for the $L\mathfrak{X}^*$ logics. We cannot go into a detailed explanation of the reasons beyond mentioning that in the absence of conjunction, the usual set-theoretic objects—“theories” of some kind or another (or various sorts of filters, algebraically speaking)—are not available. In any case, we are interested here in the sequent calculus formulations of the $L\mathfrak{X}^*$ logics and the properties that we can discover using the sequent calculi.

Sequent calculi are *preeminently suitable* for proofs of decidability (starting with the proof of the decidability of propositional intuitionistic logic). Curry [16] came up with the idea of discarding the (explicit) contraction rules in lieu of repeating the principal formulas of the connective rules in the premises—together with a more relaxed form of the axiom $p \Vdash p$ (or $\varphi \Vdash \varphi$) by allowing other formulas in the axiom as in $\Gamma, \psi \Vdash \psi, \Delta$. Curry proved that the modifications (for the logics he considered) resulted in sequent calculi that proved the same sequents, moreover, the height of the proof tree did not increase. A lemma with a similar claim for a particular logic is often referred to as *Curry’s lemma* or as height-preserving admissibility of contraction. A decidability proof then proceeds in a bottom-up fashion, so to speak. In order to determine whether a sequent is provable, a complete proof-search tree is constructed, which in fact explores all the possibilities as to how the sequent could have been proved. While the search is exhaustive (perhaps, in more than one sense:), its finiteness is guaranteed by the limitations that the cut theorem and Curry’s lemma impose (together with

an easy use of *Kőnig's lemma*). It is sufficient to look for cut-free proofs, and there is no need to seek proofs that are redundant in a sense stemming from Curry's lemma.

Taking $\Gamma, \psi \Vdash \psi, \Delta$ as an axiom has the effect of turning thinning into an admissible rule too. This is not acceptable from the point of view of many logics—from the Lambek calculi to relevance logics.⁹ Kripke [26] introduced another idea, namely, instead of *requiring* the principal formulas to be parametric in the premises, he *permits* them to be parametric. Of course, this idea is compatible with thinning as a rule, but what is really intriguing about it is that, when thinning is excluded, it still renders contraction admissible.

If thinning is not a rule, then Kripke's invention is an indispensable component of the bottom-up proof search. It reflects the insight that a formula has to be introduced in order to be contracted, hence, a limited amount of contraction in the operational rules is sufficient in place of an explicit contraction rule.

To guarantee the finiteness of the proof-search tree, Kripke introduced a lemma, which, nowadays, is called *Kripke's lemma*. Originally, this lemma is about cognate sequents, and an excellent presentation is in Dunn [19, Sect. 3.6]. In the $L\mathfrak{X}^*$ logics, a pair of sequents are cognate if their antecedent and succedent multisets comprise the same types. However, later on, it was discovered that Kripke's lemma is equivalent to various other lemmas (see [19, 31]). For example, a lemma concerning vectors is stated and proved by induction in Kopylov [25, Lemma 2.2], which also appears to be equipotent to Kripke's lemma.

REMARK 8. Here is a number-theoretic analog of Kripke's lemma that is easy to state; the claim itself is self-evident.¹⁰ Let us consider the positive integers. If we fix P , a finite set of primes, then there are finitely many numbers such that they have no other prime factors (than those in P), and they pairwise do not divide each other. For instance, if we start with $\{3\}$, then we could pick 27, but then 1, 3 and 9 are excluded (because $3 \mid 27$ and $9 \mid 27$). We can add to our collection 81 and 243, but 729 is excluded (because $27 \mid 729$), and so is any higher power of 3. The example is intended to be simple, but the case of one prime factor generalizes to the case of n prime factors without any difficulty.

NOTE 9. Before we embark on proofs of decidability for our $L\mathfrak{X}^*$ logics, it seems prudent to point out that some of our logics (possibly, in a slightly different formulation) and some closely related logics are already known to be decidable. For instance, Meyer [30] proved LR^\square decidable, which is in close proximity to bcw . Linear affine logic was proved decidable in Kopylov [25], which implies the decidability of bck^∇ . The logic that was proved decidable in Bimbó [8] is orthogonal to bci_Δ^∇ , because it has \neg but lacks \vee . For further relevant results, see [14, 31].

⁹ See, in chronological order, [28], [1], [22], [11], as well as [7] for motivations and logics that leave out the thinning rules from their sequent calculus formulations.

¹⁰ See Meyer [31] for a discussion of conceptual links that can be created between Dickson's lemma and Kripke's lemma.

Definition 17. We partition the $L\mathcal{X}^*$ group into *three subgroups*: $L\mathcal{X}_1^* = \{\text{bci}, \text{bci}_\Delta, \text{bck}\}$, $L\mathcal{X}_2^* = \{\text{bciw}, \text{bciw}_\Delta, \text{s4}\}$ and $L\mathcal{X}_3^* = \{\text{bci}^\nabla, \text{bci}_\Delta^\nabla, \text{bck}^\nabla\}$.

REMARK 10. The rationale behind the division is that we approach the question of decidability similarly for the members of the subgroups, but with some differences between the subgroups. In $L\mathcal{X}_1^*$, there is no contraction, which means that Curry’s bottom-up proof search suffices. The $L\mathcal{X}_2^*$ logics contain the $(w\mathbb{E})$ and $(\mathbb{E}w)$ rules, and we follow Kripke’s approach. For the $L\mathcal{X}_3^*$ logics, we enhance the Curry–Kripke technique with a new proof search bounded by heap numbers.

We will deal with $L\mathcal{X}_2^*$ first, where the Curry–Kripke technique is applicable.

Definition 18. The logics (bciw) , (bciw_Δ) and (s4) are defined by the axiom (I) and the following connective rules together with the thinning rules from the matching unbracketed logics. (The $(K\mathbb{E})$, $(\mathbb{E}K)$, $(\triangleright K\mathbb{E})$ and $(\mathbb{E}\triangleleft K)$ rules are unchanged, that is, they are exactly as in Definition 5. We do not repeat those rules here, though (bciw_Δ) and (s4) contain some of them.)

$$\begin{array}{c}
\frac{\alpha \Vdash \varphi; \beta}{\alpha \Vdash (\varphi \vee \psi; \beta)} \quad (\mathbb{E}\vee_1) \qquad \frac{\alpha \Vdash \psi; \beta}{\alpha \Vdash (\varphi \vee \psi; \beta)} \quad (\mathbb{E}\vee_2) \qquad \frac{\alpha; \psi \Vdash \beta \quad \alpha; \varphi \Vdash \beta}{(\alpha; \varphi \vee \psi) \Vdash \beta} \quad (\vee\mathbb{E}) \\
\\
\frac{\alpha \Vdash \varphi; \beta \quad \gamma \Vdash \psi; \delta}{(\alpha; \gamma) \Vdash (\psi \circ \varphi; \beta; \delta)} \quad (\mathbb{E}\circ) \qquad \frac{\alpha; \psi; \varphi \Vdash \beta}{(\alpha; \varphi \circ \psi) \Vdash \beta} \quad (\circ\mathbb{E}) \\
\\
\frac{\alpha; \psi \Vdash \varphi; \beta}{\alpha \Vdash (\psi \rightarrow \varphi; \beta)} \quad (\mathbb{E}\rightarrow) \qquad \frac{\alpha \Vdash \psi; \beta \quad \gamma; \varphi \Vdash \delta}{(\alpha; \gamma; \psi \rightarrow \varphi) \Vdash (\beta; \delta)} \quad (\rightarrow\mathbb{E}) \\
\\
\frac{\alpha \Vdash \psi; \varphi; \beta}{\alpha \Vdash (\varphi + \psi; \beta)} \quad (\mathbb{E}+) \qquad \frac{\alpha; \psi \Vdash \beta \quad \gamma; \varphi \Vdash \delta}{(\alpha; \gamma; \varphi + \psi) \Vdash (\beta; \delta)} \quad (+\mathbb{E}) \\
\\
\frac{\alpha^\triangleright \Vdash \varphi; \beta^\triangleleft}{\alpha^\triangleright \Vdash \triangleright \varphi; \beta^\triangleleft} \quad (\mathbb{E}\triangleright) \qquad \frac{\alpha; \varphi \Vdash \beta}{(\alpha; \triangleright \varphi) \Vdash \beta} \quad (\triangleright\mathbb{E}) \\
\\
\frac{\alpha \Vdash \varphi; \beta}{\alpha \Vdash (\triangleleft \varphi; \beta)} \quad (\mathbb{E}\triangleleft) \qquad \frac{\alpha^\triangleright; \varphi \Vdash \beta^\triangleleft}{\alpha^\triangleright; \triangleleft \varphi \Vdash \beta^\triangleleft} \quad (\triangleleft\mathbb{E})
\end{array}$$

The (\quad) notation indicates potential contractions to the following extent.

- (1) The principal formula ψ occurs in a multiset of parametric formulas α .
Then: $(\psi; \alpha)$ is either $\psi; \alpha$ or α .
- (2) A formula ψ occurs in both multisets of parametric formulas α and β .
Then: $(\alpha; \beta)$ is either $\alpha; \beta$ or $\alpha; \beta$ with an occurrence of ψ omitted.
- (3) The principal formula ψ occurs in both multisets of parametric formulas α and β .
Then: $(\psi; \alpha; \beta)$ is $\psi; \alpha; \beta$ or $\psi; \alpha; \beta$ with one or two occurrences of ψ omitted; in each case the parametric formulas are dealt with as in (2).

REMARK 11. We should emphasize that no contractions are mandatory within (\downarrow) , and whatever contractions are performed, they never lead to a loss of a type from a multiset. Sequents are finite, hence, each application of an operational rule involves finitely many contractions. However, the number of possible contractions depends on the size and shape of the premises to which a rule is applied, not simply on what the rule is.

The operational rules above *do not* introduce vagueness or indeterminacy into the concept of a proof, because in any proof, which comprises concrete sequents, the number of contractions can be determined simply by counting formulas.

REMARK 12. There are no structural rules listed in the previous definition. Contractions are omitted, because the goal is to limit the number of contractions, so that only useful contractions are considered. Thinnings are omitted from the listing, because if the application of a thinning rule would create a sequent where contraction is applicable, then the applications of that thinning rule can be retracted. But we reiterate that if some sort of thinning was in an $L\mathfrak{X}_2^*$ logic (as per Definition 6), then the same rule is in the (\downarrow) 'd version of the logic.

Note also that in the operational rules $(\models \triangleright)$ and $(\triangleright \models)$, no contractions are permitted (or possible). The principal formulas of those rules are always distinct from all the types in the multiset of parametric formulas with which they are joined.

We defined three new sequent calculi; therefore, we have to provide a cut theorem for them. (Of course, the labels for the logics express our aim of defining the same logics as before. However, we will *know* that we have reached that goal after the next two theorems.)

Definition 19. The *left rung* of the cut is the length of the longest path in the proof tree starting with the left premise of the cut in which the cut formula occurs in the succedent of each sequent on the path. The *right rung* of the cut is the length of the longest path in the proof tree starting with the right premise of the cut in which the cut formula occurs in the antecedent of each sequent on the path. The *rung of the cut* is the sum of the left and right rungs of the cut.

The notion of a rung (if not the term itself) is a parameter that is often used in proofs of cut theorems.

REMARK 13. In the proof of Theorem 15, we used the single cut rule.¹¹ However, the admissibility of the single cut is typically proved via a detour through other forms of the cut in calculi that include contraction in some form. This is so in the calculi that are designed to prove decidability using the Curry–Kripke technique.¹²

¹¹ The cut theorem is proved using the single cut in Lambek [28] and in display logics in Belnap [3] and Anderson et al. [2].

¹² The so-called *mix* rule in [23] and the *multicut* rule explicitly stated, for example, in Dunn [18] are versions of the cut that were introduced specifically to facilitate the inductive proof of the cut theorem for the single cut. An early publication that exhibits a suitable version of cut in connection to a decidability proof using the Curry–Kripke method is [4], which is a precursor of the more readily available [5].

The cut rule used in the proof of the next theorem builds in contraction, and it is formulated as

$$\frac{\alpha \Vdash \psi; \beta \quad \gamma; \psi \Vdash \delta}{(\alpha; \gamma) \Vdash (\beta; \delta)} \text{ (cut)}.$$

It is obvious that the *single cut rule* is a special instance of this rule.

Theorem 20 (Cut theorem). *The single cut rule is admissible in the three logics (bciw) , (bciw_Δ) and (s4) .*

Proof. The strategy is once again to eliminate a cut with no cut above it. The proof is by double induction on the degree of the cut formula and on the rung of the application of the cut rule. Once again, we can only include here a couple of steps as illustrations to convey the flavor of the proof.

1. Let us consider a case for \vee . The degree of a disjunction is strictly greater than the degree of the disjuncts, that is, $\mathfrak{d}(\psi \vee \varphi) = 1 + \mathfrak{d}(\psi) + \mathfrak{d}(\varphi)$.

$$\frac{\begin{array}{c} (\Vdash \vee) \quad \frac{\alpha \Vdash \varphi; \beta}{\alpha \Vdash (\psi \vee \varphi; \beta)} \\ (\text{cut}) \end{array} \quad \frac{\gamma; \psi \Vdash \delta \quad \gamma; \varphi \Vdash \delta}{(\gamma; \psi \vee \varphi) \Vdash \delta} \quad (\Vdash \vee)}{(\alpha; \gamma) \Vdash (\beta; \delta)} \quad (\text{cut}) \quad \rightsquigarrow \quad \frac{\alpha \Vdash \varphi; \beta \quad \gamma; \varphi \Vdash \delta}{(\alpha; \gamma) \Vdash (\beta; \delta)} \quad (\text{cut})$$

2. If the cut formula is parametric in the left premise, then that premise might have resulted by $(+\Vdash)$.

$$\begin{array}{c} (\Vdash +) \quad \frac{\alpha; \psi \Vdash \beta \quad \gamma; \varphi \Vdash \chi; \delta}{(\alpha; \gamma; \varphi + \psi) \Vdash (\chi; \beta; \delta)} \\ (\text{cut}) \end{array} \quad \frac{\epsilon; \chi \Vdash \eta}{(\alpha; \gamma; \epsilon; \varphi + \psi) \Vdash (\beta; \delta; \eta)} \quad \rightsquigarrow \quad \frac{\alpha; \psi \Vdash \beta \quad \frac{\gamma; \varphi \Vdash \chi; \delta \quad \epsilon; \chi \Vdash \eta}{(\gamma; \epsilon; \varphi) \Vdash (\delta; \eta)} \quad (\text{cut})}{(\alpha; \gamma; \epsilon; \varphi + \psi) \Vdash (\beta; \delta; \eta)} \quad (\Vdash +) \quad \therefore$$

Theorem 21 (Curry's lemma). *Let $\alpha' \Vdash \beta'$ be a sequent that results in bciw from $\alpha \Vdash \beta$ by finitely many applications of the $(\text{w}\Vdash)$ and $(\Vdash \text{w})$ rules. If \mathfrak{T} is a proof tree with height h of a sequent $\alpha \Vdash \beta$ in (bciw) , then there is proof tree \mathfrak{T}' of the sequent $\alpha' \Vdash \beta'$ with height h' such that $h' \leq h$. Similarly, for the two other pairs of logics: bciw_Δ and (bciw_Δ) , s4 and (s4) .*

Proof. Both parts of the claim will be important for the decidability proofs later on. The admissibility of the two contraction rules ensures that no provable sequents are lost in moving to the (\Vdash) 'd logics. Both in the inductive proof of this claim and in the proof search it is crucial that sequents that would result by the $(\text{w}\Vdash)$ and $(\Vdash \text{w})$ rules have shorter proofs than the longer sequents (from which they are obtained) have. The proof is by induction on h , the height of the given proof tree. (Once again, we omit almost all cases due to lack of space.)

1. If $\alpha \models \beta$ is an instance of (I), then $\alpha' \models \beta'$ is $\alpha \models \beta$; hence, the claim is obviously true.
2. We will abbreviate n tokens of φ in a multiset by φ^n (assuming $n \geq 1$). Given $n, n' \leq n$ and $n' \geq 1$. Let us consider the $(\models \circ)$ rule.

$$\frac{\alpha; \chi^n \models \varphi; (\varphi \circ \psi)^m; \varsigma^i; \beta \quad \gamma; \chi^j \models \psi; (\varphi \circ \psi)^k; \varsigma^l; \delta \quad \text{i.h.}}{\alpha; \gamma; \chi^{(n+j)'} \models (\varphi \circ \psi)^{(m+k)'}; \varsigma^{(i+l)'}; \beta; \delta}$$

$$\frac{\alpha; \chi^{n'} \models \varphi; (\varphi \circ \psi)^{m'}; \varsigma^{i'}; \beta \quad \gamma; \chi^{j'} \models \psi; (\varphi \circ \psi)^{k'}; \varsigma^{l'}; \delta}{\alpha; \gamma; \chi^{(n'+j')'} \models (\varphi \circ \psi)^{(m'+k')'}; \varsigma^{(i'+l')'}; \beta; \delta}$$

It is easy to see that if we want each of χ and ς to have at least 2 occurrences in the lower sequent, then $n' + j'$ and $i' + l'$ (i.e., applications of the hypothesis of the induction) suffice. Similarly, for 3 occurrences for $\varphi \circ \psi$. However, if $(n + j)' = 1$, $(m + k)' = 1$ or 2, or $(i + l)' = 1$, then we have the upper sequents by the hypothesis of the induction, and the contractions that are part of the $(\models \circ)$ rule yield the desired lower sequent. Here is the most contracted situation, in which the premises are available to us by inductive hypothesis.

$$\frac{\alpha; \chi \models \varphi; \varphi \circ \psi; \varsigma; \beta \quad \gamma; \chi \models \psi; \varphi \circ \psi; \varsigma; \delta}{\alpha; \gamma; \chi \models \varphi \circ \psi; \varsigma; \beta; \delta} \quad (\models \circ)$$

3. Let us consider an extensional rule too, namely, $(\models \vee)$.

$$\frac{\alpha; \chi^n; (\psi \vee \varphi)^m; \varphi \models \varsigma^i; \beta \quad \alpha; \chi^n; (\psi \vee \varphi)^m; \psi \models \varsigma^i; \beta \quad \text{i.h.}}{\alpha; \chi^{n'}; (\psi \vee \varphi)^{m'} \models \varsigma^{i'}; \beta}$$

$$\frac{\alpha; \chi^{n'}; (\psi \vee \varphi)^{m'}; \varphi \models \varsigma^{i'}; \beta \quad \alpha; \chi^{n'}; (\psi \vee \varphi)^{m'}; \psi \models \varsigma^{i'}; \beta}{\alpha; \chi^{n'}; (\psi \vee \varphi)^{m'} \models \varsigma^{i'}; \beta} \quad \therefore$$

Definition 22. A sequence of sequents is *irredundant* when an earlier element of the sequence is not obtainable from a latter one by finitely many applications of the contraction rules. We expand the use of the term “irredundant” to proofs. An *irredundant proof* contains no redundant sequences of sequents.

REMARK 14. The notion of irredundant sequences of sequents is in harmony with Curry’s lemma. Looking at a proof tree from its root upward, an irredundant sequence of sequents on a path in the proof tree signals an unnecessary detour in the proof.

Lemma 23 (Kripke’s lemma). *An irredundant sequence of cognate sequents is finite.*

As we already mentioned, this lemma is equivalent to various other lemmas in discrete mathematics. For a *direct proof*, we refer to Anderson and Belnap [1, Sect. 13, p. 139].

Lemma 24 (König’s lemma). *A finitely branching tree, in which all branches are finite, is finite.*

This is also a well-known lemma. For a *direct proof*, we refer to Smullyan [32].

Theorem 25. *The logics (bciw) , (bciw_Δ) and (s4) are decidable.*

Proof. The decision procedure builds a proof-search tree for the given sequent, with the property that if the sequent has a proof, then a subtree of the proof-search tree is a proof. The usual way to do this is to build the tree from its root, which is the sequent that is allegedly provable. A branch may be terminated when it would become redundant. The finiteness of the tree guarantees that an unsuccessful search will not run on indefinitely long.

The finiteness of the tree follows from several factors. Formulas and sequents are finite, with each formula having finitely many subformulas. Each rule has one or two premises, and no sequent can result from infinitely many different potential premises. These features combined with the previous two lemmas exclude infinite trees from consideration. \therefore

Corollary 26. *The logics bciw , bciw_Δ and s4 are decidable.*

Proof. The truth of the claim is a consequence of the equivalence of the logics with and without (\quad) . \therefore

NOTE 15. The decidability of s4 is also a consequence of the decidability of S4 (which is widely known) in view of Lemma 8.

Now we turn to the question of the decidability in the subgroup $L\mathfrak{X}_3^*$.

Definition 27. The logics in $L\mathfrak{X}_2^*$ and $L\mathfrak{X}_3^*$ are paired up with each other as follows: $(\text{bciw}, \text{bci}^\nabla)$, $(\text{bciw}_\Delta, \text{bci}_\Delta^\nabla)$ and $(\text{s4}, \text{bck}^\nabla)$.

Lemma 28. *If $\alpha \Vdash \beta$ is provable in an $L\mathfrak{X}_3^*$ logic, then $\alpha \Vdash \beta$ is provable in its $L\mathfrak{X}_2^*$ pair.*

Lemma 29. *If $\alpha \Vdash \beta$ is provable in bci^∇ , bci_Δ^∇ or bck^∇ , then it is provable in (bciw) , (bciw_Δ) or (s4) , respectively, by irredundant proofs.*

Proof (of Lemmas 28 and 29). It is sufficient to scrutinize the definitions of the logics together with the proof of Theorem 25. \therefore

REMARK 16. In all the calculi that we consider, the cut rule is admissible. Then, it is enough to look for cut-free proofs, for which the subformula property holds. For a formula to be contracted, it must be introduced by an axiom or rule into the proof. Compound subformulas have more than one subformula, hence, a contraction applied to a compound formula decreases the number of subformulas more than a contraction applied to one of their proper subformulas. Furthermore, a formula to which no contraction is applied remains in the sequent (possibly, as a subformula of a formula), because the subformula property holds. These observations motivate the introduction of the notion of a heap number, which is a cumulation of contractions on subformulas of a formula in a proof.

Definition 30. Let $\alpha \models \beta$ be provable in an $L\mathfrak{X}_2^*$ logic. For any subformula φ of a formula in $\alpha \sqcup \beta$, we define the *heap number* of φ , denoted by $h^\#(\varphi)$ as follows.

- (1) If φ is not of the form $\triangleright\psi$ or $\triangleleft\psi$ for some ψ , then $h^\#(\varphi) = 0$;
- (2) otherwise, $h^\#(\varphi)$ is the maximal number of contractions on φ and the ancestors of φ in any irredundant proof of $\alpha \models \beta$ in the $(\sqcup \sqcap)$ 'd $L\mathfrak{X}_2^*$ logic in question.

REMARK 17. Given a provable sequent of an $L\mathfrak{X}_2^*$ logic, we may think of all the subformulas having a number attached to them. We know that all the sequents that are provable in their $L\mathfrak{X}_3^*$ pair are among those. However, it is easy to prove that not all sequents provable in an $L\mathfrak{X}_2^*$ logic are provable in their $L\mathfrak{X}_3^*$ pair. Since contractions in the $L\mathfrak{X}_2^*$ logics are possible only on modalized formulas, we transfer all the contractions that might have happened on ancestors of a modalized formula in any irredundant proof to the formula itself.

REMARK 18. We want to emphasize that the definition of a heap number is not recursive. We simply zeroed the heap number for all non-modal formulas, whether they are or are not a subformula of a formula in the provable sequent.

For any provable sequent, there are finitely many irredundant proofs each of which is finite. Hence, the heap number requires the inspection of finitely many finite objects. As we mentioned in Remark 11, in a proof involving applications of the $(\sqcup \sqcap)$ rules, the number of contractions can be simply counted, and there is no ambiguity with respect to which formulas and how many times were contracted. In sum, the notion of a heap number is *well defined*.

Theorem 31. *The logics bci^∇ , bci_Δ^∇ and bck^∇ are decidable.*

Proof. Let $\alpha \models \beta$ be a given sequent. For any of the $L\mathfrak{X}_3^*$ logics, we can decide, by appeal to Lemma 29, whether the sequent is provable in the $L\mathfrak{X}_2^*$ pair of our $L\mathfrak{X}_3^*$ logic. If the sequent is not provable, then we may conclude that it is not provable in the $L\mathfrak{X}_3^*$ logic either.

If the sequent is provable in the $L\mathfrak{X}_2^*$ pair of our logic, then we start a new proof search using the $L\mathfrak{X}_3^*$ logic itself. The only contraction rules are $(\triangleright w \models)$ and $(\models \triangleleft w)$. We start to build a proof-search tree as usual, and for each modalized formula we limit the number of the applications of the previous two rules by the heap number for the principal formula of the rule.¹³

The proof-search tree is finite. The connective rules—looked at from the lower sequent upward—reduce the number of connectives in the sequent. So do

¹³ We defined heap numbers in a very liberal manner in order to make sure that all the necessary contractions are permitted. However, even if $h^\#(\triangleleft\varphi) > 1$, for example, it may happen that in the $L\mathfrak{X}_3^*$ logic no contraction will be applied to the formula, because it occurs on the left-hand side of the \models . (Similarly, but dually for $\triangleright\varphi$.) This does not cause any problem in the proof search, because the heap number (like the $(\sqcup \sqcap)$ notation) does not force contractions, rather, it places a limit on the number of potential applications of the contraction rules.

typically the thinning rules. The number of applications of the ($\triangleright\text{W}\Vdash$) and the ($\Vdash\triangleleft\text{W}$) rules is bounded, and there are finitely many modal formulas to start with.

The proof-search tree will contain a proof if there is one. As usual, we assume that the proof-search tree is comprehensive, that is, all the possible upper sequents are added to the tree. This guarantees—as usual—that no potential proof step is missed. We only have to scrutinize whether we have permitted all the needed applications of the ($\triangleright\text{W}\Vdash$) and ($\Vdash\triangleleft\text{W}$) rules. Let us assume that more than heap number-many contractions (i.e., some extra contractions) are required to prove a sequent. The principal formula cannot be by thinning, because the latter could be simply omitted (contradicting the necessity for extra contractions). If the principal formula is by the axiom (I) or a connective rule, then all the atomic subformulas have another occurrence introduced (possibly, on the other side of the \Vdash). If those occurrences are contracted, then the extra contractions are not necessary. If they remain in the provable sequent, then the extra contractions must have been applied in some irredundant proof; hence, they must have been counted in the heap number contradicting the starting assumption.∴

Lastly, we deal with the subgroup $L\mathfrak{X}_1^*$.

Theorem 32. *The logics bci , bci_Δ and bck are decidable.*

Proof. The proof is a simple proof-search. None of the calculi contains a contraction rule, hence, the finiteness of the sequents, of the set of subformulas of a formula and König’s lemma together guarantee the finiteness of the proof-search tree. ∴

4 Conclusions

We have selected 9 modal logics, each of which is definable as an extension of a core logic bci that includes disjunction and an implication (with two more intensional connectives), and a pair of modalities \triangleright and \triangleleft . We gave a systematic presentation of these logics as sequent calculi. From the point of view of proving their decidability, the $L\mathfrak{X}^*$ logics fall into three groups. Curry’s bottom-up approach is applicable to the $L\mathfrak{X}_1^*$ group. Kripke’s refinement delivers decidability for the $L\mathfrak{X}_2^*$ group. Finally, the concept of a heap number together with the decidability of the $L\mathfrak{X}_2^*$ logics yields the decidability of the $L\mathfrak{X}_3^*$ logics. To summarize, each of our 9 modal logics turns out to be *decidable*.

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References

1. Anderson, A.R., Belnap, N.D.: Entailment: The Logic of Relevance and Necessity, vol. I. Princeton University Press, Princeton (1975)
2. Anderson, A.R., Belnap, N.D., Dunn, J.M.: Entailment: The Logic of Relevance and Necessity, vol. II. Princeton University Press, Princeton (1992)
3. Belnap, N.D.: Display logic. *J. Philos. Logic* **11**, 375–417 (1982)
4. Belnap, N.D., Wallace, J.R.: A decision procedure for the system $E_{\bar{T}}$ of entailment with negation. Technical report 11, Contract No. SAR/609 (16), Office of Naval Research, New Haven (1961)
5. Belnap, N.D., Wallace, J.R.: A decision procedure for the system $E_{\bar{T}}$ of entailment with negation. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* **11**, 277–289 (1965)
6. Bimbó, K.: Admissibility of cut in LC with fixed point combinator. *Stud. Logica* **81**, 399–423 (2005). doi:[10.1007/s11225-005-4651-y](https://doi.org/10.1007/s11225-005-4651-y)
7. Bimbó, K.: LE_{\rightarrow}^t , LR_{Δ}^o , LK and cutfree proofs. *J. Philos. Logic* **36**, 557–570 (2007). doi:[10.1007/s10992-007-9048-0](https://doi.org/10.1007/s10992-007-9048-0)
8. Bimbó, K.: The decidability of the intensional fragment of classical linear logic. *Theoret. Comput. Sci.* **597**, 1–17 (2015). doi:[10.1016/j.tcs.2015.06.019](https://doi.org/10.1016/j.tcs.2015.06.019)
9. Bimbó, K.: Proof Theory: Sequent Calculi and Related Formalisms. *Discrete Mathematics and Its Applications*. CRC Press, Boca Raton (2015). doi:[10.1201/b17294](https://doi.org/10.1201/b17294)
10. Bimbó, K., Dunn, J.M.: Generalized Galois Logics: Relational Semantics of Non-classical Logical Calculi. *CSLI Lecture Notes*, vol. 188. CSLI Publications, Stanford (2008)
11. Bimbó, K., Dunn, J.M.: Calculi for symmetric generalized Galois logics. In: van Benthem, J., Moortgat, M. (eds.) *Festschrift for Joachim Lambek*. *Linguistic Analysis*, vol. 36, pp. 307–343. Linguistic Analysis, Vashon (2010)
12. Bimbó, K., Dunn, J.M.: New consecution calculi for R_{\rightarrow}^t . *Notre Dame J. Formal Logic* **53**(4), 491–509 (2012). doi:[10.1215/00294527-1722719](https://doi.org/10.1215/00294527-1722719)
13. Bimbó, K., Dunn, J.M.: On the decidability of implicational ticket entailment. *J. Symb. Logic* **78**(1), 214–236 (2013). doi:[10.2178/jsl.7801150](https://doi.org/10.2178/jsl.7801150)
14. Bimbó, K., Dunn, J.M.: Modalities in lattice- R (2015). (manuscript, 34 pages)
15. Cresswell, M., Mares, E., Rini, A. (eds.): *Logical Modalities from Aristotle to Carnap. The Story of Necessity*. Cambridge University Press, Cambridge (2016)
16. Curry, H.B.: A Theory of Formal Deducibility. No. 6 in *Notre Dame Mathematical Lectures*. University of Notre Dame Press, Notre Dame (1950)
17. Curry, H.B.: *Foundations of Mathematical Logic*. McGraw-Hill Book Company, New York (1963). (Dover, New York, 1977)
18. Dunn, J.M.: A ‘Gentzen system’ for positive relevant implication (abstract). *J. Symb. Logic* **38**, 356–357 (1973)
19. Dunn, J.M.: Relevance logic and entailment. In: Gabbay, D., Guenther, F. (eds.) *Handbook of Philosophical Logic*, vol. 3, 1st edn, pp. 117–224. D. Reidel, Dordrecht (1986)
20. Dunn, J.M.: Positive modal logic. *Stud. Logica* **55**, 301–317 (1995)
21. Dunn, J.M., Meyer, R.K.: Combinators and structurally free logic. *Logic J. IGPL* **5**, 505–537 (1997)
22. Dunn, J.M., Restall, G.: Relevance logic. In: Gabbay, D., Guenther, F. (eds.) *Handbook of Philosophical Logic*, vol. 6, 2nd edn, pp. 1–128. Kluwer, Amsterdam (2002)

23. Gentzen, G.: Untersuchungen über das logische Schließen. *Mathematische Zeitschrift* **39**, 176–210 (1935)
24. Gödel, K.: Eine Interpretation des intuitionistischen Aussagenkalküls. In: Feferman, S. (ed.) *Collected Works*, vol. I, pp. 300–303. Oxford University Press and Clarendon Press, New York and Oxford (1986)
25. Kopylov, A.P.: Decidability of linear affine logic. In: Meyer, A.R. (ed.) *Special issue: LICS 1995, Information and Computation*, vol. 164, pp. 173–198. IEEE (2001)
26. Kripke, S.A.: The problem of entailment (abstract). *J. Symb. Logic* **24**, 324 (1959)
27. Kripke, S.A.: Semantical analysis of modal logic I. Normal modal propositional calculi. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, pp. 67–96 (1963)
28. Lambek, J.: The mathematics of sentence structure. *Am. Math. Mon.* **65**, 154–169 (1958)
29. Lewis, C.I.: *A Survey of Symbolic Logic*. University of California Press, Berkeley (1918). (Dover Publications, Mineola, 1960)
30. Meyer, R.K.: *Topics in modal and many-valued logic*. Ph.D. thesis, University of Pittsburgh, Ann Arbor (UMI) (1966)
31. Meyer, R.K.: Improved decision procedures for pure relevant logic. In: Anderson, C.A., Zelény, M. (eds.) *Logic, Meaning and Computation: Essays in Memory of Alonzo Church*, pp. 191–217. Kluwer Academic Publishers, Dordrecht (2001)
32. Smullyan, R.M.: *First-Order Logic*. Springer, New York (1968). doi:[10.1007/978-3-642-86718-7](https://doi.org/10.1007/978-3-642-86718-7). (Dover, New York 1995)