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Communication

Characterization of grid graphs

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Abstract

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In this paper we are mainly interested in the characterization of grid graphs i.e. products of paths.

Introduction

A graph is called an (n-dimensional) p_1–p_2–$\cdots$–p_n-grid if it is the product of n paths $P_{p_1}, P_{p_2}, \cdots, P_{p_n}$. Those graphs are of special interest because they can be used for practical implementation of parallel algorithms. Fig. 1 shows the 3–4 grid $P_3 \Box P_4$.

A cycle $C \leqslant G$ is called a *gap* iff it is an isometric subgraph of G, i.e. for any two of its vertices x, y their distances $d_C(x, y)$ and $d_G(x, y)$ are equal.

Let $\mathcal{I}_n(G)$ denote the set of the different subgraphs induced (up to isomorphism) in G by intervals of length $\leqslant n$.

2-dimensional grids

Theorem 1. *A simple connected graph G is a 2-dimensional grid or a tree iff*

(i) $\mathcal{I}_3(G) \subseteq$ ,

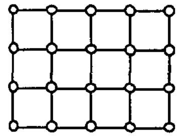
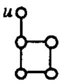
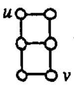
$P3\square P4$ = 

Fig. 1.

(ii) *Any edge in G belongs to at most two 4-cycles in G,*
(iii) *In G are no other gaps G than 4-cycles.*

Sketch of the proof. It starts with easy observations and a lemma:

(a)  $\subseteq G \Rightarrow d_G(u, v) = 3$ *and*  $\subseteq G$ *as induced subgraph*;

(b) $K_{2,3} \nsubseteq G$;

(c) *every edge of G is contained in at least one 4-cycle*;

(d) $\forall x \in V(G) \quad d_G(x) \leqslant 4$.

Lemma. *If G is a connected graph satisfying* (i), (ii) *and* (iii) *then for any cycle $C \subseteq G$ there exist adjacent edges e, $e' \subseteq C$ and edges e'', $e''' \in E(G)$ such that e, e', e'', e''' form a 4-cycle in G.*

The proof of this lemma is rather long, since many cases have to be considered. Now suppose G fulfills the conditions of the theorem and consider a maximal tree T such that there are no adjacent edges e, e' from the tree contained in the same 4-cycle. We prove that either $G = T$ or there are pairwise disjoint isomorphic trees $T = T_1, T_2, \cdots, T_r$ in G with $V(G) = \bigcup V(T_i)$ and $G = T\square P_n$ where P_n is a path. From (ii) T must be a path too and we are done.

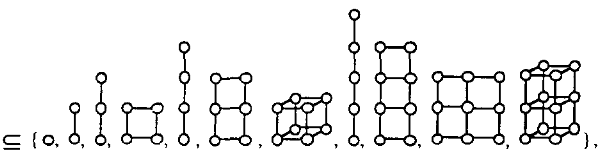
Theorem 2. *2-dimensional grids can be recognized in linear time.*

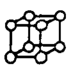
Proof. Assume the graph is given as a list of n records representing the neighbors of successively all vertices. When reading those data we check that there are just 2, 3 or 4 neighbors in each record, making the input time at most $4n$. If no record corresponds to a degree 2 we are done and G cannot be a grid. The procedure consists in a progressive embedding of G in a grid $\subseteq \mathbb{N} \times \mathbb{N}$. We start with a vertex s of degree 2, which we assign coordinates $(0, 0)$. We now determine the unique square σ_0 containing s, assigning its other vertices, on a standard way, the coordinates $(1, 0)$, $(0, 1)$ and $(1, 1)$. If such a square does not exist or is not uniquely determined we return the answer NO, unless G turns out to be a path (this of course can be checked in linear time). This leads us to distinguish a horizontal direction along $(0, 0) \rightarrow (1, 0)$ and a vertical one along $(0, 0) \rightarrow (0, 1)$. We can now 'translate' σ_0 horizontally—as long as it is possible—and obtain a ladder λ_0 having p 'steps'. In such a translation we assign at each step coordinates to two new involved vertices of G and check their degrees and adjacency relations with the already embedded vertices. The computation time for λ_0 is bounded by some $k_1 p + k_1'$. We now translate the ladder vertically—as long as possible—and obtain a p–q-grid. Here coordinates are assigned to $p+1$ new involved vertices and the corresponding degrees and adjacency relations are to be checked. The computation time is $k_2 q + k_2'$ where $k_2 = kp + k'$. For the whole computation we do not need more than $(kp + k')q + k_2' = kpq + k'q + k_2' < Cn + C'$. For connectedness we verify $n = (p+1)(q+1)$. □

n-dimensional grids

The preceding result can be quite easily extended to 3-dimensional case, giving

Theorem 1′. *A connected simple bipartite graph G is a 3-dimensional grid iff*:

(i) $\mathcal{I}_4(G) \subseteq \{$  $\}$,

(i′)  $\in \mathcal{I}_4(G)$,

(ii) *Any induced square in G 'belongs' to at most two cubes in G,*

(iii) *In G are no other gaps than 4-cycles and 6-cycles.*

Various considerations support however the following:

Conjecture. For $n \geqslant 1$ connected simple bipartite graph G is a n-dimensional grid iff

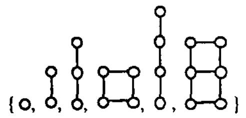
(i) $\{n\text{-cube}\} \subseteq \mathcal{I}_{n+1}(G) \subseteq \{$grids of diameter $\leqslant n+1$ excluding the $(n+1)$-cube$\}$,

(ii) any $(n-1)$-dimensional induced hypercube in G 'belongs' to at most two n-dimensional hypercubes in G,

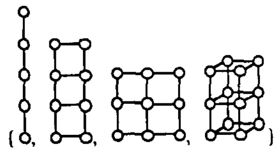
(iii) in G are no other gaps than p-cycles for $p \in [4, 2n]$.

As suggested by Horst Sachs (ii) may be replaced in the last conjecture by:

(ii′) Any edge in G belongs to p 4-cycles where $2^{n-2} \leqslant p \leqslant 2^{n-1}$.

Remark 1. '*Grids of diameter* $\leqslant n+1$' can be obtained from '*grids of diameter* $\leqslant n$' in adding them to the grids corresponding to the partitions of $n+1$: when upgrading from 2 to 3 we have to add to the grid set  the grid  corresponding to $1+1+1=3$

and those corresponding to $4=4+0=3+1=2+2=2+1+1$ but not to $1+1+1+1+1$:



Remark 2. The conjecture holds for $n = 1, 2, 3$.

Remark 3. The procedure presented by the proof of Theorem 2 extends obviously to the n-dimensional case:

Theorem 2′. *For a given n, there exists a linear algorithm for deciding if a graph is an (at most) n-dimensional grid.*