

CONSISTENCY STRENGTH OF HIGHER CHANG'S CONJECTURE, WITHOUT CH

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ABSTRACT. We prove that $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$ implies there is an inner model with a weak repeat measure.

1. INTRODUCTION

The original Chang's Conjecture states that for every structure \mathcal{A} on ω_2 in a countable language, there is a substructure $\mathcal{B} \prec \mathcal{A}$ where $\mathcal{B} \cap \omega_1$ is countable, yet $|\mathcal{B}| = \omega_1$; this statement is abbreviated by $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$. Chang's Conjecture is a strengthening of the Downward Löwenheim-Skolem Theorem, since it places more constraints on the elementary substructures.

$(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$ is equiconsistent with an ω_1 -Erdős cardinal (see [3]), but shifting the cardinals in the statement upward results in a stronger statement in terms of consistency strength. The generalizations $(\omega_{n+2}, \omega_{n+1}) \twoheadrightarrow (\omega_{n+1}, \omega_n)$ for $n \geq 1$ are consistent relative to a huge cardinal (arguments due to Laver and Kunen; see [4]). The known lower bounds for consistency strength are considerably lower: Schindler [12] proved that “CH and $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$ ” implies there is an inner model of $o(\kappa) = \kappa^{+\omega}$, and obtained stronger results in [10] when the bottom cardinal of the conjecture is $\geq \omega_2$ (also with an assumption on cardinal arithmetic). Without the CH assumption, Vickers [13] obtained 0-sword (a mouse with a measure of order 1) from $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$; a similar lower bound was obtained in [2] for a weaker form of Chang's Conjecture. Jónsson cardinals and algebras are also closely related to Chang's Conjecture: Vickers and Welch [14] used covering arguments with the core model to show that K correctly computes successors of Jónsson cardinals.

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In this paper we use covering arguments to improve Vickers' result about $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$ to:

Theorem 1. *Assume $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$. Then there is an inner model with a weak repeat measure (this has consistency strength between $o(\kappa) = \kappa^+$ and $o(\kappa) = \kappa^{++}$).*

The paper is organized as follows. Sections 2 and 3 provide background on Chang's Conjecture and repeat measures, respectively. The proof of Theorem 1 is split into two sections: Section 4 shows there are mice with repeat measures, and Section 5 shows there is an inner model with repeat measures. Section 6 has some final remarks.

2. CHANG STRUCTURES AND THE CHANG FILTER

2.1. Chang structures. In this section we give some basic facts about Chang structures and the Chang ideal. For a more general treatment, and many other variations of Chang's Conjecture, see Foreman [3]. In this paper we deal with only the following special instances of Chang's Conjecture:

Definition 2. *Let $\mu < \lambda < \kappa$ be regular cardinals. $(\kappa, \lambda) \twoheadrightarrow (\lambda, \mu)$ means that for every structure \mathcal{A} with domain κ in a countable language, there is an $X \prec \mathcal{A}$ such that $|X| = \lambda$ and $|X \cap \lambda| = \mu$. Such a set X will be called a $(\kappa, \lambda) \twoheadrightarrow (\lambda, \mu)$ Chang Structure, or simply Chang Structure if it is clear from the context.*

Next are some standard arguments to show $(\kappa, \lambda) \twoheadrightarrow (\lambda, \mu)$ implies some apparently stronger statements. First, it is not necessary to require the domain of \mathcal{A} to be κ : suppose \mathcal{A} is an \mathcal{L} -structure with domain $H \supset \kappa$; WLOG assume $\mathcal{A} = (H, (h_n)_{n \in \omega})$ is fully Skolemized and the collection of h_n 's are closed under compositions (so for every $X \subset H$, $Sk^{\mathcal{A}} := \bigcup_{n \in \omega} h_n[X^{<\omega}]$ is a fully elementary substructure of \mathcal{A}). Let $D_n := \{\vec{\xi} \in [\kappa]^{<\omega} \mid h_n(\vec{\xi}) \in \kappa\}$, and let $X \prec (\kappa, (h_n \upharpoonright D_n)_{n \in \omega})$ be a Chang structure. Then $Y := Sk^{\mathcal{A}}(X)$ is an elementary substructure of \mathcal{A} of cardinality λ , and $Y \cap \kappa = X$ so $|Y \cap \lambda| = \mu$.

For the remainder of the paper we will always deal with structures of the form $\mathcal{A} = (H_\theta, \in, \Delta, \kappa, \lambda, \mu, \dots)$ where $\theta \geq \kappa$ is regular and Δ is a well-ordering of H_θ . Such structures are convenient because they have definable Skolem functions and model ZFC^- in the expanded language (i.e. where formulas in the language of \mathcal{A} are allowed in the Separation and Replacement Schema). Unless stated otherwise, all structures are in a countable language.

Claim 3. *Assume $(\kappa, \lambda) \twoheadrightarrow (\lambda, \mu)$ where $cf(\mu) < cf(\lambda)$. Let $\theta \geq \kappa$ be regular. Then for every structure \mathcal{A} on H_θ there is an $X \prec \mathcal{A}$ such that:*

- $\mu \subset X$.
- $|X| < \lambda$
- If $\lambda = \mu^+$ then $X \cap \lambda$ is transitive.

Proof. WLOG \mathcal{A} expands $(H_\theta, \in, \Delta, \kappa, \lambda, \mu)$. Let Z be a Chang substructure of \mathcal{A} , and let $X := Sk^{\mathcal{A}}(Z \cup \mu)$. Clearly $\mu \subset X$; we show that $sup(X \cap \lambda) = sup(Z \cap \lambda)$. Let x be an arbitrary element of $X \cap \lambda$; then $x = h(p, \bar{\eta})$ for some Skolem function h which is definable in \mathcal{A} , some $p \in Z$, and some $\bar{\eta} \in \mu$. Now $s := sup(\{h(p, \eta) \mid \eta < \mu\} \cap \lambda)$ is definable in \mathcal{A} from parameters p, μ, λ , so s is an element of Z ; and since $cf(\mu) < cf(\lambda)$ then $s < \lambda$. Thus $s < sup(Z \cap \lambda)$; and $x \leq sup(\{h(p, \eta) \mid \eta < \mu\} \cap \lambda)$, so $x < sup(Z \cap \lambda)$. This shows that $sup(X \cap \lambda) \leq sup(Z \cap \lambda)$. The other inequality is trivial.

If $\lambda = \mu^+$ then for any $\beta \in X \cap \mu^+$ there is a surjection $\phi : \mu \rightarrow \beta$ such that $\phi \in X$; so if $\mu \subset X$ (e.g. as we just constructed) then $\beta \subset X$. So $X \cap \mu^+$ is transitive. \square

Corollary 4. *Assume $(\kappa, \lambda) \twoheadrightarrow (\lambda, \mu)$. Let \mathcal{L} be a language of cardinality $\leq \mu$. Then every \mathcal{L} -structure \mathcal{A} on κ has a Chang substructure.*

Proof. Again for simplicity we work in H_κ ; say $\mathcal{A} = (H_\kappa, \in, (f_i)_{i < \mu})$ and is fully Skolemized (and the collection of (f_i) are closed under compositions). Let $R := \{(i, x, f_i(x)) \mid i < \mu \text{ and } x \in H_\kappa\} \subset H_\kappa$. By Claim 3 there is a Chang substructure X of (H_κ, \in, R) such that $\mu \subset X$. Then X is an elementary substructure of \mathcal{A} . \square

Note that if X is a Chang structure witnessing $(\mu^{++}, \mu^+) \twoheadrightarrow (\mu^+, \mu)$ and $X \prec (H_{\mu^{++}}, \in)$ then

$$(1) \quad otp(X \cap \mu^{++}) = \mu^+$$

To see (1): otherwise there would be a μ^+ -th element β of $X \cap \mu^{++}$; by elementarity there is an $f \in X$ which is an injection of β into μ^+ . But then $|f[X \cap \beta]| = \mu^+$ and $f[X \cap \beta] \subset X \cap \mu^+$, which contradicts that $|X \cap \mu^+| = \mu$.

The following first appeared in [6]:

Theorem 5. (Foreman, Magidor). *Assume $(\mu^{++}, \mu^+) \twoheadrightarrow (\mu^+, \mu)$. Then there is a structure \mathcal{A} on μ^{++} such that whenever X is a $(\mu^{++}, \mu^+) \twoheadrightarrow (\mu^+, \mu)$ Chang substructure of \mathcal{A} , then $sup(X \cap \mu^+)$ has cofinality $\geq cf(\mu)$.*

Lemma 6. *If X is as in the conclusion of Theorem 5, then:*

- (2) $X \cap \mu^{++}$ is closed under increasing sequences of length $< cf(\mu)$.

Proof. Let $\beta < \mu^{++}$ be a limit of X of cofinality $< cf(\mu)$. By (1), $otp(X \cap \mu^{++}) = \mu^+$; in particular $cf(\sup(X \cap \mu^{++})) = \mu^+$ so there is an element of $X \cap \mu^{++}$ above β ; let η be the least such element. Let $\alpha_X := X \cap \mu^+$. Since $X \prec (H_\theta, \in)$ there is a bijection $f \in X$ between μ^+ and η , and so $f[\alpha_X] \subseteq (X \cap \eta) = (X \cap \beta)$. Since $cf(\alpha_X) = cf(\mu)$ and $cf(\beta) < cf(\mu)$ (by assumption) then there is a $\delta < \alpha_X$ such that $f[\delta]$ intersects β cofinally often; note $f[\delta]$ is both an element and a subset of X . So in fact $f[\delta] \subset \beta$ and $\sup(f[\delta]) = \beta$, so β is an element of X . \square

2.2. The strongly closed unbounded filter. Now we recall the “strongly closed unbounded filter”; see Foreman [3] for a more general treatment of the subject. Fix a large regular θ . For any structure \mathcal{A} on H_θ , let $C_{\mathcal{A}}$ denote the collection of all $X \subset H_\theta$ such that $X \prec \mathcal{A}$. The *strongly closed unbounded filter* on H_θ is the filter generated by sets of the form $C_{\mathcal{A}}$; it is clearly countably closed (recall we assume \mathcal{A} is in a countable language unless otherwise stated). If $S \subset P(H_\theta)$ intersects every set in the strongly closed unbounded filter, then S is called *weakly stationary*. For example, the collection $[H_\theta]^\omega$ of countable subsets of H_θ is weakly stationary by the Downward Löwenheim-Skolem Theorem. The strongly closed unbounded filter is normal; i.e. whenever S is weakly stationary and $F : S \rightarrow \bigcup S$ is regressive, then F is constant on a weakly stationary set.

Furthermore, if $S \subset P(H_\theta)$ is weakly stationary then the *restriction of the strongly closed unbounded filter to S* is the collection generated by sets of the form $C_{\mathcal{A}} \cap S$ (we require S to be weakly stationary so that this restriction will be proper). This restriction is normal.

2.3. The Chang filter. Under the assumption $(\mu^{++}, \mu^+) \twoheadrightarrow (\mu^+, \mu)$, the facts from section 2 guarantee that S is a weakly stationary set, where:

- (3) S is defined to be the collection of Chang $X \prec (H_\theta, \in, \Delta)$ such that $X \cap \mu^+$ is transitive and $cf(X \cap \mu^+) = cf(\mu)$.

So the strongly closed unbounded filter can be restricted to S ; this restriction is called the *Chang Filter* and we will denote this filter by \mathcal{F} . \mathcal{F} is normal and, by Corollary 4, is $< \mu^+$ -complete. The expression “almost every $X \in S$ has property Q ” will mean that $\{X \in S \mid X \text{ does not have property } Q\}$ is in the dual of \mathcal{F} . For the

rest of the paper, the term “stationary” will be used in reference to the Chang Filter \mathcal{F} .

Definition 7. For a $(\kappa, \lambda) \twoheadrightarrow (\lambda, \mu)$ Chang structure X , $X \uparrow$ will denote the collection of Chang elementary substructures of $(H_\theta, \in, \Delta, X \cap \kappa)$ such that $\mu \subset Y$.

Since $(\mu^{++}, \mu^+) \twoheadrightarrow (\mu^+, \mu)$ -Chang structures all have the same order-type below μ^{++} (namely, μ^+), we will not be able to build \subset -increasing chains of Chang Structures and form directed systems in the usual way; e.g. if X, Y are Chang and $X \cap \mu^{++} \in Y$ then $X \cap \mu^{++}$ cannot be contained in Y . The next lemma provides a remedy:

Lemma 8. Assume $(\mu^{++}, \mu^+) \twoheadrightarrow (\mu^+, \mu)$ and let X be a Chang structure. For each $Y \in X \uparrow$ (see Definition 7) let $\lambda_{XY} := \sup(X \cap Y \cap \mu^{++})$; then $X \cap \lambda_{XY} \subset Y$.

Proof. Let $\eta \in X \cap \lambda_{XY}$; since λ_{XY} is clearly a limit ordinal, then there is an $\eta' \in (\eta, \lambda^{XY}) \cap X \cap Y$. Since $\text{otp}(X \cap \mu^{++}) = \mu^+$, then $|X \cap \eta'| = \mu$; furthermore, $X \cap \mu^{++}$ and η' are both elements of Y , so there is an $F \in Y$ which maps μ onto $X \cap \eta'$. Since $\mu \subset Y$ then $\text{range}(F) \subset Y$ and so $\eta \in Y$. \square

Lemma 9. Assume $(\mu^{++}, \mu^+) \twoheadrightarrow (\mu^+, \mu)$ where $\text{cf}(\mu) > \omega$. Let T be a stationary subset of S where S is defined in (3), and for every $X \in T$ let $A_X \subset X \cap \mu^{++}$ be a set of cardinality $< \text{cf}(\mu)$. Then there is a set B of cardinality $< \mu$ and a stationary $T' \subset T$ such $A_X \subset B \in X$ for every $X \in T'$.

Proof. WLOG assume A_X is a set of ordinals. First, note that for each X there is a $B_X \in X$ which covers A_X . To see this: since $\text{cf}(\sup(X \cap \mu^{++})) = \mu^+$, then $\sup(A_X) < \sup(X \cap \mu^{++})$. So there is an $f \in X$ with domain μ^+ such that $A_X \subset \text{range}(f)$. Since $\text{cf}(X \cap \mu^+) = \text{cf}(\mu) > |A_X|$, then $f^{-1}[A_X] \subseteq \bar{\alpha}$ for some $\bar{\alpha} < X \cap \mu^+$. Similarly there is a $g \in X$ which maps μ onto $\bar{\alpha}$, and $g^{-1} \circ f^{-1}[A_X]$ is contained in some ordinal $\bar{\beta} < \mu$. Then $f \circ g[\bar{\beta}] \in X$ is the covering set B_X we seek.

Since the Chang ideal is normal, the regressive function $X \mapsto B_X$ can be used to obtain the stationary T' and the set B as in the statement of the lemma. \square

2.4. Using normal filters to build extenders over K . In this section we point out a simple fact about normal filters (e.g. the strongly closed unbounded filter), but which is very useful in building extenders over K . If F is an extender on a sufficiently closed algebra P and $\eta < lh(F)$, then $(F)_\eta$ denotes the ultrafilter $\{z \in P \mid \eta \in F(z)\}$ (see [15] for a detailed introduction to extenders).

Lemma 10. *Assume $\kappa < \nu$ are ordinals, $P \subset P(\kappa)$ is a reasonably closed algebra of subsets of κ , $\theta \geq (2^\nu)^+$ is regular, S is a weakly stationary collection of X such that $X \prec (H_\theta, \in, \dots)$, and \mathcal{F} is some normal filter on S . For each $X \in S$ let $\sigma_X : H_X \rightarrow H_\theta$ be the inverse of the Mostowski collapse of X . For any $b \in X$, b_X will denote $\sigma_X^{-1}(b)$.*

Suppose for each $X \in S$ there is an extender $F^X : P_X \rightarrow P(\nu_X)$. For each $\eta < \nu$, define the collection $G_\eta \subset P$ by:

$$(4) \quad z \in G_\eta \text{ iff } z \in P \text{ and } \{X | z_X \in (F^X)_{\eta_X}\} \in \mathcal{F}$$

Let $G := \langle G_\eta | \eta < \nu \rangle$. Note since $\theta \geq (2^\nu)^+$ then $G \in X$ for almost every X (so G_X makes sense for such X , of course however X does not have access to the definition in (4)). If G_η is an ultrafilter on P for each $\eta < \nu$, then for \mathcal{F} -almost every $X \in S$, $G_X = F^X$.

Proof. By elementarity $(G_X)_\eta$ is an ultrafilter on P_X for every $\eta < \nu_X$. Now suppose for a contradiction that there are \mathcal{F} -stationarily many X such that $G_X \neq F^X$. So for such X there is an $\eta_X < \nu_X$ and a $z_X \in P_X$ such that $z_X \in (G_X)_{\eta_X} - (F^X)_{\eta_X}$ (so $(z_X)^c \in (F^X)_{\eta_X}$). By normality of \mathcal{F} there is a pair $\bar{z} \in P$, $\bar{\eta} < \nu$ and an \mathcal{F} -stationary T such that $\sigma_X(z_X, \eta_X) = (\bar{z}, \bar{\eta})$ for every $X \in T$. Since $G_{\bar{\eta}}$ is an ultrafilter (by assumption) and there are \mathcal{F} -stationarily many X with $(z_X)^c \in (F^X)_{\bar{\eta}_X}$, then $z^c \in G_{\bar{\eta}}$. But this means that $(z_X)^C \in (F^X)_{\bar{\eta}_X}$ for \mathcal{F} -many X , which contradicts the fact that $z_X \in (G_X)_{\bar{\eta}_X}$ for every $X \in T$. \square

For example: if $P = P^K(\kappa)$ and F^X is some $(\kappa_X, o^{K_X}(\kappa_X))$ -extender on $P^{K_X}(\kappa_X)$ (typically arising on the K side of the K vs. K_X coiteration) and the definition of G_η in (4) always yields an ultrafilter on $P^K(\kappa)$,¹ then F^X is an element of H_X (for \mathcal{F} -almost every X).

3. REPEAT MEASURES

The notion of a repeat measure was introduced by Radin [9] (to preserve measurability under Radin forcing) and refined by Mitchell [8]. Here we discuss only weak repeat points and up-repeat points; see Gitik [7] for more information.

Definition 11. *A normal measure \mathcal{U} on κ is a weak repeat measure iff for every $A \in \mathcal{U}$ there is a normal measure \mathcal{W} below \mathcal{U} in the Mitchell order such that $A \in \mathcal{W}$.*

¹Note this is equivalent to: for each $\eta < o^K(\kappa)$ and $z \in P^K(\kappa)$, $T_z := \{X | z_X \in (F^X)_{\eta_X}\}$ and $T_{z^c} := \{X | (z_X)^c \in (F^X)_{\eta_X}\}$ are not both \mathcal{F} -stationary.

The Mitchell order of a weak repeat measure on κ is much larger than κ^+ . For example, if the order of \mathcal{V} is some $\nu < \kappa^+$ then there is no smaller measure in the Mitchell order which concentrates on the set $\{\xi < \kappa \mid o(\xi) = h_\nu(\xi)\}$, where h_ν is the ν -th canonical function on κ . Similarly if $o(\mathcal{V}) = \kappa^+$ then there is no smaller measure in the Mitchell order which concentrates on $\{\xi < \kappa \mid o(\xi) = \xi^+\}$. More generally, if ν is an ordinal such that there is a function $h : \kappa \rightarrow Ord$ which represents ν in every normal ultrapower, then ν cannot be the order of a weak repeat measure. Such ordinals ν are called *uniformly representable*.

We also recall the stronger notion of an *up repeat measure*:

Definition 12. *A normal measure \mathcal{U} on κ is an up repeat measure iff for every $A \in \mathcal{U}$ there is a \mathcal{W} above \mathcal{U} in the Mitchell order such that $A \in \mathcal{W}$.*

Lemma 13. *Every up repeat measure is a weak repeat measure.*

Proof. Suppose $A \in \mathcal{U}$ witnesses that \mathcal{U} is *not* a weak repeat measure. Let $B_A := \{\xi < \kappa \mid \text{There is no normal measure concentrating on } A \cap \xi\}$. Then clearly $B_A \in \mathcal{U}$. Now suppose \mathcal{W} is any normal measure above \mathcal{U} in the Mitchell order. Then $ult(V, \mathcal{W})$ has a normal measure concentrating on A (namely \mathcal{U}) so \mathcal{W} concentrates on $\kappa - B_A$. So $B_A \notin \mathcal{W}$. \square

Corollary 14. *If \mathcal{U} is a normal measure on κ which is not a weak repeat measure, then there is an $A \in \mathcal{U}$ which is not an element of any other normal measure on κ which is comparable to \mathcal{U} in the Mitchell order.*

Proof. Let $X \in \mathcal{U}$ witness that \mathcal{U} is not a weak repeat measure. By Lemma 13, \mathcal{U} is also not an up-repeat measure, so let $Y \in \mathcal{U}$ witness this fact. Then $A := X \cap Y$ satisfies the conclusion of the claim. \square

Since we will only deal with coherent sequences \vec{E} of extenders (on some premouse N), then E_ν and E_ζ are always comparable in the Mitchell order. Suppose for simplicity that each extender E_ν on \vec{E} is generated by a single normal measure U_ν . We will call U_ν a *weak repeat measure on N* iff every $A \in U$ is an element of some U_ξ for $\xi < \nu$. So if U_ν is *not* a weak repeat measure, there is an $A \in U_\nu$ which distinguishes U_ν from all other U_ξ .

4. PART 1 OF THE PROOF OF THEOREM 1: GETTING MICE WITH REPEAT MEASURES

Assume $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$. Throughout the rest of the paper, K denotes the core model for non-overlapping extenders, built under the assumption that 0-pistol does not exist. Basic facts about 0-pistol and

K can be found in Chapter 8 of [15]. In particular, K is capable of having a strong cardinal, but comparisons of mice are still *linear*. Note that if 0-pistol exists, then there is a sharp for an inner model with a strong cardinal, so the conclusion of Theorem 1 would hold and we'd be finished.

Let S be the Chang-stationary collection of subsets of H_θ (for some $\theta \gg \omega_3$) defined in (3); so for every $X \in S$, $\alpha_X := X \cap \omega_2$ has cofinality ω_1 . WLOG we assume $X \prec (H_\theta, \in, E \upharpoonright \theta)$ for all $X \in S$. Let $\sigma_X : H_X \rightarrow X \prec H_\theta$ be the inverse of the Mostowski collapse of X , and $K_X = \sigma_X^{-1}(K \upharpoonright \omega_3)$. Note that $\alpha_X = cr(\sigma_X)$, $\sigma_X(\alpha_X) = \omega_2$, $\sup(\sigma_X[\omega_2]) = \sup(X \cap \omega_3) =: \lambda_X$, and $\sigma_X(\omega_2) = \omega_3$.

For $X \in S$, consider the coiteration of K_X with K . Let Ω_X denote the length of the K_X vs. K coiteration, and

$$\langle N_i^X, \pi_{i,j}^X, \nu_i^X, \kappa_i^X, \tau_i^X, \delta_i^X, (N_i^X)^* \mid i \leq j \leq \Omega_X \rangle$$

denote the objects on the K side of the coiteration; i.e. N_i^X is the i -th iterate, $\pi_{i,j}^X$ the iteration map, ν_i^X the iteration index, κ_i^X the critical point, τ_i^X the smaller of the cardinal successors of κ_i^X in the i -th iterates, δ_i^X the maximal segment of N_i^X where τ_i^X is a cardinal, and $(N_i^X)^* = N_i^X \parallel \delta_i^X$.

By (2), each $X \in S$ has an ω -closed intersection with ω_3 , so Lemma 44 from [1] applies.² So $\alpha_X^{+K_X}$ is not a cardinal in K , the K side of the coiteration truncates to an ω_1 -sized mouse either at stage 0 or at stage 1, and the K_X side of the coiteration is trivial. Let $\iota_0^X \in \{0, 1\}$ denote the first truncation stage. So $|(N_{\iota_0^X}^X)^*| < \omega_2$. Since $|(N_{\iota_0^X}^X)^*| < \omega_2$, and $|K_X| < \omega_2$, then $|(N_i^X)^*| < \omega_2$ for every $i \in [\iota_0^X, \Omega_X)$, and $\Omega_X < \omega_2$. Also, since the K_X side is trivial in the coiteration, we have:

$$(5) \quad \nu_i^X = o^{K_X}(\kappa_i^X)$$

for every stage i of the coiteration.

Since the K side must win the coiteration and $|K_X| = \omega_2$, then the length of the K vs. K_X coiteration is at least ω_2 . Since $|(N_i^X)^*| < \omega_2$ for each $i \in [\iota_0^X, \omega_2)$, the usual pressing down and pigeonhole arguments

²This was based on an argument of Mitchell.

yield:³

- (6) For any stationary $T \subset \omega_2$, there is a stationary $T' \subset T$ such that for all $i < j$ in T' , $\pi_{i,j}^X(\nu_i^X) = \nu_j^X$.

By (6) and basic properties of direct limits, $D_X := \{\kappa_i^X \mid \pi_{i,\omega_2}^X(\kappa_i^X) = \omega_2\}$ is a club in ω_2 . Furthermore for every $i \geq 1$ $(N_i^X)^*$ projects to and is sound above κ_i^X , and continuity of σ_X on $\text{cof}(\omega)$ along with the Weak Covering Lemma implies that τ_i^X has uncountable cofinality. Thus by Lemma 16 of [1], for every $i < \omega_2$, $\sigma_X \upharpoonright \tau_i^X$ can be lifted to $(N_i^X)^*$. Pick a stage $i_0^X < \omega_2$ such that there are no truncations at stages in the interval $[i_0^X, \omega_2)$; so $N_i^X = (N_i^X)^*$ for all $i \in [i_0^X, \omega_2)$. For such i let n_i^X denote the degree of κ_i^X in N_i^X , i.e. the maximal $n \in \omega$ such that κ_i^X is strictly less than the n -th projectum of N_i^X (this is the degree of the fine-structural ultrapower used at stage i). Then $\langle n_i^X \mid i_0^X \leq i < \omega_2 \rangle$ is a nonincreasing sequence of natural numbers; WLOG suppose i_0^X was chosen so that n_i^X has constant value n_X for every $i \in [i_0^X, \omega_2)$. Since the Chang ideal is countably complete (in fact $< \omega_2$ complete) we will WLOG assume that n_X is fixed at n for every $X \in S$. Let $\tilde{\sigma}_i^X : \widehat{N_i^X} \rightarrow_{\Sigma(n)} \widehat{\tilde{N}_i^X}$ denote the canonical lifting of $\sigma_X \upharpoonright \tau_i^X$. This ultrapower $\widehat{\tilde{N}_i^X}$ really is a premouse (not a protomouse), since κ_i^X is measurable in N_i^X and so the top extender on N_i^X (if there is one) has critical point $\geq \kappa_i^X$ (recall we assume 0-pistol does not exist so there are no overlapping extenders). By Lemma 43 of [1], $\widehat{\tilde{N}_i^X}$ is an initial segment of K (the collapsing segment of K for $\tilde{\tau}_i^X := \sup[(\sigma_X \upharpoonright \tau_i^X)]$).

Since D_X is club in ω_2 for every $X \in S$, for almost every stage $i < \omega_2$ we have $\kappa_i^X = i$. This will simplify the following notation a bit. For each such i , let $F_i^X := E_{\nu_i^X}^{N_i^X}$, i.e. the extender used at stage i (with critical point i). Let $U_i^X := (F_i^X)_i$; i.e. U_i^X is the (normal) ultrafilter on the hypermeasure F_i^X indexed by i .

From now on:

- (7) Assume all mice extenders have only one generator

³Under the additional assumption of CH , Schindler [12] showed that if T' is a stationary subset of $\text{cof}(\omega)$ as in (6) and $i \in T' \cap \text{Lim}(T')$, then $\bar{E} := E_{\nu_i^X}^{N_i^X}$ must have order $\geq (\kappa_i^X)^{+\omega}$ in the mouse N_i^X . This used the fact that under CH , $[X]^\omega \subset X$ for every $X \in S$ (see the discussion following Theorem 62 of [5]), so $[H_X]^\omega \subset H_X$. If the order of \bar{E} were $< (\kappa_i^X)^{+\omega}$, Schindler showed that the collection of generating sequences for \bar{E} could be reconstructed inside the countably closed structure H_X , and thus $\bar{E} \in H_X$. This would imply that \bar{E} is on the K_X sequence, a contradiction.

If (7) fails, then there are mice with much stronger cardinals than repeat points. If F is a mouse extender with two generators, then iterating F Ord -many times yields an inner model W which models GCH , has a coherent sequence \vec{E} of extenders, and $o^{\vec{E}}(\kappa) = \kappa^{++}$ for a proper class of κ . For such κ , all sufficiently large measures on κ in the Mitchell order are weak repeat measures.

Now we prove Theorem 1; in fact we prove:

Theorem 15. *For almost every $X \in S$ and almost every $i \in S_0^2$, U_i^X is a weak repeat measure in the mouse N_i^X .*

Proof. Suppose for a contradiction that there is an \mathcal{F} -stationary set $S' \subset S$ of Chang structures such that for every $X \in S'$:

$$(8) \quad \begin{aligned} T_X &:= \{i \in S_0^2 \mid U_i^X \text{ is NOT a weak repeat measure in } N_i^X\} \\ &\text{is stationary in } \omega_2. \end{aligned}$$

By (6) there is a stationary $T'_X \subset T_X$ such that the extenders used at stages in T'_X lie on a common thread; i.e. for every pair $i < j$ in T'_X , $\pi_{i,j}^X(\nu_i^X) = \nu_j^X$. WLOG assume $\min(T'_X) > i_0^X$; recall i_0^X was chosen to be a sufficiently large stage so there are no truncations at stages in the interval $[i_0^X, \omega_2)$.

For each $X \in S'$ pick a μ_X which is an element and limit of the stationary set T'_X ; recall that since D_X is club in ω_2 we can WLOG assume that $\kappa_{\mu_X}^X = \mu_X$. Let

$$(9) \quad \begin{aligned} &\bullet N_X := N_{\mu_X}^X, F_X := F_{\mu_X}^X, \text{ and } U_X := U_{\mu_X}^X; \text{ by} \\ &\quad (7) U_X \text{ generates } F_X \\ &\bullet \tau_X \text{ be the cardinal successor of } \mu_X \text{ in } K_X; \\ &\bullet \tilde{\sigma}_X := \tilde{\sigma}_{\mu_X}^X \text{ and } \tilde{N}_X := \tilde{N}_{\mu_X}^X \text{ (so } \tilde{\sigma}_X : N_X \rightarrow \\ &\quad \tilde{N}_X \text{ is the canonical lifting of } \sigma_X \upharpoonright \tau_X \text{ to } N_X). \end{aligned}$$

Then

$$(10) \quad \mu_X \cap T'_X \text{ is a generating sequence for the measure } U_X$$

Since $\mu_X \in T'_X \subset T_X$, U_X is not a weak repeat measure in N_X . So by Corollary 14 there is a set $a_X \in U_X$ which uniquely identifies U_X ; i.e. $a_X \in U_X$ but is not an element of any other normal measure on N_X 's sequence.⁴

Since stage μ_X is past all truncations on the K side of the coiteration, then $P^{N_X}(\mu_X) = P^{K_X}(\mu_X)$ (so $a_X \in K_X$). Since $\nu_{\mu_X}^X = o^{K_X}(\mu_X)$ (see

⁴More precisely: a_X is not an element of any normal measure which generates an extender on N_X 's coherent extender sequence.

5), then:

- (11) No extender on K_X 's extender sequence is generated by a measure which concentrates on a_X .

By applying the Fodor Lemma to the Chang-positive collection S' , we may WLOG assume that $\sigma_X(a_X)$ and $\sigma_X(\mu_X)$ have fixed values for every $X \in S'$; say $(a, \mu) = \sigma_X(a_X, \mu_X)$ for every $X \in S'$. So by elementarity of σ_X and (11):

- (12) No extender on K 's extender sequence is generated by a measure which concentrates on a .

We will construct a normal K -measure on μ which concentrates on a and yields a wellfounded ultrapower. By Corollary 29 of [1], such a measure would generate an extender on K 's extender sequence, which will contradict (12) and complete the proof of Theorem 15.

For each $X \in S'$ let E_X be a countable subset of $T'_X \cap \mu_X$ which is cofinal in μ_X ; note by (10), E_X is a generating sequence of U_X . By Lemma 9, there is a stationary $S'' \subset S'$ and a countable set E such that for every $X \in S''$, $E'_X := \sigma_X[E_X] \subset E \in X$. For every $X \in S''$ and $Y \in S'' \cap X \uparrow$, let $\sigma_{XY} := \sigma_Y^{-1} \circ \sigma_X \upharpoonright \tau_X$; note this map is defined on all of τ_X because $Y \in X \uparrow$. So for such pairs X, Y we have $E_Y \subset \text{range}(\sigma_{XY})$.

The following construction, through Claim 16, is similar to the construction which begins at the bottom of page 85 in [1]; but for the reader's convenience we include most details here. Consider any pair X, Y such that $X \in S''$ and $Y \in S'' \cap X \uparrow$. Using an interpolation-like argument with the map $\tilde{\sigma}_X$, and the fact that $\widehat{N_X}$ is sound above μ_X , construct the following maps (here we omit the “hats” on the book-keeping preimage):

- $\tilde{\sigma}_{XY} : N_X \rightarrow_{\Sigma_0^{(n)}} \tilde{N}_{XY}$ which is n -cofinal and extends $\sigma_{XY} \upharpoonright \tau_X$;⁵ the ultrapower really is a premouse since the top extender on N_X (if there is one) has critical point $\geq \mu_X$.
- $\sigma'_{XY} : \tilde{N}_{XY} \rightarrow_{\Sigma_0^{(n)}} \tilde{N}_X$ which extends $\sigma_Y \upharpoonright \tau_Y$.⁶

⁵Since we are working past all truncations—i.e. μ_X is not a truncation stage—then τ_X is a cardinal in N_X (similarly for Y) and lifting $\sigma_{XY} \upharpoonright \tau_X$ to N_X is possible. For a similar argument where there may be truncations, see [1].

⁶ σ'_{XY} is defined by $[\xi, f]_{\sigma_{XY}} \mapsto \tilde{\sigma}_X(f)(\sigma_Y(\xi))$, where f is a good $\Sigma_1^{(n-1)}(\widehat{N_X})$ function with $\text{dom}(f) = \mu_X$ and $\xi \in \sigma_{XY}(\mu_X)$. It can be shown this is well-defined and has the desired properties.

The following diagram depicts the situation; here Q_X refers to $K_X|_{\tau_X}$.

$$\begin{array}{ccc}
 & \tilde{N}_X & \\
 \tilde{\sigma}_X \nearrow & & \nwarrow \sigma'_{XY} \\
 N_X & \xrightarrow{\tilde{\sigma}_{XY}} & \tilde{N}_{XY} \\
 \\
 Q_X & \xrightarrow{\sigma_{XY}} & Q_Y
 \end{array}$$

The expansions of \tilde{N}_{XY} and N_Y are both sound above μ_Y and coiterate above μ_Y . It is straightforward to show that \tilde{N}_{XY} is a (proper) initial segment of N_Y (note $\mu_Y^{+\tilde{N}_{XY}} = \sup(\sigma_{XY}[\tau_X]) < \tau_Y$).

Recall that E_Y is a cofinal subset of $\text{range}(\sigma_{XY}) \cap \mu_Y$, so $\sigma_{XY}^{-1}[E_Y]$ is a cofinal subset of μ_X .

Claim 16. $\sigma_{XY}^{-1}[E_Y]$ is eventually contained in $D_X \cap \mu_X$.

Proof. Every element of $\widehat{N_X}$ is of the form $\tilde{h}_{N_X}^{n+1}(\xi, p_{\widehat{N_X}})$ for some $\xi < \mu_X$, and similarly for $\widehat{N_Y}$. Let \tilde{h}_X^{n+1} and \tilde{h}_Y^{n+1} denote the good uniform $\Sigma_1^{(n)}$ Skolem functions $\tilde{h}_{N_X}^{n+1}(-, p_{N_X})$ and $\tilde{h}_{N_Y}^{n+1}(-, p_{N_Y})$, respectively (see [15] for the definition). N_{XY} is an element of N_Y , so there is an $\eta_{XY} < \mu_Y$ from which the partial function $\tilde{h}_{XY}^{n+1} := \tilde{h}_{N_{XY}}^{n+1}(-, \tilde{\sigma}_{XY}(p_{\widehat{N_X}}))$ is defined in N_Y ; i.e. $\tilde{h}_{XY}^{n+1} = \tilde{h}_Y^{n+1}(\eta_{XY})$. Let β be any element of E_Y such that $\beta > \sigma_{XY}(\min(D_X))$ and $\beta > \eta_{XY}$. Then $\bar{\beta} := \sigma_{XY}^{-1}(\beta) \in D_X$. If not, then by Lemma 24 of [1], there is some $\bar{\xi} < \bar{\beta}$ such that $\tilde{h}_X^{n+1}(\bar{\xi})$ is in the interval $[\bar{\beta}, \mu_X)$ (note μ_X is past all truncations, so N_X is the direct limit). Let $\bar{\zeta} := \tilde{h}_X^{n+1}(\bar{\xi})$, $\xi := \tilde{\sigma}_{XY}(\bar{\xi})$, and $\zeta := \tilde{\sigma}_{XY}(\bar{\zeta})$. Since $\tilde{\sigma}_{XY}$ is $\Sigma_0^{(n)}$ preserving and n -cofinal, then it is $\Sigma_1^{(n)}$ preserving, so $\tilde{h}_{XY}^{n+1}(\xi)$ is defined and equals ζ , and ζ is in the interval $[\beta, \mu_Y)$. But $\zeta = \tilde{h}_{XY}^{n+1}(\xi) = \tilde{h}_Y^{n+1}(\eta_{XY})(\xi)$ and $\prec \xi, \eta_{XY} \succ \beta$. So ζ witnesses that $\tilde{h}_Y^{n+1}[\beta] \cap [\beta, \mu_Y) \neq \emptyset$. This contradicts Lemma 24 of [1] and the fact that $\beta \in D_Y \cap \mu_Y$. \square (Claim 16)

Let $VG_X := \{\kappa_i^X \in D_X \cap \mu_X \mid \pi_{i, \mu_X}^X(\nu_i^X) = \nu_{\mu_X}^X\}$; we will call these the “very good critical points for X .” Note that $E_X \subseteq T'_X \subseteq VG_X$ and VG_X generates the ultrafilter U_X (i.e. for every $z \in P^{N_X}(\mu_X)$, $z \in U_X \iff z$ contains a tail end of $VG_X \iff z \cap VG_X$ is unbounded in μ_X).

For the remainder of the proof, the notation $z =^* w$ will denote eventual equality; i.e. $\sup(z) = \sup(w) =: s$ and there is some $\xi < s$

such that $z - \xi = w - \xi$. The notation $z \subseteq^* w$ will mean that z is eventually contained in w .

Claim 17. *For every $X \in S''$, $VG_X =^* D_X \cap a_X$.*

Proof. $VG_X \subset D_X$ by definition. Pick \hat{i} so that $\nu_{\mu_X}^X$ and a_X are in $\text{range}(\pi_{\hat{i}, \mu_X}^X)$. Let j be any stage $\geq \hat{i}$ such that $\kappa_j^X \in D_X$. So for such j , $a_X = \pi_{j, \mu_X}^X(a_X \cap \kappa_j^X)$. Let π denote π_{j, μ_X}^X .

If $\kappa_j^X \in VG_X$ then $\nu_j^X = \pi^{-1}(\nu_{\mu_X}^X)$ and by elementarity, U_j^X concentrates on $a_X \cap \kappa_j^X$. So $\kappa_j^X \in \pi(a_X \cap \kappa_j^X) = a_X$. This shows $VG_X \subseteq^* D_X \cap a_X$.

Now assume $j \geq \hat{i}$ is such that $\kappa_j^X \in D_X \cap a_X$. Then U_j^X (the measure applied at stage j) concentrates on $a_X \cap \kappa_j^X$ (note $a_X \in \text{range}(\pi)$). Then $\pi(U_j^X)$ concentrates on a_X ; since U_X is the only normal measure in N_X which concentrates on a_X then $\pi(U_j^X) = U_X$. Then $\pi(\nu_j^X) = \nu_{\mu_X}^X$ and so $\kappa_j^X \in VG_X$. \square (Claim 17)

Now by Claims 16 and 17, for each $X \in S''$ and $Y \in S'' \cap X \uparrow$:

$$(13) \quad \begin{aligned} &E_Y \subset \text{range}(\sigma_{XY}), \ E_Y \subseteq VG_Y =^* D_Y \cap a_Y, \text{ and} \\ &\sigma_{XY}^{-1}[E_Y] \subseteq^* D_X \cap a_X =^* VG_X \end{aligned}$$

Define a collection $W \subset P^K(\mu)$ by:

$$(14) \quad \begin{aligned} &z \in W \text{ iff } \{X \in S'' \mid z \in X \text{ and } z_X := \sigma_X^{-1}(z) \in U_X\} \\ &\text{is an element of the Chang filter } \mathcal{F}. \end{aligned}$$

Note that, for a given $z \in P^K(\mu)$ and $X \in S''$ with $z \in X$, one of z_X or z_X^c must be in U_X ; this is because $P^{K_X}(\mu_X) = P^{N_X}(\mu_X)$ and U_X is an ultrafilter on $P^{N_X}(\mu_X)$.

Claim 18. *W is normal with respect to K .*

Proof. Suppose not; so there is a $g : \mu \rightarrow P^K(\mu)$ such that $g \in K$ and $g(\xi) \in W$ for every $\xi < \mu$, yet $\Delta g \notin W$. By the definition of W , this means that $T := \{X \in S'' \mid g \in X \text{ and } (\Delta g)_X \notin U_X\}$ is \mathcal{F} -stationary. Now $g \in K \upharpoonright \tau$ so for each $X \in T$, g_X is an element of $K_X \upharpoonright \tau_X = N_X \upharpoonright \tau_X$. U_X is normal with respect to N_X , so since $\Delta g_X \notin U_X$ there is some ξ_X such that $g_X(\xi_X) \notin U_X$. The map $X \mapsto \sigma_X(\xi_X)$ is regressive on T , so by the normality of \mathcal{F} there is an \mathcal{F} -stationary $T' \subseteq T$ and an ordinal $\hat{\xi}$ such that $\sigma_X(\xi_X) = \hat{\xi}$ for every $X \in T'$. In other words, $\sigma_X^{-1}(g(\hat{\xi})) \notin U_X$ for every $X \in T'$. But this contradicts that $g(\hat{\xi}) \in W$. \square (Claim 18)

Claim 19. *W is an ultrafilter on $P^K(\mu)$.*

Proof. It is clear that W is a filter, since \mathcal{F} is a filter and each U_X is an ultrafilter over $P^{K_X}(\mu_X)$. The issue is showing that W is maximal.

Suppose to the contrary, and let $z \in P^K(\mu)$ be a counterexample; so neither z nor $\mu - z$ is an element of W . Let $T := \{X \in S'' \mid z \in X\}$ and for $X \in T$ let z_X denote $\sigma_X^{-1}(z)$. So both $T_z := \{X \in T \mid z_X \in U_X\}$ and $T_{z^c} := \{X \in T \mid (z_X)^c \in U_X\}$ are \mathcal{F} -stationary.

Fix an $X \in T_z$ and a $Y \in T_{z^c} \cap X \uparrow$ (recall $X \uparrow$ is an element of the Chang Filter \mathcal{F}). Since $(z^c)_Y \in U_Y$ then it contains a tail end of U_Y 's generating sequence E_Y . This fact, combined with (13), implies that $(z^c)_X = \sigma_{XY}^{-1}((z^c)_Y)$ intersects VG_X cofinally often in μ_X . Since VG_X generates U_X , then $(z^c)_X \in U_X$. This is a contradiction because $X \in T_z$. \square (Claim 19)

Claim 20. *ult(K, W) is wellfounded.*

Proof. Note $o^K(\eta) < \mu$ for every $\eta < \mu$; this follows from the fact that for every $X \in S''$, μ_X is measurable in N_X and so for every $\eta < \mu_X$, $o^{N_X}(\eta) = o^{K_X}(\eta) < \mu_X$ (recall we assume there are no overlapping extenders).

So by Corollary 19 of [1], it suffices to show that $ult(K|\tau, W)$ is wellfounded, where $\tau = \mu^{+K}$. Suppose not; then there is a $\langle g_n \mid n \in \omega \rangle$ such that $g_n \in K|\tau$ and $A_n := \{\xi < \mu \mid g_{n+1}(\xi) < g_n(\xi)\} \in W$ for every $n \in \omega$. Since $\tau = \mu^{+K}$ we can WLOG assume $g_n : \mu \rightarrow \mu$. Let $C_n \in \mathcal{F}$ witness that $A_n \in W$. Let C be the collection of Chang structures Z with $\langle g_n \mid n \in \omega \rangle \in Z$. Pick any $X \in C \cap \bigcap_{n \in \omega} C_n$. Then $\langle (g_n)_X \mid n \in \omega \rangle$ witnesses that $ult(N_X, U_X)$ is illfounded (note each $(g_n)_X$ is an element of $K_X|\tau_X = N_X|\tau_X$). This is a contradiction since U_X generates an extender on the mouse N_X .

\square (Claim 20)

So W is normal with respect to K and $ult(K, W)$ is wellfounded. Note from the proofs that the normality of W was essentially due to the normality of the Chang Filter \mathcal{F} , and the wellfoundedness of $ult(K, W)$ was essentially due to the countable completeness of \mathcal{F} . By Corollary 29 of [1], W generates an extender on K 's extender sequence. Clearly W concentrates on a (since each U_X concentrates on a_X). This contradicts (12) and completes the proof of Theorem 15. \square (Theorem 15)

Finally, we note that if i is any of the elements of S_0^2 where U_i^X is a repeat measure, then $\tilde{\sigma}_i^X(U_i^X)$ is a repeat measure in \tilde{N}_i^X . As noted before, \tilde{N}_i^X is an initial segment of K (it is the collapsing segment for $\sup(\sigma_X[\tau_i^X])$).

5. PART 2 OF THE PROOF OF THEOREM 1: GETTING A PROPER
CLASS MODEL WITH A REPEAT MEASURE

In this section we prove that there is an inner model of $ZFC +$ “There is a repeat measure.” Note that if there exists a mouse N which has a repeat measure \mathcal{U} on κ and one more measure \mathcal{W} above \mathcal{U} in the Mitchell order, then iterating \mathcal{W} out of the universe yields an inner ZFC model with many repeat measures.

So for the rest of the paper:

- (15) Assume that for every mouse N , if ν indexes a repeat measure in N then ν is the largest measure in N (in the Mitchell order).

Under this assumption we will show that K has a repeat measure on ω_3 .

By Theorem 15, for almost every $X \in S$ there is an ω -club $C_X \subset \omega_2 \cap \text{cof}(\omega)$ such that for every $i \in C_X$, ν_i^X indexes a repeat measure in N_i^X . For each such X pick a $\mu_X = \kappa_{\mu_X}^X$ such that $\mu_X \in \text{Lim}(C_X) \cap \text{cof}(\omega)$ (so $\mu_X \in C_X$). Similarly to the proof of Theorem 15 let $VG_{\mu_X} := \{\kappa_i^X \in D_X \cap \mu_X \mid \pi_{i, \mu_X}^X(\nu_i^X) = \nu_{\mu_X}^X\}$ (the “very good stages” for X corresponding to μ_X). Then VG_{μ_X} is a generating sequence for the measure (which generates) $U_{\nu_{\mu_X}^X}^{N_{\mu_X}^X}$. By assumption (15), $\nu_{\mu_X}^X$ indexes the *only* repeat measure in $N_X := N_{\mu_X}^X$. This allows VG_{μ_X} to be characterized as follows:

- (16) VG_{μ_X} is the set of $\kappa_i^X \in D_X$ such that for every $a \in U_{\nu_i^X}^{N_i^X}$ there is a measure in N_i^X indexed below ν_i^X which concentrates on a ; and this is the same as the set of $\kappa_i^X \in D_X$ such that for every $a \in U_{\nu_i^X}^{N_i^X}$ there is a measure in K_X which concentrates on a .

Now consider any choice $X \mapsto \mu_X$ (for $X \in S$) where μ_X is an arbitrary element of $\text{Lim}(C_X) \cap \text{cof}(\omega)$ (for $X \in S$). As in the proof of Theorem 15, for each $X \in S$ pick a countable E_X which is a cofinal subset of VG_X . By Lemma 9 there is a stationary S' and a fixed set F such that $E_X \subset F \in X$ for every $X \in S'$. Then whenever $X \in S'$ and $Y \in S' \cap X \uparrow$, then $E_Y \subset \text{range}(\sigma_{XY})$ and as before, $\sigma_{XY}[E_Y]$ is eventually contained in $D_X \cap \mu_X$ (note the pressing down automatically fixes $\sigma_X(\mu_X)$ at some μ for all $X \in S'$).

Recall from the proof of these facts that $\tilde{\sigma}_{XY} : N_X \rightarrow \tilde{N}_{XY}$ is the canonical lifting of $\sigma_{XY} \upharpoonright \tau_X$ to N_X , \tilde{N}_{XY} is an element of N_Y , and N_Y is sound above μ_Y . Let $\eta_{XY} < \mu_Y$ be the parameter from which \tilde{N}_{XY} is

defined in N_Y , i.e. $\tilde{N}_{XY} = \tilde{h}_{\tilde{N}_Y}^{n+1}(\eta_{XY}, p_{N_Y})$. Given any element b of N_X , let $i_b < \mu_X$ be a stage such that $b \in \text{range}(\pi_{i_b, \mu_X}^X)$ and such that $N_{i_b}^X$ is sound above $\kappa_{i_b}^X$ (in our case, any $i_b \geq 1$ will suffice since there is a truncation by stage 1). Then $b = \tilde{h}_{N_X}^{n+1}(\xi_b, p_{N_X})$ for some $\xi_b < \kappa_{i_b}^X$. Since $\tilde{\sigma}_{XY}$ is $\Sigma_1^{(n)}$ preserving, then $\tilde{b} := \tilde{\sigma}_{XY}(b) = \tilde{h}_{\tilde{N}_{XY}}^{n+1}(\sigma_{XY}(\xi_b), \sigma_{XY}(p_{N_X}))$. So \tilde{b} is an element of $\tilde{h}_{N_Y}^{n+1}[\{\eta_{XY}\} \cup \{\sigma_{XY}(\xi_b)\} \cup \{p_{N_Y}\}]$. In particular, if $\kappa_j^Y > \max(\eta_{XY}, \sigma_{XY}(\xi_b))$ then $\tilde{b} \in \tilde{h}_{N_Y}^{n+1}[\kappa_j^Y \cup \{p_{N_Y}\}]$, and the latter is just the range of the iteration map π_{j, μ_Y}^Y . So we have just shown:

$$(17) \quad \begin{aligned} &\text{If } b \in \text{range}(\pi_{i_b, \mu_X}^X) \text{ and } \kappa_j^Y \text{ is such that } \kappa_j^Y \geq \\ &\sigma_{XY}(\kappa_{i_b}^X) > \eta_{XY}, \text{ then } \tilde{\sigma}_{XY}(b) \in \text{range}(\pi_{j, \mu_Y}^Y). \end{aligned}$$

Note also that if $b \in N_X \upharpoonright \tau_X$ then $\tilde{\sigma}_{XY}(b) = \sigma_{XY}(b)$.

Claim 21. *For every $X \in S'$ and every $Y \in X \uparrow \cap S'$: $\sigma_{XY}^{-1}[E_Y]$ is eventually contained in VG_X .*

Note this will imply that $U_X = U_{\nu_{\mu_X}^X}^{N_X^X}$ and $U_Y = U_{\nu_{\mu_Y}^Y}^{N_Y^Y}$ cohere, since $E_Y \subset VG_Y$ is a generating sequence for U_Y and VG_X is a generating sequence of U_X

Proof. Pick any $\kappa_\ell^Y \in E_Y$ which is $> \eta_{XY}$ and such that $\sigma_X^{-1}[E_Y - \kappa_\ell^Y]$ is contained in D_X .

Pick any $\kappa_j^Y \in E_Y$ above κ_ℓ^Y . Suppose for a contradiction that $\kappa_i^X := \sigma_{XY}^{-1}(\kappa_j^Y) \in D_X - VG_X$.

By (16) there is some $\hat{a}_X \subset \kappa_i^X$ which witnesses that U_i^X is *not* a repeat measure (i.e. U_i^X is the only measure on the N_i^X sequence which concentrates on \hat{a}_X). Let $a_X := \pi_{i, \mu_X}^X(\hat{a}_X)$. Then:

- K_X has no measure concentrating on \hat{a}_X ;
- $\kappa_i^X \in a_X$

By elementarity of σ_{XY} , K_Y has no measure concentrating on $\hat{a}_Y := \sigma_{XY}(\hat{a}_X) (\subset \kappa_j^Y)$. Since $\kappa_j^Y \in VG_Y$, then U_j^Y is a repeat measure so (recall $\nu_j^Y = o^{K_Y}(\kappa_j^Y) = \sigma_{XY}(o^{K_X}(\kappa_i^X)) = \sigma_{XY}(\nu_i^X)$):

$$(18) \quad U_j^Y \text{ does not concentrate on } \hat{a}_Y$$

By (17), $\tilde{a}_Y := \tilde{\sigma}_{XY}(a_X) = \sigma_{XY}(a_X)$ is an element of $\text{range}(\pi_{j, \mu_Y}^Y)$. So $\pi_{j, \mu_Y}^Y(\hat{a}_Y) = a_Y$; also, since $\kappa_i^X \in a_X$ then $\kappa_j^Y \in a_Y$. But then the measure applied at stage \hat{j} must have concentrated on a_Y , contradicting (18). \square (Claim 21)

Define a filter W on $P^K(\mu)$ by:

$$(19) \quad \begin{aligned} z \in W & \text{ iff } \sigma_X^{-1}(z) \in U_X \text{ for almost every } X \in S' \text{ with} \\ z \in X. & \text{ (here } U_X = U_{\nu_{\mu_X}^X}^{N_{\mu_X}^X}). \end{aligned}$$

Note that if $z \in K \cap X$ then $z_X \in K_X|_{\tau_X}$, so $z_X \in N_{\mu_X}^X$; so one of z_X or its complement is an element of U_X .

The proof that W defines an ultrafilter and is on K 's sequence is exactly the same as the proof of Claims 18, 19, and 20. Now we just need to see that W is a repeat measure. If not, there is some $A \in P^K(\mu)$ which uniquely identifies W . Let B_A be the collection of $\xi < \mu$ such that no K measure concentrates on $A \cap \xi$; then $B_A \in W$. Pull this down to some N_X so $A_X \in U_X$ and $(B_A)_X \in U_X$ too; but by elementarity of σ_X , $(B_A)_X$ is the collection of $\xi < \mu_X$ such that no K_X measure concentrates on $A_X \cap \xi$; since K_X and N_X agree below μ_X , this is the same as the collection of ξ such that no N_X measure concentrates on $A_X \cap \xi$. So:

$$(20) \quad \{\xi < \mu_X \mid \text{No } N_X \text{ measure concentrates on } A_X \cap \xi\} \in U_X$$

But $A_X \in U_X$ and U_X is a weak repeat measure in N_X , so there is some U' below U_X in N_X 's Mitchell order such that $A_X \in U'$. Thus:

$$(21) \quad \{\xi < \mu_X \mid \text{There is an } N_X \text{ measure which concentrates on } A_X \cap \xi\} \in U_X$$

which contradicts (20).

6. FINAL REMARK

There is still a tremendous gap between a weak repeat measure and the best known upper bound for the consistency of $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$, which is a huge cardinal (due to arguments of Kunen and Laver). The known model obtained from a huge cardinal, however, also has a precipitous (in fact saturated) ideal on ω_2 . Now $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$ together with a precipitous ideal on ω_2 implies at least that there is an inner model with a Woodin cardinal. This is because:

- $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$ implies that \square_{ω_2} fails
- If there is a precipitous ideal on ω_2 and no inner model with a Woodin cardinal, then by a result of Schindler [11] the core model K computes ω_2^+ correctly. Since K has a $\square_{\omega_2^V}$ sequence, this will then be a $\square_{\omega_2^V}$ sequence in V .

So it is natural to ask:

Question. *Is it possible, starting with significantly less than a huge cardinal, to obtain a model of $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$ which does not have a precipitous ideal on ω_2 ?*

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