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Cyclic group and knapsack facets

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Abstract. Any integer program may be relaxed to a group problem. We define the master cyclic group problem and several master knapsack problems, show the relationship between the problems, and give several classes of facet-defining inequalities for each problem, as well as a set of mappings that take facets from one type of master polyhedra to another.

1. Introduction

1.1. Motivation

Consider the integer programming problem

$$\min\{cx \mid Ax = b, x \geq 0 \text{ and integer}\} \tag{1.1}$$

and its linear programming relaxation

$$\min\{cx \mid Ax = b, x \geq 0\}. \tag{1.2}$$

Let (B, N) represent the sets of basic and non-basic variables for an optimal basic solution to (1.2). We may rewrite (1.1) as

$$\min\{c_B x_B + c_N x_N \mid Bx_B + Nx_N = b, x_B, x_N \geq 0 \text{ and integer}\}. \tag{1.3}$$

A *group relaxation* of (1.3) may be found by removing the non-negativity restrictions on the basic variables x_B:

$$\min\{c_B x_B + c_N x_N \mid Bx_B + Nx_N = b, x_N \geq 0, x_B, x_N \text{ integer}\}. \tag{1.4}$$

From this formulation, we can see that given a non-negative and integer vector x_N, x_B is integer if and only if

$$Nx_N \equiv b \pmod{B}. \tag{1.5}$$

In other words, if $Nx_N - b$ gives some integer combination of the columns of B. This is the traditional *group problem*.

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We can derive an equivalent formulation of the group problem by first multiplying the constraints of (1.4) by the inverse of the basis matrix B:

$$\min\{c_B x_B + c_N x_N | x_B + (B^{-1}N)x_N = B^{-1}b, x_N \geq 0, x_B, x_N \text{ integer}\}. \tag{1.6}$$

From this formulation, we can see that given any non-negative and integer vector x_N, x_B is integer if and only if

$$(B^{-1}N)x_N \equiv B^{-1}b \pmod 1. \tag{1.7}$$

A single row i of (1.7) has the form

$$\sum_{j \in N} \bar{a}_{ij} x_j \equiv \bar{b}_i \pmod 1. \tag{1.8}$$

Because we are taking both sides of (1.8) (mod 1), we only need to include the fractional part of each coefficient $\bar{a}_{ij}$ and $\bar{b}_i$:

$$\sum_{j \in N} \hat{a}_{ij} x_j \equiv \hat{b}_i \pmod 1, \tag{1.9}$$

where $\hat{a}_{ij} \equiv \bar{a}_{ij} \pmod 1$ for all j, and $\hat{b}_i \equiv \bar{b}_i \pmod 1$.

When generating cutting planes for the original integer programming problem, the practical way to find group characters [9] uses the updated rows of the optimal linear programming relaxation tableau as derived above. Subadditive functions on the unit interval can then be used to derive cutting planes for the integer programming problem. However, when the entries of A and b are integral, $D = \det(B)$ is a common denominator for all entries $\hat{a}_{ij}$ and $\hat{b}_i$. Therefore, we can multiply the relation in (1.9) by D to find

$$\sum_{j \in N} (D\hat{a}_{ij}) x_j \equiv (D\hat{b}_i) \pmod D, \tag{1.10}$$

where $(D\hat{a}_{ij})$ and $(D\hat{b}_i)$ are all integers. (1.10) with the added conditions $x_N \geq 0$ and integer, is a *cyclic group problem*. Let $C_n = \{0, 1, ..., n-1\}$ represent the cyclic group of order n. The generic version of a cyclic group problem is

$$\sum_{j \in S} a_j x_j \equiv r \pmod n, \tag{1.11}$$
$$x_S \geq 0, \text{ and integer}$$

where S is a set of variables indices, $a_j \in C_n$ for all $j \in S$ and $r \in C_n$. (1.10) is therefore the group problem over the cyclic group C_D. Typically, only a subset of the elements of C_n are represented in the cyclic group problem derived for an IP. In other words, there are some elements $a \in C_n$ that do not appear as coefficients in (1.10). The master cyclic group problem corresponding to a given cyclic group problem has each group element represented exactly once:

$$\sum_{j=1}^{n-1} jx_j \equiv r \pmod{n}, \tag{1.12}$$

$$x_j \geq 0, \text{ and integer for } j = 1, ..., n-1.$$

Gomory [6] showed that the facets of the convex hull of solutions to the cyclic group problem (1.11) may be obtained from a subset of the facets for the convex hull of solutions to the master problem (1.12) by simply deleting the facet coefficients corresponding to elements that are not present in (1.11). The remaining facets for the master problem give valid inequalities for the cyclic group problem. Therefore, master cyclic group problems can provide cutting planes for any integer program [6–8, 10].

The procedure given by Gomory and Johnson [9] mentioned above for generating cutting planes uses the updated rows of the tableau directly and does not require knowledge of which cyclic group is actually present for a given basis, therefore avoiding the need for exact arithmetic with large integers and the computation of D. This method is based on theory about the infinite group problem over the unit interval modulo 1 [7, 8].

The cut-generation method using master cyclic group facets that we just described is essentially a finite and discrete version of this theory. If the relation of the master cyclic group problem is divided by the size of the group, then the congruence is $\pmod{1}$ instead of $\pmod{n}$ and the group elements may be represented by *grid points* on the unit interval.

There are several intersections of these continuous and discrete theories. For example, classes of seed facets we will develop in later sections for finite master cyclic group problems give extreme subadditive functions [9]. Conversely, extreme subadditive (and piecewise-linear) functions in the continuous interval problem also give facets for the discrete master problem when the break points of the function fall on grid points.

By studying the facets of the discrete master cyclic group problem, we gain intuition and knowledge about extreme subadditive functions for the continuous problem, which leads to new ways to generate cutting planes for general integer programs.

1.2. Structure of paper

We conducted a detailed study of the facets of the master cyclic group polyhedra, as well as the master knapsack partitioning and covering polyhedra. We can now explain many of the facets of these problems using several classes of *seeds* and *mappings*. Seeds are methods that generate facets directly for any given master polyhedra. Mappings take facets from one type of master polyhedra to another, and often give a sequence of facets for infinitely larger master problems.

Gomory and Johnson [7] showed that the number of facets of master cyclic group polyhedra grows exponentially. It is therefore impractical to assume we can explain all facets of master cyclic group polyhedra. However, we may hope to explain the "important" facets of these polyhedra. Gomory [10] developed a *shooting experiment* to try to estimate which facets are largest on the surface of the polyhedra. Using the shooting experiment results, we may now try to explain the facets that we think are most important to the structure of the master polyhedra. In fact, these facets tend to have nicer structure than the facets that appear to be small on the surface of the polyhedra, and we are able to explain many of them. As we introduce classes of facets throughout this paper, we will note those that performed strongly in the shooting experiment.

In the next section, we define the master cyclic group problem and review a subadditive characterization that gives the facets of its polyhedra. The remainder of the paper develops theory and machinery to understand and explain facets and give methods to generate facets for master polyhedra. Section 3 shows that for small problems, all facets may be explained using only a few facets as the foundation for lifting-type methods. For larger problems, the facets explained include those that the shooting experiment shows are relatively important. In sections 4, 6, and 7, two master knapsack problems are introduced and many of their facets are classified. The relationship between these problems and the master cyclic group problem is discussed in section 5, and a method for obtaining cyclic group facets from knapsack facets is introduced.

Appendix B summarizes the classes of facets we developed for the master polyhedra we consider. Essentially, we give a core of ten classes of facets and five different ways to get facets from these classes for other master polyhedra. Tables of all facets for small problems are also given in appendix B. Complete tables of facet-defining inequalities for some larger problems are available at http://www.tli.gatech.edu/AEGJ. An explanation of the double-description method [12] and a parallel implementation that we used to generate the facets may also be found there, and some supplemental proofs are available.

2. The master cyclic group problem

Recall the *master cyclic group problem* of order n with right-hand-side r:

$$\min\{\sum_{i=1}^{n-1} c_i x_i \mid \sum_{i=1}^{n-1} i x_i \equiv r \bmod n, x \geq 0 \text{ and integer}\}. \qquad (C_{n,r})$$

Since we are interested in the convex hull of feasible solutions, we will typically ignore the objective function.

Without loss of generality, we may assume $0 \leq r \leq n-1$. In the case of the *zero-rhs problem* where $r = 0$, we must add the constraint $\sum_{i=1}^{n-1} x_i \geq 1$ to eliminate the trivial solution $x_i = 0$ for $i = 1, ..., n-1$.

The *cyclic group polyhedron* $P(C_{n,r})$ is the convex hull of feasible solutions to the cyclic group problem $C_{n,r}$. For any n and r, the recession cone of $P(C_{n,r})$ is the non-negative orthant of R^{n-1} [6].

Let (π, γ), where π is an $(n-1)$-row vector and γ a scalar, represent the inequality $\sum_{i=1}^{n-1} \pi_i x_i \geq \gamma$. Gomory [6] gave a subadditive characterization of all facet-defining inequalities for $P(C_{n,r})$:

Theorem 2.1 (Non-zero rhs). *For integers r and n, where $1 \leq r < n$, an inequality (π, γ) is facet defining for the cyclic group polyhedron $P(C_{n,r})$ if and only if it is a non-negativity constraint or its coefficients are given by the vectors $\pi \in R^{n-1}$ and $\gamma \in R$*

that represent the extreme rays of the cone

$$S_{n,r} = conv \left\{ \begin{array}{lll} \pi_i \geq 0, & i = 1, ..., n-1 & \textit{(Nonnegativity)} \\ \pi_i + \pi_j \geq \pi_k, & 1 \leq i, j, k < n \textit{ and} & \textit{(Subadditivity)} \\ & (i+j) \equiv k \pmod{n} & \\ \pi_i + \pi_j = \gamma, & 1 \leq i, j < n \textit{ and} & \textit{(Complementarity)} \\ & (i+j) \equiv r \pmod{n} & \\ \pi_r = \gamma & & \end{array} \right\}.$$

Since γ always equals π_r, we will often write the facet-defining inequalities as (π, π_r). However, the zero-rhs problem does not have $\gamma = \pi_r$, so we give its characterization separately:

Theorem 2.2 (Zero rhs). *For integer $n \geq 2$, an inequality (π, γ) is facet defining for the cyclic group polyhedron $P(C_{n,0})$ if and only if it is a non-negativity constraint or its coefficients are given by the vectors $\pi \in R^{n-1}$ and $\gamma \in R$ that represent the extreme rays of the cone*

$$S_{n,0} = conv \left\{ \begin{array}{lll} \pi_i \geq 0 & i = 1, ..., n-1 & \textit{(Nonnegativity)} \\ \pi_i + \pi_j \geq \pi_k & 1 \leq i, j, k < n \textit{ and} & \textit{(Subadditivity)} \\ & (i+j) \equiv k \bmod n & \\ \pi_i + \pi_j = \gamma & 1 \leq i, j < n \textit{ and} & \textit{(Complementarity)} \\ & (i+j) \equiv 0 \bmod n & \end{array} \right\}.$$

The subadditive cones introduced here are contained in the non-negative orthant and are, therefore, pointed. They can be seen to be full dimensional. The facet-defining inequalities given this way are unique subject to multiplication by a constant. Throughout the paper, we will scale inequalities to have all integer coefficients with no common divisors, unless otherwise noted. Also, we will often refer to a facet-defining inequality for a polyhedron as simply a facet of that polyhedron.

3. Master cyclic group facets

Sections 3.2–3.3, except theorem 3.4, review previous results about master cyclic group polyhedra. Section 3.4 gives a new class of facet-defining inequalities for this problem.

3.1. Classifying cyclic group facets

The following definition will be useful throughout the remainder of the paper:

Definition 3.1. *For an inequality (π, γ) and $1 \leq i, j, k \leq n-1$ with $k \equiv (i+j) \bmod n$, we will call $\pi_i + \pi_j \geq \pi_k$ an* **additive relation** *if it is satisfied at equality.*

Note that all complementarity constraints are additive relations. We will use theorems 2.1 and 2.2 to prove an inequality (π, γ) is facet-defining for $P(C_{n,r})$ in two steps:

1. Show (π, γ) satisfies non-negativity, subadditivity, and complementarity
2. Construct $n-2$ linearly independent additive relations satisfied by (π, γ).

3.2. Mappings

Using the group structure of the master cyclic group polyhedra, Gomory [6] gave the following two theorems that use automorphisms and homomorphic liftings to explain facets.

Theorem 3.2 (auto). *For the cyclic group C_n of order n and $0 \leq r, s \leq n-1$, let $\phi : C_n \to C_n$ be an automorphism where $r = \phi(s)$. Then there is a one-to-one mapping of facets from $P(C_{n,r})$ to $P(C_{n,s})$. If*

$$\sum_{i=1}^{n-1} \pi_i x_i \geq \pi_r$$

is a facet for $P(C_{n,r})$, then

$$\sum_{i=1}^{n-1} \pi_i' x_i \geq \pi_s'$$

is the corresponding facet for $P(C_{n,s})$, where $\pi_i' = \pi_{\phi(i)}$.

When the facets are known for $P(C_{n,r})$, the facets for $P(C_{n,\phi(r)})$ for an automorphism ϕ are essentially the same. Thus, table B.2 only lists facets for one of the right hand side elements r in the set of group elements that map onto each other under automorphisms.

Theorem 3.3 (homo). *For integers r, d, and n, $1 \leq r, d < n$, where d divides n but does not divide r, and $s \equiv r \bmod d$, any facet of $P(C_{d,s})$ can be lifted to a facet for $P(C_{n,r})$ using the homomorphism $\psi : C_n \to C_d$ given by $\psi(i) \equiv i \bmod d$. Explicity, if*

$$\sum_{i=1}^{d-1} \pi_i x_i \geq \pi_s$$

is a facet for $P(C_{d,s})$, then the facet for $P(C_{n,r})$ from homomorphic lifting is:

$$\sum_{i=1}^{n-1} \pi_i' x_i \geq \pi_s',$$

where $\pi_i' = \pi_{\psi(i)}$. Here π_0 is considered to be 0.

Essentially the new facet is obtained by cyclically repeating the original facet n/d times. That is, the new facet is:

$$\pi' = (\pi_1, \pi_2, ..., \pi_{d-1}, 0, \pi_1, \pi_2, ..., \pi_{d-1}, 0, ..., 0, \pi_1, \pi_2, ..., \pi_{d-1}).$$

In table B.2, we omit these facets because they are derived from listed facets of smaller groups.

Every lifting of this type will have at least one coefficient among $\pi_1', ..., \pi_{n-1}'$ equal to 0. Gomory [6] also showed the converse theorem: any facet of a master cyclic group polyhedron with some coefficient $\pi'_i = 0$ comes from such a homomorphic lifting. Facets from homomorphic liftings were consistently among the most hit facets in the shooting experiment.

The previous theorem requires that $r > 0$, which means the right hand side is not in the kernel of the homomorphism. The next theorem discusses a different type of homomorphic lifting when the right-hand side is in the kernel of the homomorphism. Instead of setting $\pi'_i = 0$ when $\psi(i) = 0$, we will set $\pi'_i = \sigma_{\frac{i}{d}}$, where (σ, σ_0) is a facet for the zero-rhs problem $P(C_{\frac{n}{d},0})$. Gomory's original paper [6] on the group problem had a lifting result of this type using a "special" facet for the kernel. Gastou [4] had a different version of that result that recognized the role of self-inverse elements. Our result is a strengthening of both results and, empirically, seems to be the strongest possible for cyclic groups.

Theorem 3.4 (0lifting). *For integers r, d, and n, $1 < d < n$, where d divides both n and r, let*

$$\sum_{i=1}^{d-1} \sigma_i x_i \geq \gamma$$

be a facet of $P(C_{d,0})$ such that either

1. *d is odd, or*
2. *d is even and there is some $i \neq \frac{d}{2}$ such that the subadditive relation*

$$\sigma_{\frac{d}{2}} + \sigma_i \geq \sigma_{(\frac{d}{2}+i) \bmod d}$$

holds at equality.

Let

$$\sum_{i=1}^{k-1} \pi_i x_i \geq \pi_s$$

be any facet of $P(C_{k,s})$ where $k = \frac{n}{d}$ and $s = \frac{r}{d}$. Letting $\phi : C_n \to C_d$ denote the homomorphism given by $\phi(i) \equiv i \pmod{d}$, a lifted facet for $P(C_{n,r})$ is:

$$\sum_{i=1}^{n-1} \pi'_i x_i \geq \pi'_s,$$

where

$$\pi'_i = \begin{cases} \alpha\sigma_{\phi(i)} \text{ if } \phi(i) \neq 0 \\ \beta\pi_{i/d} \text{ otherwise} \end{cases}$$

where $\alpha = \frac{\pi_s}{gcd(d,\pi_s)}$ and $\beta = \frac{d}{gcd(d,\pi_s)}$.

The case $r = 0$ is allowed. Essentially the new facet is obtained as in theorem 3.3, and then the 0's are replaced by the appropriate elements of π, giving

$$\pi' = (\alpha\sigma_1, \alpha\sigma_2, ..., \alpha\sigma_{d-1}, \beta\pi_1, \alpha\sigma_1, ..., \alpha\sigma_{d-1}, \beta\pi_2, ..., \beta\pi_{k-1}, \alpha\sigma_1, ..., \alpha\sigma_{d-1}).$$

The lengthy proof of this theorem is in appendix A.

3.3. *Mixed integer cut*

The first cyclic group facet we use as a facet-generating seed is the mixed integer cut *(mic)*. For the polyhedron $P(C_{n,r})$, $r \neq 0$, the mixed integer cut (π, γ) is:

$$\pi = (\frac{1}{r}, ..., \frac{i}{r}, ..., \frac{r-1}{r}, 1, \frac{n-r-1}{n-r}, ..., \frac{n-i}{n-r}, ..., \frac{1}{n-r}).$$

and $\gamma = 1$. This cut is a facet for every cyclic group problem ([8], theorem 3.3). When $r = 0$, the cut is

$$\pi = (\frac{1}{n}, ..., \frac{i}{n}, ..., \frac{n-1}{n}),$$

and again $\gamma = 1$. Mappings given by automorphisms and homomorphisms are powerful tools when used with the mixed integer cut. For example, for $n \leq 7$, every cyclic group facet, except one, is either a mixed integer cut itself or comes from one or more mappings of a mixed integer cut. In every shooting experiment conducted to this point, the most hit facet is either a mixed integer cut itself, or a homomorphic lifting of a mixed integer cut for a smaller group problem, or an automorphism of a mixed integer cut for a problem with a different right hand side. Additionally, all facets in these three sets consistently were among the most hit facets.

3.4. *Patterns for cyclic groups*

We will often refer to the *slopes* and *lines* of a facet (π, γ). Recall that in the defining relation of the non-zero right hand side master cyclic group problem $C_{n,r}$, the coefficients of both x_0 and x_n are zero. Therefore, we can think of the coefficients of π_0 and π_n as equal to zero in a facet (π, γ) of $P(C_{n,r})$. We consider the slope $s(i) = \pi_{i+1} - \pi_i$ to be the difference between two consecutive coefficients π_i and π_{i+1} for $i = 0, ..., n-1$. If we let $S = \{s(i) : 1 \leq i \leq n-2\}$ denote the set of all unique slopes of a facet (π, γ), then we may say (π, γ) has $|S|$ slopes. Let a line $L(i, i')$ refer to a set of consecutive elements from i to i' with constant slope; for example, if $s(i) = s(i+1)$ then $L(i, i+2) = \{i, i+1, i+2\}$ is a line. If $L(i, k) = \{i, i+1, ..., k\}$ is a line that satisfies both conditions:

1. either $i = 0$ or $s(i-1) \neq s(i)$, and
2. either $k = n$ or $s(k) \neq s(k-1)$

then we call $L(i, k)$ a *maximal line*.

The mixed integer cut for $r > 0$ described in section 3.3 has two slopes: $S = \{\frac{1}{r}, -\frac{1}{n-r}\}$, and two maximal lines: $L(0, r) = \{0, ..., r-1, r\}$, and $L(r, n) = \{r, ..., n\}$.

We introduce two classes of cyclic group facets with four maximal lines. The first has two slopes (α, β), and the second has three slopes (α, β, δ). In the theorems, we describe the facets scaled to have $\pi_r = 1$. In our examples and tables, we will give the facets scaled to have all integer coefficients.

Theorem 3.5 below is a discrete version of the two-slope theorem described in theorem 3.3 of [8]. That theorem described a class of piecewise linear functions on the unit interval that have two slopes.

Theorem 3.5 (2slope). *For positive integers n, r, and d, with*

$$\max\{r+1, \lceil \frac{n+1}{2} \rceil\} \le d \le \lfloor \frac{n+r-1}{2} \rfloor\},$$

(π, π_r) defined as follows gives a facet of $P(C_{n,r})$:

$$\pi_i = \begin{cases} \alpha i & i \in J_1 = \{1, ..., r-1\} \\ 1 + \beta(i-r) & i \in J_2 = \{r, ..., d\} \\ \frac{2d-n}{2r} + (i-d)\alpha & i \in J_3 = \{d+1, ..., n+r-d\} \\ (i-n)\beta & i \in J_4 = \{n+r-d+1, ..., n-1\}, \end{cases}$$

where $\alpha = \frac{1}{r}$ and $\beta = \frac{2(r-d)-n}{2r(r-d)}$.

Proof. Complementarity follows from the construction of π, and subadditivity follows from the condition on d, which guarantees that $2\pi_d = \pi_{2d-n}$. There are $r-1$ triangular relations $\pi_1 + \pi_i = \pi_{i+1}$ for $i \in J_1$. Similarly, there are $d-r-1$ triangular relations $\pi_{n-1} + \pi_i = \pi_{i-1}$ for $i \in J_4$. Finally, there are $n-d$ triangular relations $\pi_i + \pi_d = \pi_{i+d}$ for $i = 1, ..., n+r-2d$ and for $i = r, ..., d-1$. With the exception of π_d, all of these relations either contain one element from J_1 and one from J_3, or one from J_2 and one from J_4, so they are independent from the previous relations. □

Example 3.6. $P(C_{12,5})$ has the following 2slope facets:

$$d = 7 : (\pi, \gamma) = ((1, 2, 3, 4, 5, 3, 1, 2, 3, 4, 2), 5)$$
$$d = 8 : (\pi, \gamma) = ((1, 2, 3, 4, 5, 4, 3, 2, 3, 2, 1), 5)$$

The facets described in the following theorem have three slopes and four maximal lines. In the proof, we use lemma 6.2, which we will state and prove in section 6. Again, this theorem gives a special case of a general theorem about functions on the unit interval from Gomory and Johnson [9].

Theorem 3.7 (3slope). *For positive integers n, r, and d, with $r+1 \le d \le \lfloor \frac{n+r}{4} \rfloor$, (π, π_r) defined as follows gives a facet of $P(C_{n,r})$:*

$$\pi_i = \begin{cases} i\alpha & i \in J_1 = \{1, ..., r\} \\ 1 + (i-r)\beta & i \in J_2 = \{r+1, ..., d-1\} \\ i\delta & i \in J_3 = \{d, ..., n+r-d\} \\ (i-n)\beta & i \in J_4 = \{n+r-d+1, ..., n-1\}, \end{cases}$$

where $\alpha = \frac{1}{r}$, $\delta = \frac{1}{n+r}$, and $\beta = \frac{n+r-d}{(n+r)(r-d)}$.

Proof. Complementarity follows from the construction of π, and subadditivity follows from the condition on d, which guarantees $d, 2d \in J_3$, so $2\pi_d = \pi_{2d}$. As in the previous proof, there are $r-1$ triangular relations $\pi_1 + \pi_i = \pi_{i+1}$ for $i \in J_1$. Similarly, there are $d-r-1$ triangular relations $\pi_{n-1} + \pi_i = \pi_{i-1}$ for $i \in J_4$. There are $d-r$ triangular relations $\pi_i + \pi_d = \pi_{i+d}$ for $i = r, ..., d-1$. With the exception of π_d, all of these relations either contain one element from J_2 and one from J_4, so they are independent from our previous relations. Finally, the condition on d guarantees that $3d \le n+r-d$, so by lemma 6.2, the maximal line $L(d, n+r-d)$, gives $n+r-2d$ linearly independent relations with all coefficients in J_3. □

Example 3.8. $P(C_{12,1})$ has the following 3slope facets, scaled to have integer coefficients:

$$d = 2 : (\pi, \gamma) = ((13, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11), 13)$$
$$d = 3 : (\pi, \gamma) = ((13, 8, 3, 4, 5, 6, 7, 8, 9, 10, 5), 13)$$

4. The master knapsack problem

4.1. The master equality knapsack problem

The *master equality knapsack problem* of size n is:

$$\min\{\sum_{i=1}^{n} c_i x_i \,|\, \sum_{i=1}^{n} i x_i = n, x \geq 0 \text{ and integer}\}. \qquad (K_n)$$

Again, we refer to this as the master problem because for each $i = 1, ..., n$, there is a variable x_i with coefficient i in the constraint. The equality knapsack polyhedron $P(K_n)$ is bounded because for each $i = 1, ..., n, 0 \leq x_i \leq \frac{n}{i}$ must be satisfied. Furthermore, the polyhedron has dimension at most $n - 1$ because the defining knapsack equality must be satisfied for every solution. Define a *polytope* to be a bounded polyhedron. For the remainder of this paper, we will refer to the master equality knapsack polytope as simply the knapsack polytope. Most previous knapsack polyhedral studies, such as [11],[3], looked only at 0-1 problems, unlike the knapsack over general integers we are concerned with. It is clear to see that the knapsack polytope $P(K_n)$ has dimension $n - 1$: for i = 2,...,n, $\{x_1 = n - i, x_i = 1, x_j = 0$ for $j \neq 1, i\}$ is a solution, as is $\{x_1 = n, x_j = 0$ for $j = 2, ..., n\}$; and these n solutions are affinely independent.

In section 5 we will show that the master equality knapsack polytopes are actually facets of certain cyclic group polyhedra. Therefore, studying the facets of these polytopes gives us information about these cyclic group polyhedra. In section 6 we will give several classes of facets for this problem.

4.2. The master knapsack packing problem

The *master knapsack packing problem* is:

$$\min\{\sum_{i=1}^{n} c_i x_i \,|\, \sum_{i=1}^{n} i x_i \leq n, x \geq 0 \text{ and integer}\}.$$

However, we do not need to study this problem independently: if we introduce a slack variable s to this inequality, then, x_1 and s are identical in the constraint. Therefore, the master packing knapsack polyhedron is the same as the master equality knapsack polyhedron.

4.3. Subadditive characterization of facets

Let (ρ, ρ_n) refer to the inequality $\sum_{i=1}^{n} \rho_i x_i \geq \rho_n$. The following subadditive characterization of facets (ρ, ρ_n) of the master equality knapsack polytope $P(K_n)$ is given in [1]:

Theorem 4.1. *The facets (ρ, ρ_n) of the knapsack polytope $P(K_n)$ are the extreme rays of the cone*

$$T_{n,r} = \left\{ \begin{array}{lll} \rho_i + \rho_j \geq \rho_{(i+j)}, & 1 \leq i, j, i+j \leq n & \textit{(Subadditivity)} \\ \rho_i + \rho_{n-i} = \rho_n, & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor & \textit{(Complementarity)} \end{array} \right\}.$$

The defining knapsack equation, given by $\rho_i = i$, is a basis for the lineality of this cone.

As with cyclic group facets, we will typically prove an inequality (ρ, ρ_n) gives a facet for $P(K_n)$ by first showing complementarity and subadditivity are satisfied, and then giving $n-2$ linearly independent additive relations.

5. Cyclic group facets from knapsack facets

In section 5.1, we describe the relationship between the master equality knapsack polytope and the master cyclic group polyhedron. Sections 5.2 and 5.3 describe and prove a method for obtaining facets for cyclic group polyhedra from facets of master knapsack polytopes.

5.1. The knapsack polytope is a facet for cyclic group polyhedra $P(C_{n+1,n})$ and $P(C_{n,0})$

The convex hull $P(K_n)$ of master knapsack solutions is a subset of the solutions for the two cyclic group polyhedra $P(C_{n+1,n})$ and $P(C_{n,0})$ because any solution that satisfies the defining knapsack equation must satisfy the corresponding defining congruence for these cyclic group problems.

As discussed in section 3.3, the mixed integer cut

$$x_1 + 2x_2 + ... + nx_n \geq n$$

is a facet for the cyclic group polyhedron $P(C_{n+1,n})$. Every integer solution of the cyclic group problem $C_{n+1,n}$ for which the mixed integer cut holds with equality is a solution to the knapsack problem K_n and visa versa. Thus, the knapsack polytope $P(K_n)$ is precisely the facet of the cyclic group polyhedron $P(C_{n+1,n})$ given by the intersection of the mixed integer cut with the polyhedron $P(C_{n+1,n})$.

Also discussed in section 3.3,

$$x_1 + 2x_2 + ... + (n-1)x_{n-1} \geq n$$

is a facet for the zero-rhs cyclic group problem $C_{n,0}$. It may seem that the knapsack polytope $P(K_n)$ cannot be a face of the polyhedron $P(C_{n,0})$ because the dimensions of the problems are different. However, x_n is involved in the master knapsack problem trivially: it is only positive in the solution $x' = (0, 0, ..., 0, 1)$. Thus, the intersection of $P(K_n)$ with the hyperplane $x_n = 0$ contains all solutions to $K(n)$ except x', and adjoining that vertex may be done by taking the convex combination of it with the vertices in the $x_n = 0$ hyperplane. Thus, the polytope $P(K_n) \cap \{x_n = 0\}$ is a facet of $P(C_{n,0})$.

5.2. *Tilting knapsack facets*

Because the knapsack polytope is not full dimensional, adding a non-zero multiple of the defining knapsack equation $\sum_{i=1}^{n} ix_i = n$ to any facet-defining inequality of $P(K_n)$ gives an equivalent facet. We refer to this operation as *tilting* the facet. Precisely, if (ρ, ρ_n) is a knapsack facet and α is any non-zero constant, then

$$\sum_{i=1}^{n} (\rho_i + i\alpha)x_i \geq \rho_n + n\alpha$$

is an equivalent facet given by tilting. We will refer to multiplying an inequality by a constant as *scaling* the inequality. Note that the tilted facets we will refer to do not necessarily have integer coefficients.

The subadditivity constraints of theorem 4.1 that define the knapsack polytope are a subset of the subadditivity constraints for the description of the cyclic group polyhedra in theorems 2.1 and 2.2. Given a facet for $P(K_n)$, a facet of $P(C_{n+1,n})$ or $P(C_{n,0})$ may be derived by tilting the knapsack facet just enough to satisfy the additional subadditivity constraints. The details and proof are given in the next section.

In this way, every facet of a knapsack polytope is identified with a corresponding tilted facet for $P(C_{n+1,n})$, and the same facet of $P(K_n)$ is identified with a facet for $P(C_{n,0})$. Thus, there is a one-to-one correspondence between a subset of facets for $P(C_{n+1,n})$ and $P(C_{n,0})$ through the polytope $P(K_n)$. When combined with Gomory's automorphic and homomorphic lifting theorems in section 3.2, this construction identifies facets for many cyclic group polyhedra, not just these two cases, as liftings of tilted knapsack facets.

5.3. *Tilting using the mixed integer cut*

We may generalize the idea of tilting so that facets for a knapsack problem $K(r)$ give facets for larger cyclic group problems $C(n, r)$ where $n > r$. For convenience, we will represent the mixed integer cut for $C(n, r)$ as a vector μ defined as follows:

$$\mu_i = \begin{cases} \frac{i}{r} & \text{for } 1 \leq i \leq r \\ \frac{n-i}{n-r} & \text{for } r \leq i \leq n-1 \end{cases}$$

Lemma 5.1. *For n, r such that $1 \leq r < n$, choose an $(n-1)$-vector*

$$\rho = (\rho_1, \rho_2, ..., \rho_r = 1, \frac{n-r-1}{n-r}, ..., \frac{n-i}{n-r}, ..., \frac{1}{n-r})$$

such that the subadditivity and complementarity conditions are satisfied for all pairs $(i, j) \in \{1, ..., r\}^2$ *with* $i + j \leq r$. *Set*

$$\alpha = \max \begin{cases} \frac{r}{n}[\rho_k - \rho_i - \rho_j] & \text{for } 1 \leq i, j, k \leq r \quad (1) \\ \frac{r}{n(k-r)}[-(\rho_i + \rho_j)(n-r) + (n-k)] & \text{for } 1 \leq i, j \leq r \quad (2) \\ & \text{and } r \leq k < n \\ \frac{r}{n}[(\rho_k - \rho_i)\frac{n-r}{n-j} - 1] & \text{for } 1 \leq i, k \leq r \quad (3) \\ & \text{and } r \leq j < n \end{cases}$$

where $k \equiv (i + j) \bmod n$. *Then the inequality* (π, π_r) *defined by*

$$\pi = \rho + \alpha\mu,$$

satisfies all complementarity and subadditivity conditions of $P(C_{n,r})$.

The complete proof may be found at www.tli.gatech.edu/AEGJ. Essentially, choosing α to satisfy these conditions ensures that enough of μ is added to ρ so that all additional cyclic group subadditivity relations are satisfied by π, and at least one new subadditivity relation holds with equality.

Theorem 5.2 (Tilt). *Assume* (ρ, ρ_r) *is a facet of the knapsack polytope* $P(K_r)$ *which is tilted so that* $\rho_i \geq 0$ *for all i and at least one* $\rho_i = 0$, *and scaled so that* $\rho_r = 1$ *(this may be done without loss of generality). For a given* $n > r$, *if the condition in lemma 5.1 gives* $\alpha > 0$, *then* (π, π_r) *constructed as follows is a facet of* $P(C_{n,r})$, *where* $1 \leq r < n$.

1. Extend the vector ρ to the appropriate dimension:

$$\rho = (\rho_1, \rho_2, ..., \rho_r = 1, \frac{n-r-1}{n-r}, \frac{n-r-2}{n-r}, ..., \frac{1}{n-r}).$$

Choose α as in lemma 5.1. Add α times the mixed integer cut for $C(n, r)$ to (ρ, ρ_r).

$$\pi_i = \begin{cases} \rho_i + \alpha\frac{i}{r} & \text{if } 1 \leq i \leq r \\ \rho_i + \alpha\frac{n-i}{n-r} & \text{if } r+1 \leq i \leq n-1 \end{cases}$$

Proof. By lemma 5.1 and the fact that (ρ, ρ_r) is a knapsack facet, we know that (π, π_r) satisfies complementarity and subadditivity. By assumption, it also satisfies non-negativity. (ρ, ρ_r) has $r - 2$ linearly independent additive relations

$$\rho_i + \rho_j = \rho_{i+j}$$

for $i, j, i + j \in \{1, ..., r\}$ given by the knapsack additive relations, and $n - r - 1$ of the form

$$\rho_i + \rho_{n-1} = \rho_{i-1}$$

for $i \in \{r + 1, ..., n - 1\}$ by construction, and these relations are still satisfied by π. The additional additive relation is $\pi_i + \pi_j = \pi_{(i+j) \bmod n}$ for any pair (i, j) that defines α. If this relation was dependent on the others, then it would also be satisfied by ρ and $\alpha = 0$ would be true; therefore, it must be linearly independent. □

The tilting procedure for finding facets of the zero-rhs cyclic group problem is similar, but α is chosen differently because the subadditive constraints for the zero-rhs problem are different.

Lemma 5.3. *For $n \geq 2$, choose an n-vector*

$$\rho = (\rho_1, \rho_2, ..., \rho_n)$$

such that subadditivity and complementarity is satisfied for all pairs $(i, j) \in \{1, ..., n\}^2$ with $i + j \leq n$. Then the inequality (π, π_n) defined by

$$\pi = \rho + \alpha\mu,$$

where μ refers to the mixed integer cut for $C(n, 0)$: $\mu_i = \frac{i}{n}$ for $i = 1, ..., n$ and

$$\alpha = \max\{\rho_{i+j-n} - \rho_i - \rho_j | 1 \leq i, j \leq n - 1 \text{ and } i + j > n\}$$

satisfies all complementarity and subadditivity constraints of $P(C_{n,0})$.

Proof. Complementarity for $P(C_{n,0})$ requires $\pi_i + \pi_{n-i} = \pi_n$ for $1 \leq i \leq n - 1$.

$$\begin{aligned} \pi_i + \pi_{n-i} &= \pi_n \\ \rho_i + \alpha\frac{i}{n} + \rho_{n-i} + \alpha\frac{n-i}{n} &= \rho_n + \alpha \\ \rho_i + \rho_{n-i} &= \rho_n, \end{aligned}$$

which is satisfied by assumption on ρ. Similarly, for $i + j < n$, subadditivity of (π, π_n) follows from subadditivity of ρ. When $i + j > n$, subadditivity is equivalent to:

$$\begin{aligned} \pi_i + \pi_j &\geq \pi_{(i+j) \bmod n} \\ \rho_i + \alpha\frac{i}{n} + \rho_j + \alpha\frac{j}{n} &\geq \rho_{i+j-n} + \alpha\frac{i+j-n}{n} \\ \alpha &\geq \rho_{i+j-n} - \rho_i - \rho_j, \end{aligned}$$

which is satisfied by α given in the lemma. □

Theorem 5.4. *Assume (ρ, ρ_n) is a facet of the knapsack polytope $P(K_n)$, which is tilted so that $\rho_i \geq 0$ for all i and at least one $\rho_i = 0$, and scaled so that $\rho_n = 1$ (this may be done without loss of generality). Then (π, π_n) constructed as in lemma 5.3 is a facet of $P(C_{n,0})$ if $\alpha > 0$.*

Proof. By lemma 5.3, (π, π_n) satisfies complementarity and subadditivity, and non-negativity is satisfied by assumption. Because (ρ, ρ_n) is a knapsack facet, there are $n - 2$ linearly independent additive relations with $i + j \leq n$. The final necessary relation is $\pi_i + \pi_j = \pi_{(i+j) \bmod n}$ for any pair (i, j) that satisfies the definition of α at equality; again, this relation is linearly independent from the others because $\alpha > 0$. □

6. Knapsack facets

Section 6.2 introduces two lemmas that are useful in proving many classes of facets for the master knapsack equality polytope $P(K_n)$. In each of sections 6.2–6.4, a set of related classes of facets for the master knapsack equality problem is given. Unless otherwise noted, all inequalities $\sum_{i=1}^{n} \rho_i x_i \geq \rho_n$ in this section are tilted so that $\rho_n = 0$.

6.1. Linear segments

We first give two lemmas that will be useful in proving later theorems.

Lemma 6.1. *If ρ is a vector of length n, the inequality (ρ, γ) is subadditive, and there is a set $J \subseteq \{1, ..., n\}$ for which there are $|J|-1$ linearly independent additive relations*

$$\rho_i + \rho_j = \rho_{i+j},$$

with $i, j, i+j \in J$, then ρ must be linear on J; i.e. for some constant σ,

$$\rho_j = j\sigma$$

for all $j \in J$.

Proof. A homogeneous system of $n-1$ linearly independent equations in n variables has solution space that is one-dimensional, i.e. every solution is a multiple of some non-zero solution. Since $\rho_j = j \quad \forall j \in J$ is a solution to the set of equations $\{\rho_i + \rho_j = \rho_{i+j} | i, j, i+j \in J\}$, every other solution to this system must be a multiple of it. □

The next lemma is a special case of the converse. It gives conditions sufficient to ensure that (ρ, γ) must be linear on some subset J of the coefficients.

Lemma 6.2. *For a subadditive valid inequality $\sum_{i=1}^{n} \rho_i x_i \geq \gamma$, if there is a set J satisfying either of the following conditions:*

1. *$J = \{j\kappa | j = 1, ..., j'\}$ for some constants j' and κ with $j'\kappa \leq n$, or*
2. *$J = \{j | d \leq j \leq D\}$ where $3d \leq D$,*

where $\rho_j = j\sigma$ for all $j \in J$ and some constant σ, then there are $|J|-1$ linearly independent relations

$$\rho_i + \rho_j = \rho_{i+j},$$

with $i, j, i+j \in J$.

Proof. Case 1: The additive relations $\rho_\kappa + \rho_{j\kappa} = \rho_{(1+j)\kappa}$ for $j = 1, ..., j'-1$ are lower triangular in columns 2 through j' and are therefore linearly independent.
Case 2: We construct d lower triangular relations for columns $d+1, ..., 2d$ as follows:

$$\begin{aligned}
(d+1)\rho_d &= d\rho_{d+1} \\
d\rho_d &= (d-2)\rho_{d+1} + \rho_{d+2} \\
&\vdots \\
(2d+2-i)\rho_d &= (2d-i)\rho_{d+1} + \rho_i \\
&\vdots \\
3\rho_d &= \rho_{d+1} + \rho_{2d-1} \\
2\rho_d &= \rho_{2d}
\end{aligned}$$

Except for the last, these relations are the triangularized form of the simpler additive relations $\rho_{d+1} + \rho_i = \rho_{d+1+i}$. The next to last equation is the sum of the relations:

$$
\begin{array}{llllll}
2\rho_d & & = & & \rho_{2d} & \\
\rho_d & +\rho_{2d} & = & & & +\rho_{3d} \\
 & \rho_{3d} & = & \rho_{d+1} & +\rho_{2d-1} & \\
\hline
3\rho_d & & = & \rho_{d+1} & +\rho_{2d-1} &
\end{array}
$$

Now for $i = d+1, ..., 2d-2$, we may derive the equation for i from that for $i+1$ as follows:

$$
\begin{array}{lllllll}
(2d-i+1)\rho_d & & = & (2d-i-1)\rho_{d+1} & & +\rho_{i+1} & \\
\rho_d & +\rho_{i+1} & = & & & & +\rho_{d+i+1} \\
 & \rho_{d+i+1} & = & \rho_{d+1} & +\rho_i & & \\
\hline
(2d+2-i)\rho_d & & = & (2d-i)\rho_{d+1} & +\rho_i & &
\end{array}
$$

The remaining relations are simply

$$\rho_d + \rho_{i-d} = \rho_i$$

for $i = 2d+1, ..., D$. □

6.2. (1,0,-1) Facets

Theorem 6.3 (1,0,–1). *Choose n and d such that either*

1. *n and d are even and $d \leq \frac{n}{2}$, or*
2. *n is odd and d is even, and $d \leq \frac{n-2}{3}$.*

Then (ρ, ρ_n) defined as follows is a facet for $P(K_n)$:

$$\rho_i = \begin{cases} 1 & \text{for } i < d, \text{ and } i \text{ odd} \\ -1 & \text{for } n-i < d, \text{ and } n-i \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Proof. In both cases, complementarity and subadditivity follow from the definition of ρ. The following constructions give $n-2$ linearly independent additive relations: For case 1, lemma 6.2 with $J_{even} = \{2, 4, ..., n\}$ and $\sigma = 0$ gives $\frac{n}{2} - 1$ relations. To find $\frac{n}{2} - 1$ relations that are independent in columns $J_{odd} = \{3, 5, ..., n-1\}$, we use the following:

$$
\begin{array}{lllllllllll}
0 = & & \rho_2 & +\rho_1 & -\rho_3 & & & & & & \\
0 = & & \rho_4 & +\rho_1 & & -\rho_5 & & & & & \\
\vdots & & & & & \ddots & & & & & \\
0 = & \rho_{d-2} & & +\rho_1 & & & -\rho_{d-1} & & & & \\
0 = & \rho_{2(d+1)} & & & & & & -2\rho_{d+1} & & & \\
0 = & & \rho_2 & & & & & +\rho_{d+1} & -\rho_{d+3} & & \\
\vdots & & & & & & & & \ddots & & \\
0 = & & \rho_2 & & & & & & +\rho_{\frac{n}{2}-3} & -\rho_{\frac{n}{2}-1} & \\
0 = & & \rho_n & & & & & & & -\rho_{\frac{n}{2}-1} & -\rho_{\frac{n}{2}+1} \\
\vdots & & & & & & & & & & \ddots \\
0 = & & \rho_n & -\rho_1 & & & & & & & -\rho_{n-1}
\end{array}
$$

For case 2, we partition indices 3, ..., n into six sets:

$$
\begin{aligned}
J_1 &= \{3, ..., d\} \\
J_2 &= \{d+1\} \\
J_3 &= \{d+2, ..., 2d-1\} \\
J_4 &= \{2d, ..., n-d\} \\
J_5 &= \{n-d+1, ..., n-1\} \\
J_6 &= \{n\}.
\end{aligned}
$$

We now construct a set of relations for i in each set that is almost lower-triangular, and give a simple argument for linear independence. By a set of lower-triangular relations, we mean for a relation i, all coefficients ρ_j for $j > i$ are zero in the relation.

J_1: If i odd: $0 = \rho_1 + \rho_{i-1} - \rho_i$.
If i even: $0 = \rho_2 + \rho_{i-2} - \rho_i$.
J_2: $0 = 2\rho_{d+1} - \rho_{2(d+1)}$ (this relation violates the lower-triangular property, and we will return to it later).
J_3: $0 = \rho_2 + \rho_{i-2} - \rho_i$
J_4: $0 = \rho_d + \rho_{i-d} - \rho_i$
J_5: $0 = \rho_{n-i} + \rho_i + \rho_n$
J_6: $0 = \rho_{\frac{n-1}{2}} + \rho_{\frac{n+1}{2}} - \rho_n$

To make the relations for J_2 lower triangular, we subtract the relations for $i = 2d+2$ and $d+2$ to find the new relation $0 = -\rho_2 - 2\rho_d + 2\rho_{d+1}$. Although the relations for J_5 are not lower triangular, clearly they are linearly independent from the relation for J_6 because they only have the variable ρ_n in common. □

Corollary 6.4. *Given a facet (ρ, ρ_n) of $P(K_n)$ tilted so that $\rho_n = 0$, if $\rho_2 = 0$, then (ρ, ρ_n) must be a $(1, 0, -1)$ facet, up to multiplication by a constant.*

Proof. Assume ρ is scaled so that $\rho_1 = 1$. By subadditivity and $\rho_2 = 0$, $0 \geq \rho_{2k} \geq \rho_{2(k+1)}$ for $k = 1, ..., \lfloor \frac{n}{2} \rfloor - 1$. Also by subadditivity and $\rho_1 = 1$, $1 \geq \rho_{2k-1} \geq \rho_{2k+1}$ for

$k = 1, ..., \lfloor \frac{n}{2} \rfloor - 1$. Using these facts, $\rho_n = 0$, and complementarity, for $1 \le i \le \lfloor \frac{n}{2} \rfloor$, $\rho_i = 0$ when i is odd and $\rho_i \in \{0, 1\}$ when i is even. Furthermore, using these observations and subadditivity again, if $\rho_i = 0$ for $i \le \lfloor \frac{n}{2} \rfloor - 2$ and odd, then $\rho_i + 2 = 0$ also. The remaining coefficients follow from complementarity. □

6.3. *Knapsack inequalities with linear pieces*

In this section, we will use the notion of slopes and maximal lines of inequalities from section 3.4. Theorems 6.5 and 6.7 give facets with 3 maximal lines with positive slope. Theorem 6.8 gives a facet with several maximal lines with positive slope.

Theorem 6.5 (2lin). *For any integers n and k such that $n \ge 4(k + 1)$, (ρ, ρ_n) defined by*

$$\rho_i = \begin{cases} i & \text{for } i = 1, ..., k \\ 0 & \text{for } i = k + 1, ..., n - k - 1 \\ i - n & \text{for } i = n - k, ..., n \end{cases}$$

is a facet for $P(K_n)$.

Proof. Complementarity and subadditivity are clear from construction. Three maximal lines of ρ are

$$L(0, k) = \{0, 1, 2, ..., k\}$$
$$L(k + 1, n - k - 1) = \{k + 1, ..., n - k - 1\}$$
$$L(n - k, n) = \{n - k, ..., n\}$$

Lemma 6.2 with $J_1 = \{1, ..., k\}$ gives $k-1$ additive relations. By the condition on n and k, we may again apply lemma 6.2 with $J_2 = L(k+1)$ for an additional $n-2k-2$ relations. Complementarity gives us k independent relations $\rho_n = \rho_i + \rho_{n-i}$ for $i = n-k, ..., n-1$. Finally, the complementarity relation $\rho_n = \rho_{\lfloor \frac{n}{2} \rfloor} + \rho_{\lceil \frac{n}{2} \rceil}$ is independent from the previous relations because all previous relations with $\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil$ have all coefficients in $L(k + 1, n - k - 1)$, and $n \notin L(k + 1, n - k - 1)$. □

We will use the next lemma in proving theorems 6.7 and 6.8. It essentially says that if certain conditions are satisfied by the linear pieces of an inequality, then that inequality is a facet.

Lemma 6.6. *Assume (ρ, ρ_n) is subadditive and complementary and has the set of maximal lines $L(i_0 = 0, i_1 - 1), L(i_1, i_2 - 1), ..., L(i_m, n)$ such that the slopes of these lines are equal to ρ_1: $s(i_j) = \rho(i_j + 1) - \rho(i_j) = \rho_1$ for $j = 0, ..., m$. Let $K = \{\kappa\lambda | \rho(\kappa\lambda) = \kappa\tau, \kappa = 1, ..., n\}$ be a set of* **knapsack points** *for some constants τ and λ. Define j^* such that $L(i_{j^*}, i_{j^*+1} - 1)$ is the line containing the middle element of ρ: $\lceil \frac{n}{2} \rceil \in L(i_{j^*}, i_{j^*+1} - 1)$.*

If $L(i_j, i_{j+1} - 1) \cap K \ne \emptyset$ for $j = 1, ..., j^* - 1$ and either

1. *$L(i_{j^*}, i_{j^*+1} - 1) \cap K \ne \emptyset$, or*

2. $L(i_{j^*+1}, i_{j^*+2} - 1) \cap K \neq \emptyset$,

then (ρ, ρ_n) *is a facet of* $P(K_n)$.

Proof. By lemma 6.2 with $J = 1, ..., i_1 - 1$ there are $|L(0, i_1 - 1)| - 2$ relations with coefficients within $L(1)$. For the lines $L(i_j, i_{j+1} - 1)$ for $j = 1, .., j^* - 1$, there are $|L(i_j, i_{j+1} - 1)| - 1$ lower triangular relations $\rho(1) + \rho(i) = \rho(i + 1)$ for $i = i_j, ..., i_{j+1} - 1$. These relations hold because of the condition on the slopes of the lines.

When condition 1 holds, define K' to be subset of K which contains exactly one knapsack element for lines $L(i_1, i_2 - 1), ..., L(i_{j^*}, i_{j^*+1} - 1)$, and no knapsack elements from the remaining lines. By lemma 6.2, there are $|K'| - 1 = j^* - 1$ relations within these elements. Because K' contains only one element from each line, these relations are linearly independent from those above that each contain 2 from a line, so we now have

$$\sum_{j=0}^{j^*}(|L(i_j, i_{j+1} - 1)| - 1) - 1 + (j^* - 1) = \sum_{j=0}^{j^*} |L(i_j, i_{j+1} - 1)| - 3$$

linearly independent relations with coefficients in $L(i_0, i_1 - 1) \cup L(i_1, i_2 - 1) \cup ... \cup L(i_{j^*}, i_{j^*+1} - 1)$. There are

$$\sum_{j=j^*+1}^{m} |L(i_j, i_{j+1} - 1)| - 1$$

lower triangular complementarity relations

$$\rho_i + \rho_{n-i} = \rho_n$$

for $i \in \{i_{j^*+1}, i_{j^*+1} + 1, ..., n - 1\}$. Finally, the complementarity relation

$$\rho_{\lfloor \frac{n}{2} \rfloor} + \rho_{\lceil \frac{n}{2} \rceil} = \rho_n$$

is linearly independent from the previous relations because $\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil \in L(i_{j^*}, i_{j^*+1} - 1)$ and $n \notin L(i_0, i_1 - 1) \cup L(i_1, i_2 - 1) \cup ... \cup L(i_{j^*}, i_{j^*+1} - 1)$.

In the case when condition 1 fails, we define K' to contain exactly one knapsack element for lines $L(i_0, i_1 - 1), ..., L(i_{j^*-1}, i_{j^*} - 1)$ and $L(i_{j^*+1}, i_{j^*+2} - 1)$ and use similar construction of relations. □

Theorems 6.7 and 6.8 give cases where the lemma applies when $\rho_n = 0$.

Theorem 6.7 (3lin1slp). *Let* n *and* k *be positive integers such that* $\frac{n-2}{3} \leq k \leq \frac{n-1}{2}$. *Then* (ρ, ρ_n) *defined as follows is a facet of* $P(K_n)$:

$$\rho_i = \begin{cases} i & \text{for } i = 1, ..., k \\ i - \frac{n}{2} & \text{for } i = k + 1, ..., n - k - 1 \\ i - n & \text{for } i = n - k, ..., n \end{cases}$$

Proof. Complementarity follows from construction. Three maximal lines in ρ are

$$\begin{aligned} L(0,k) &= \{0,...,k\} \\ L(k+1,n-k-1) &= \{k+1,...,n-k-1\} \\ L(n-k,n) &= \{n-k,...,n\}. \end{aligned}$$

Subadditivity is clear for pairs (i, j) such that $i, j \in L(0,k)$. The condition on k from the theorem guarantees that $2(k+1) \geq n-k$, so if $i+j \in L(k+1,n-k-1)$ then either $i \in L(0,k)$ or $j \in L(1)$, and subadditivity is satisfied for these pairs. If $i+j \in L(n-k,n)$ then without loss of generality $i \in L(0,k)\setminus\{0\} \cup L(k+1,n-k-1)$ and $\rho_i + \rho_j = i+j-n/2 \geq i+j-n$ or $\rho_i + \rho_j = i+j-n = i+j-n$. Therefore, subadditivity is satisfied.

We may apply lemma 6.6 with $\lambda = 1$ and $\tau = 0$. $j^* = 2$ and $n \in K \cap L(n-k,n)$ satisfy the conditions of the lemma, so (ρ, ρ_n) is a knapsack facet. $\square$

Theorem 6.8 (mod1slp). *Choose n, d, and k such that d divides n, $3 \leq d < \frac{n}{2}$, and $\frac{n-d-1}{2} \leq k \leq \frac{n-1}{2}$. Then (ρ, ρ_n) defined as follows is a facet for $P(K_n)$:*

$$\rho_i = \begin{cases} i \bmod d & \textit{for } i = 1,...,k \\ i - \frac{n}{2} & \textit{for } i = k+1,...,n-k-1 \\ -1 * [(n-i) \bmod d] & \textit{for } i = n-k,...,n \end{cases}$$

Proof. Let $\beta = \max\{i | id \leq k, i = 1,...,k\}$. The maximal lines with positive slope are

$$\begin{aligned} L(0,d-1) &= \{0,...,d-1\} \\ L(d,2d-1) &= \{d,...,2d-1\} \\ &\vdots \\ L(\beta d,k) &= \{\beta d,...,k\} \\ L(k+1,n-k-1) = L(i_{j^*}) &= \{k+1,...,n-k-1\} \\ L(n-k,n-\beta d) = L(i_{j^*+1}, i_{j^*+2}-1) &= \{n-k,...,n-\beta d\} \\ &\vdots \\ L(n-d,n) &= \{n-d+1,...,n\}. \end{aligned}$$

(Notice that if n is odd and $k = \frac{n-1}{2}$, $n-k = k+1$, then $L(k+1,n-k-1)$ is empty). Complementarity follows from construction of ρ. Subadditivity is clear for $1 \leq i, j \leq k$. If $k < j, i+j < n-k$, then by the conditions on d and k, $i < d$. Therefore,

$$\begin{aligned} \rho_i + \rho_j &= i + (j - \frac{n}{2}) \\ &= (i+j) - \frac{n}{2} \\ &= \rho_{i+j} \end{aligned}$$

For $k+1 \leq i, j \leq n-k \leq i+j$, by the conditions on n and d, $i+j > n-d$, so

$$\begin{aligned}\rho_i + \rho_j &= (i - \frac{n}{2}) + (j - \frac{n}{2}) \\ &= i + j - n \\ &= \rho_{i+j}\end{aligned}$$

For $1 \leq i \leq k < j < n-k$ and $n-k \leq i+j \leq n$, the conditions on n, d, and k ensure

$$\min\{\rho_j | k+1 \leq j \leq n-k-1\} \geq \min\{\rho_j | j \geq n-k\}.$$

Also, $\rho_i \geq 0$ for $i \in \{1, ..., k\}$. Thus, subadditivity trivially holds for these pairs (i, j).

Therefore, subadditivity holds for all pairs (i, j). We now apply lemma 6.6 with $\lambda = d$ and $\tau = 0$. There is a knapsack point in each line $L(i_j, i_{j+1} - 1)$ for $i_j = 1, d, 2d, ..., \beta d$ and $L(n-k, n-\beta d)$, so (ρ, ρ_n) is a facet. □

6.4. Facets from cyclic groups

The following theorems show how facets for a knapsack polytope may be obtained from facets for master cyclic group polyhedra in lower dimension.

Theorem 6.9 (cyc). *A facet (ρ, ρ_n) of the knapsack polytope $P(K_n)$ is given from a facet (π, π_r) of the master cyclic group polyhedron $P(C_{d,r})$ by*

$$\rho_i = \pi_k,$$

where $k \equiv i \bmod d$ and $\pi_0 = 0$, provided that

1. *d does not divide n*
2. *$d \leq \frac{n+1}{2}$.*
3. *$r \equiv n \bmod d$.*

Proof. Complementarity and subadditivity follow from the group complementarity and subadditivity of π. Because π is a cyclic group facet, there are $d-2$ linearly independent additive relations $\pi_i + \pi_j = \pi_{(i+j) \bmod d}$ for $1 \leq i, j \leq d-1$. By condition (2), these relations also hold for the knapsack facet as $\rho_i + \rho_j = \rho_{i+j} = \rho_{(i+j) \bmod n}$.

The remaining $n-d$ relations are lower triangular and define each element in terms of the first d elements. For $i = d+1, ..., n$, the relation

$$\rho_i = \rho_{i-d} + \rho_d$$

holds. These relations are linearly independent from the previous relations because no previous relation included ρ_d. □

The following theorem is similar to theorem 6.9. It gives a second class of knapsack facets, tilted so that $\rho_n = 0$, that may be derived from cyclic group facets.

Theorem 6.10 (cyc0).
Choose n and d such that d divides n and $d \leq \frac{n}{4}$. Let (π, π_r) be a facet of $P(C_{d,r})$ for any $1 \leq r \leq d$. Then (ρ, ρ_n) defined as follows is a facet of $P(K_n)$:

$$\rho_i = \begin{cases} \pi_k & \textit{for } i \leq \lfloor \frac{n-1}{2} \rfloor \textit{ and } k \equiv i \bmod d \\ 0 & \textit{for } i = \frac{n}{2} \textit{ if } n \textit{ is even} \\ -\pi_k & \textit{for } i \geq \lceil \frac{n+1}{2} \rceil \textit{ and } k \equiv (n-i) \bmod d, \end{cases}$$

where $\pi_0 = 0$ for convenience.

Proof. Complementarity follows from construction, and subadditivity follows from the cyclic group subadditivity of π.

By lemma 6.2 with $J_1 = \{d, 2d, ..., n\}$, there are $\frac{n}{d} - 1$ additive relations within those columns. For $J_2 = \{d+1, ..., \lfloor \frac{n-1}{2} \rfloor\} \setminus J_1$, the relations

$$\rho_i = \rho_d + \rho_{i-d}$$

hold for $i = \in J_2$ and are lower triangular. Additionally, for $i \in J_3 = \{\lceil \frac{n}{2} \rceil, ..., n-1\} \setminus J_1$, the complementarity relations

$$\rho_n = \rho_{n-i} + \rho_i$$

hold and are independent from all previous relations. We have $n - d$ relations so far. The remaining $d - 2$ additive relations come from the $d - 2$ necessary cyclic group relations

$$\pi_i + \pi_j = \pi_{(i+j) \bmod d}$$
$$\Rightarrow \rho_i + \rho_j = \rho_{(i+j) \bmod d} = \rho_{i+j}$$

by the condition on d that ensures π is repeated at least twice at the beginning of ρ. □

7. Knapsack cover facets

7.1. The master covering knapsack problem

Another related problem is the *master covering knapsack problem* of size n, which is defined as:

$$\min\{\sum_{i=1}^{n} c_i x_i \mid \sum_{i=1}^{n} i x_i \geq n, x \geq 0 \text{ and integer}\}. \qquad (G_n)$$

Section 7.2 gives two subadditive characterizations for facet-defining inequalities of the master knapsack covering problem. Section 7.3 describes the relationship between the covering polyhedra and knapsack equality polytopes. Finally, section 7.4 gives two new classes of facets for this problem.

7.2. Subadditive characterization of covering facets

Denote the convex hull of solutions to the master covering knapsack problem by $P(G_n)$. This polyhedron is unbounded and its recession cone is the non-negative orthant. Aráoz [1] gave two subadditive characterizations of its facet-defining inequalities.

Theorem 7.1. *The facet-defining inequalities (σ, σ_n) of the knapsack covering polyhedron $P(G_n)$ are exactly the extreme rays of the cone*

$$U_{n,r} = \left\{ \begin{array}{rlr} \sigma_i \geq 0, & i = 1, \ldots, n & \textit{(Nonnegativity)} \\ \sigma_i + \sigma_j \geq \sigma_{i+j}, & 1 \leq i, j, i+j \leq n & \textit{(Subadditivity 1)} \\ \sigma_i + \sigma_j \geq \sigma_n, & 1 \leq i, j \leq n < i+j & \textit{(Subadditivity 2)} \\ \sigma_i + \sigma_{n-i} = \sigma_0, & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor & \textit{(Complementarity)} \end{array} \right\}.$$

Theorem 7.2. *The facet-defining inequalities (σ, σ_n) of the knapsack covering polyhedron $P(G_n)$ are exactly the extreme rays of the cone*

$$V_{n,r} = \left\{ \begin{array}{rlr} \sigma_i \geq 0, & i = 1, \ldots, n & \textit{(Nonnegativity)} \\ \sigma_i + \sigma_j \geq \sigma_{i+j}, & 1 \leq i, j, i+j \leq n & \textit{(Subadditivity 1)} \\ \sigma_i \leq \sigma_{i+1}, & 1 \leq i \leq n-1 & \textit{(Monotonicity)} \\ \sigma_i + \sigma_{n-i} = \sigma_0, & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor & \textit{(Complementarity)} \end{array} \right\}.$$

The cones described by the two theorems are actually identical, so either theorem may be used when proving inequalities are facet-defining for the master knapsack covering polyhedron. As with the previous problems, we will show (σ, σ_n) is a facet for $P(G_n)$ by showing it satisfies the necessary complementarity, subadditivity, monotonicity, and non-negativity conditions, and then constructing $n-1$ linearly independent additive relations.

7.3. Relationship with equality knapsack and cyclic group polyhedra

The equality knapsack problem shares a relationship with the covering problem similar to the relationship we showed in section 5 for cyclic group problems. The equality knapsack polytope is a facet of the knapsack cover polyhedron. The facet in question is formed by intersecting the polyhedron with the defining knapsack cover inequality, which also gives a facet for the knapsack cover polyhedron. Every knapsack cover solution that satisfies the defining inequality with equality is a solution to the equality knapsack problem, and visa versa.

The implication is that every equality knapsack facet may be tilted to give a facet for three different polyhedra: two cyclic group polyhedra and the knapsack cover polyhedron. Furthermore, it gives all the facets of the knapsack packing polytope.

There is another relationship between the knapsack cover polyhedron $P(G_n)$ and the two cyclic group polyhedra $P(C_{n+1,n})$ and $P(C_{n,0})$: both polyhedra are contained in $P(G_n)$. Any non-negative solution x to the problem $C_{n,0}$ satisfies

$$\sum_{i=1}^{n-1} i x_i \equiv 0 \bmod n \quad \text{and} \quad \sum_{i=1}^{n-1} x_i > 0,$$

so $\sum_{i=1}^{n-1} ix_i > n$, and x is also a solution to the knapsack covering problem. A similar proof holds for $C_{n+1,n}$.

Given an equality knapsack facet (ρ, ρ_n), note that it satisfies the non-negativity, subadditivity (1), and complementarity conditions of theorem 7.2. Therefore, to find a covering facet by tilting ρ, we must ensure monotonicity is satisfied. $\alpha = \max\{\rho_i - \rho_{i+1} | i = 1, ..., n-1\}$ guarantees that $\rho + \alpha\mu$ gives a facet for $P(G_n)$.

7.4. Inequalities with coefficients 1,...,G

Lemma 7.3. *If a vector $(\sigma_1, ..., \sigma_n = G)$ satisfies complementarity, subadditivity, and monotonicity, and has some i satisfying $\sigma_i = j$ for each $j = 1, ..., G$ then (σ, σ_n) is facet-defining for $P(G_n)$.*

Proof. For $k = 1, ..., G$, let $\kappa_k = \max\{i | \sigma_i = k\}$. By the conditions of the lemma, $\kappa_k < \kappa_{k+1}$ for $k = 1, ..., G-1$. By theorem 7.2, we must find $n-1$ linearly independent additive relations and monotonicity relations at equality. There are $n - G$ monotonicity conditions at holds at equality for $i \neq \kappa_k$:

$$\sigma_i = \sigma_{i+1}.$$

For $i \in \{\kappa_1, ..., \kappa_{G-1}\}$, the additive relation:

$$\sigma_1 + \sigma_{i-1} = \sigma_i \tag{7.1}$$

holds. These $n-1$ relations are lower triangular and clearly linearly independent. □

Theorem 7.4 (allG). *If a vector $(\sigma_1, ..., \sigma_n = G)$ satisfies monotonicity and has some i satisfying $\sigma_i = j$ for each $j = 1, ..., G$ then (σ, σ_n) is facet-defining for $P(G_n)$ if and only if*

1. $\kappa_{i+j} \geq \kappa_i + \kappa_j$ *for* $1 \leq i, j, i+j \leq G$ *and*
2. $\kappa_{G-i} = n - (\kappa_{i-1} + 1)$ *for* $i = 1, ..., \lfloor \frac{G}{2} \rfloor$.

Proof. We will apply lemma 7.3 by showing condition 1 is sufficient for subadditivity and condition 2 is sufficient for complementarity.
Subadditivity: For $1 \leq i, j, i+j \leq n$ and

$$\begin{aligned}\sigma(i) + \sigma(j) &= \sigma(\kappa_{\sigma(i)}) + \sigma(\kappa_{\sigma(j)}) \\ &\geq \sigma(\kappa_{\sigma(i)+\sigma(j)}) \text{ (because } \kappa_{\sigma(i)} + \kappa_{\sigma(j)} \leq \kappa_{\sigma(i)+\sigma(j)}) \\ &\geq \sigma(i+j) \text{ (because monotonicity and } \sigma(i) + \sigma(j) \leq \kappa_{\sigma(i)+\sigma(j)})\end{aligned}$$

Complementarity: The condition $\kappa_{G-i} = n - (\kappa_{i-1} + 1)$ ensures that the first vector element with value i, $\sigma(\kappa_{i-1} + 1)$, is the complementarity pair with the last element with value $G - i$, $\sigma(\kappa_{G-i})$, which implies complementarity. □

Notice that no facets for even n may have odd G if all coefficients are integer. To consider odd G, we look at a class of facets for even n scaled so that $\sigma_1 = 1$ and $\sigma_{\frac{n}{2}}$ is half-integer. When these inequalities are scaled to have all integer coefficients, then all coefficients except $\sigma_{\frac{n}{2}}$ are even, as in V23 of table B.5.

Theorem 7.5 (allGhalf). *For odd G and even n, if $(\sigma_1, ..., \sigma_n = G)$ satisfies monotonicity, has some i satisfying $\sigma_i = j$ for each $j = 1, ..., G$, and $(\sigma_{\frac{n}{2}-1}, \sigma_{\frac{n}{2}}, \sigma_{\frac{n}{2}+1}) = (\frac{G-1}{2}, \frac{G}{2}, \frac{G+1}{2})$, then (σ, σ_n) is facet-defining for $P(G_n)$ if and only if*

1. $\kappa_{i+j} \geq \kappa_i + \kappa_j$ *for* $1 \leq i, j, i + j \leq G$,
2. $\kappa_{G-i} = n - (\kappa_{i-1} + 1)$ *for* $i = 1, ..., \lfloor \frac{G}{2} \rfloor$, *and*
3. $\kappa_1 \geq 2$.

Proof. The proof is the same as the previous theorem, with a minor alteration to lemma 7.3: (7.1) no longer holds for $i = \kappa_{\frac{G}{2}}$ and $\kappa_{\frac{G+1}{2}}$. Instead, use the two relations $2\sigma_{\frac{n}{2}} = \sigma_n$ and $\sigma_2 + \sigma_{\frac{n}{2}-1} = \sigma_{\frac{n}{2}-1}$, which is guaranteed by the third condition of the theorem. □

A. Proof of 0lifting theorem

A.1. Notation

Let C_n^+ denote the set of elements $\{1, 2, ..., n - 1\}$.

Given an inequality (π, γ) scaled so that $\gamma = 1$ for a master cyclic group polyhedron $P(C_{n,r})$, define the matrix

$$M = M(C_{n,r}, \pi, \gamma)$$

corresponding to the set of additive relations of (π, γ). M has a row for each non-zero element $i \in C_n$. If $r = 0$, M also has an n-th row to represent the right hand side element. M has a column $\delta_i + \delta_j - \delta_h$ for each additive relation $\pi_i + \pi_j \geq \pi_h$ satisfied at equality for $i, j \in C_n^+$ and $h \equiv (i + j) \bmod n$ not equal to zero. If $r = 0$, then M also has a column $\delta_i + \delta_{n-i} = \delta_0$ for $i \in C_n^+$. Finally, M has a column δ_i for each zero coefficient $\pi_i = 0$ for $i \in C_n^+$. It is well known that (π, γ) is an extreme ray of $P(C_{n,r})$ if and only if the set $L = \{\lambda | \lambda M(C_{n,r}, \pi, \gamma) = \vec{\mathbf{0}}\}$ has dimension 1, since that implies that (π, γ) satisfies a set of $n - 1$ linearly independent additive relations.

For some d that divides n, let $\phi : C_n \to C_d$ represent the homomorphism defined by $\phi(i) \equiv i \pmod d$. Let

$$K = \{i \in C_n | \phi(i) = 0\} = \{0, d, 2d, ..., n - d\}$$

and, for each $i \in C_n$, let

$$\bar{i} = \{j \in C_n | \phi(j) = \phi(i)\}.$$

(K is the kernel of ϕ and the sets $\bar{i}$ are the cosets in $C_n \setminus K$).

A.2. The theorem

The following is a restatement of theorem 3.4, altered so that all facets (π, γ) are scaled to have $\gamma = 1$. For simplicity of notation, we will sometimes refer to the coefficient π_i as $\pi(i)$.

Theorem A.1 (0lifting). *Let*

$$\sum_{i=1}^{k-1} \pi_i^1 x_i \geq 1$$

be any facet of $P(C_{k,s})$, and let

$$\sum_{i=1}^{d-1} \pi_i^2 x_i \geq \gamma^2$$

be any facet of $P(C_{d,0})$ such that either

- *d is odd, or*
- *there is some additive relation $\pi^2_{\frac{n}{2}} + \pi^2_j = \pi^2_h$ where $\frac{n}{2} + j \equiv h \pmod d$ and neither j nor h is equal to $\frac{d}{2}$.*

Let $n = kd$, $r = sd$, and $K = \{0, d, 2d, ..., (k-1)d\}$. Define the $(n-1)$-length vector π as follows:

$$\pi_i = \begin{cases} \pi^1(\frac{i}{d}) & \text{if } i \in K \setminus \{0\} \\ \pi^2(i \bmod d) & \text{if } i \notin K \end{cases}$$

Then (π, π_r) is a facet of $P(C_{n,r})$.

Proof. Define the homomorphism $\phi : C_n \to C_d$, the sets $\bar{i}$ for $i \in C_n$, and the matrix $M = M(C_{n,r}, \pi, \gamma)$ as in the previous section.

We will prove the theorem in three steps:

1. Show (π, π_r) satisfies complementarity, subadditivity, and non-negativity.
2. Show that $\lambda M = 0$ implies that for any $i_0 \notin K$, $\lambda(i_0) = \lambda(i)$ for any $i \in \bar{i_0}$.
3. Show that (1) and (2) imply that (π, π_r) is a facet of $P(C_{n,r})$.

1. π^1 and π^2 must satisfy non-negativity because they are both facets of master cyclic group polyhedra. Therefore, by construction, π satisfies non-negativity.

To show complementarity, choose any two elements $i, j \in C_n^+$ so that $(i + j) \equiv r \pmod n$. Because $r \in K$, either both i and j are in K or neither is. If both are, then

$$\begin{aligned} \pi(r) &= \pi(sd) \\ &= \pi^1(s) \\ &= \pi^1(\frac{i}{d}) + \pi^1(\frac{j}{d}) \\ &= \pi(i) + \pi(j) \end{aligned}$$

because $i + j \equiv r \pmod n$ implies $\frac{i}{d} + \frac{j}{d} \equiv s \pmod k$, and π^1 satisfies complementarity for $P(C_{k,s})$. If neither i nor j is in K, then

$$\begin{aligned} \pi(r) &= 1 \\ &= \gamma^1 \\ &= \pi^2(i \bmod d) + \pi^2(j \bmod d) \\ &= \pi(i) + \pi(j) \end{aligned}$$

The third equality follows from the fact that $i + j \equiv sd \pmod{kd}$ implies that $i + j \equiv 0 \pmod{d}$.

To show subadditivity, choose any triple $(i, j, h) \in (C_n^+)^3$ so that $i+j \equiv h \pmod{n}$.

If $i, j \in K$, then h must also be in K, and subadditivity follows from the subadditivity of π^1 using an argument similar to the complementarity argument. Similarly, if $i, j, h \notin K$, then subadditivity follows from the subadditivity of π^2.

If $i \in K$ and $j, h \notin K$, then $((i + j) \bmod kd) \bmod d \equiv (i + j) \bmod d \equiv j \bmod d$. Therefore

$$\begin{aligned}
\pi(h) &= \pi((i + j) \bmod kd) \\
&= \pi^2(((i + j) \bmod kd) \bmod d) \\
&= \pi^2(j \bmod d) \\
&\leq \pi^1(\frac{i}{d}) + \pi^2(j \bmod d) \text{ (by non-negativity of } \pi^1) \\
&= \pi(i) + \pi(j).
\end{aligned}$$

If $i, j \notin K$ and $h \in K$, then there is some $j' \equiv j \pmod{d}$ such that $(i + j') \equiv r \pmod{n}$. Therefore

$$\begin{aligned}
\pi(i) + \pi(j) &= \pi(i) + \pi(j') \\
&= \pi(r) \\
&\geq \pi(h).
\end{aligned}$$

Any other triples (i, j, h) are either impossible or equivalent to those listed above. Therefore, (π, π_r) satisfies subadditivity, and (1) is proven.

2. Let $\lambda \in R_+^{n-1}$ satisfy $\lambda M = 0$. Choose any $i_0 \in C_n \setminus K$. By the condition of the theorem and complementarity of π_2, there exists some j_0, h_0 such that $\pi(i_0) + \pi(j_0) = \pi(h_0)$ where $(i_0 + j_0) \equiv h_0 \pmod{d}$ and i_0 is not equal to either j_0 or h_0.

For any $i \in \bar{i}_0$, $j \in \bar{j}_0$, $h \equiv (i + j) \bmod d \in \bar{h}_0$

$$\pi(i) + \pi(j) - \pi(h) = \pi(i_0) + \pi(j_0) - \pi(h_0) = 0,$$

so

$$\lambda(i) + \lambda(j) - \lambda(h) = 0.$$

Furthermore, for any element $\kappa \in K$ and positive integer α,

$$\lambda((i + \alpha\kappa) \bmod n) + \lambda(j) = \lambda((h + \alpha\kappa) \bmod n)$$

and

$$\lambda((i + \alpha\kappa) \bmod n) + \lambda((j + \kappa) \bmod n) = \lambda((h + \alpha\kappa + \kappa) \bmod n).$$

By subtracting the second equality from the first, we find

$$\lambda(j) - \lambda((j + \kappa) \bmod n) = \lambda(h + \alpha\kappa) - \lambda((h + \alpha\kappa + \kappa) \bmod n).$$

Choose an integer $\mu > 0$ so that $\mu\kappa = 0$. Such a μ must exist because $\kappa \in K$ and K is a finite group (it is isomorphic to the cyclic group C_k). Then

$$\begin{aligned}
&\sum_{\alpha=0}^{\mu-1} \lambda((h+\alpha\kappa) \bmod n) - \lambda((h+(1+\alpha)\kappa) \bmod n) \\
&\quad = \lambda(h) - \lambda((h+\mu\kappa) \bmod n) \\
&\quad = 0.
\end{aligned}$$

Therefore,

$$\sum_{\alpha=0}^{\mu-1} (\lambda(j) - \lambda((j+\kappa) \bmod n)) = 0,$$

which implies $\lambda(j) = \lambda((j+\kappa) \bmod n)$ for all $\kappa \in K$, which proves (2).

3. Let $L = \{\lambda \in R^{n-1} | \lambda M = 0\}$, $\bar{L} = \{\bar{\lambda} \in R^{k-1} | \bar{\lambda} M(C_{k,s}, \pi^1, \gamma^1) = 0\}$, and $\hat{L} = \{\hat{\lambda} \in R^d | \hat{\lambda} M(C_{d,0}, \pi^2, \gamma^2) = 0\}$. Because (π^1, γ^1) is a facet of $P(C_{k,s})$, $dim(\bar{L}) = 1$. Similarly, $dim(\hat{L}) = 1$.

Choose any $\lambda \in L$. Define $\bar{\lambda}(i) = \lambda(di)$ for $i = 1, ..., k-1$. For every column $\delta_{i_0} + \delta_{j_0} - \delta_{h_0}$ in $M(C_{k,s}, \pi^1, \gamma^1)$, there is a column $\delta_{di_0} + \delta_{dj_0} - \delta_{dh_0}$ in M by definition of π. Therefore, $\bar{\lambda}(i_0) + \bar{\lambda}(j_0) - \bar{\lambda}(h_0) = 0$, so $\bar{\lambda} \in \bar{L}$.

Similarly, define $\hat{\lambda}(i) = \lambda(i)$ for $i = 1, ..., d-1$, and $\hat{\lambda}(d) = \lambda(r)$. For every column $\delta_{i_0} + \delta_{j_0} - \delta_{h_0}$ in $M(C_{d,0}, \pi^2, \gamma^2)$, there is a column $\delta_i + \delta_j - \delta_h$ in M for some $i \in \bar{i}_0$, $j \in \bar{j}_0$, and $h \in \bar{h}_0$. Using (2), this implies $\hat{\lambda}(i_0) + \hat{\lambda}(j_0) - \hat{\lambda}(h_0) = 0$, so $\hat{\lambda} \in \hat{L}$.

Therefore, $\lambda' \in L$ must have the form

$$\lambda'(i) = \begin{cases} \alpha\lambda(i) & \text{for } i \in K \setminus \{0\} \\ \beta\lambda(i) & \text{for } i \notin K. \end{cases}$$

But then $\hat{\lambda}'(i) = \beta\hat{\lambda}(i)$ for $i = 1, ..., d-1$, and $\hat{\lambda}'(d) = \alpha\hat{\lambda}(d)$. Since $dim(\hat{L}) = 1$, this implies $\alpha = \beta$. Therefore, $dim(L) = 1$, which proves (π, π_r) is a facet of $P(C_{n,r})$.

B. Summary of classes: seeds and mappings

Table B.1 summarizes the seeds and mappings used for explaining facets of cyclic group and knapsack problems. Tables B.2-B.5 list facets for some small problems. Each facet is given an identification number, which is listed in the first column. If a facet belongs to more than one class, the most simple class is listed. Cyclic group facets explained by homomorphisms are not included in table B.2. If a group or covering facet may be derived by tilting a knapsack equality facet, the identification number of the knapsack equality facet is listed.

Table B.1. Summary of facet classes

SEEDS

Abbrev	For	Condition	Reference
mic	$P(C_{n,r})$	None	Section 3.3
2slope	$P(C_{n,r})$	None	Theorem 3.5
3slope	$P(C_{n,r})$	$r \leq \frac{n-4}{3}$	Theorem 3.7
1,0,-1	$P(K_n)$	n even or $n \geq 7$	Theorem 6.3
2lin	$P(K_n)$	$n \geq 8$	Theorem 6.5
3lin1slp	$P(K_n)$	$n \geq 5$	Theorem 6.7
mod1slp	$P(K_n)$	$n \geq 5$	Theorem 6.8
allG	$P(G_n)$	None	Theorem 7.4
allGhalf	$P(G_n)$	n even	Theorem 7.5

MAPPINGS

Abbrev	From	To	Condition	Reference
auto (a)	$P(C_{n,r})$	$P(C_{n,s})$	r, s in same coset of C_n	Theorem 3.2
homo	$P(C_{d,r})$	$P(C_{n,s})$	$d\|n$ and $s \equiv r \pmod{d}$	Theorem 3.3
0lifting	$P(C_{d,0})$, $P(C_{n/d,r/d})$	$P(C_{n,r})$	$d\|n$, $d\|r$	Theorem 3.4
cyc	$P(C_{d,r})$	$P(K_n)$	$d \leq \frac{n+1}{2}$, $d \nmid n$, $r \equiv n \pmod{d}$	Theorem 6.9
cyc0	$P(C_{d,r})$	$P(K_n)$	$d \leq \frac{n}{4}$ and $d\|n$	Theorem 6.10
tilt	$P(K_r)$	$P(C_{n,r})$	$n \geq r+1$	Theorem 5.2
tilt	$P(K_r)$	$P(C_{r,0})$	None	Theorem 5.4
tilt	$P(K_r)$	$P(G_r)$	None	Section 7.3

Table B.2. Non-zero rhs cyclic group facets $\sum_{i=1}^{n-1} \pi_i x_i \geq \pi_r$

Id #	n	r	Class	Tilt From	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_r
C1	2	1	mic		1								1
C2	3	2	mic		1	2							2
C3	4	2	mic		1	2	1						2
C4	4	3	mic		1	2	3						3
C5	5	4	mic	K1	1	2	3	4					4
C6	5	4	a. mic	K2	4	3	2	6					6
C7	6	2	mic		2	4	3	2	1				4
C8	6	3	mic		1	2	3	2	1				3
C9	6	3	2slope		1	2	3	1	2				3
C10	6	3	2slope		2	1	3	2	1				3
C11	6	5	mic		1	2	3	4	5				5
C12	7	6	mic	K2	1	2	3	4	5	6			6
C13	7	6	a. mic	K5	4	8	5	2	6	10			10
C14	7	6	3slope	K1	6	5	4	3	2	8			8
C15	7	6	a. mic	K6	9	4	6	8	3	12			12
C16	8	2	mic	K6	3	6	5	4	3	2	1		6
C17	8	2	a. mic	K6	3	6	1	4	3	2	5		6
C18	8	4	mic		1	2	3	4	3	2	1		4
C19	8	4	a. mic		3	2	1	4	1	2	3		4
C20	8	4	2slope	K2	1	2	3	4	1	2	3		4
C21	8	4	2slope	K2	3	2	1	4	3	2	1		4
C22	8	7	mic		1	2	3	4	5	6	7		7
C23	8	7	a. mic		9	10	3	12	5	6	15		15
C24	8	7	a. 2slope	K10	1	2	1	2	1	2	3		3
C25	8	7	a. 2slope	K8	3	2	1	4	3	2	5		5
C26	8	7	3slope	K3	7	6	5	4	3	2	9		9
C27	9	3	mic		2	4	6	5	4	3	2	1	6
C28	9	3	a. mic		2	4	6	2	1	3	5	4	6
C29	9	3	a. mic		5	1	6	2	4	3	2	4	6
C30	9	3	a. 2slope	K1	4	8	12	7	2	6	10	5	12
C31	9	3	2slope	K1	7	5	12	10	8	6	4	2	12
C32	9	3	a. 2slope	K1	10	2	12	4	5	6	7	8	12
C33	9	3	0lifting	K6	2	4	6	2	4	3	2	4	6
C34	9	3	0lifting		4	2	6	4	2	3	4	2	6
C35	9	8	mic	K7	1	2	3	4	5	6	7	8	8
C36	9	8	a. 2slope	K3	2	1	3	2	1	3	2	4	4
C37	9	8	a. mic	K12	4	8	12	7	2	6	10	14	14
C38	9	8	3slope		8	7	6	5	4	3	2	10	10
C39	9	8	tilt 1,0,-1	K4	11	4	6	8	10	12	5	16	16
C40	9	8	a. mic		16	5	12	10	8	15	4	20	20

Table B.3. Zero rhs cyclic group facets $\sum_{i=1}^{n-1} \pi_i x_i \geq \gamma$

Id #	n	Class	Tilt From	π_1	π_2	π_3	π_4	π_5	π_6	π_7	γ
Z1	3	mic		1	2						3
Z2	3	auto mic	K1	2	1						3
Z3	4	mic		1	2	3					4
Z4	4	auto mic	K2	3	2	1					4
Z5	5	mic		1	2	3	4				5
Z6	5	auto mic	K4	2	4	1	3				5
Z7	5	auto mic	K3	3	1	4	2				5
Z8	5	auto mic		4	3	2	1				5
Z9	6	mic		1	2	3	4	5			6
Z10	6	auto mic		5	4	3	2	1			6
Z11	6	tilt	K5	2	4	3	2	4			6
Z12	6	tilt	K6	4	2	3	4	2			6
Z13	7	mic		1	2	3	4	5	6		7
Z14	7	auto mic	K9	2	4	6	1	3	5		7
Z15	7	auto mic	K10	3	6	2	5	1	4		7
Z16	7	auto mic	K7	4	1	5	2	6	3		7
Z17	7	auto mic	K8	5	3	1	6	4	2		7
Z18	7	auto mic		6	5	4	3	2	1		7
Z19	8	tilt	K12	1	2	3	2	1	2	3	4
Z20	8	auto tilt		3	2	1	2	3	2	1	4
Z21	8	mic		1	2	3	4	5	6	7	8
Z22	8	auto mic	K11	3	6	1	4	7	2	5	8
Z23	8	auto mic	K13	5	2	7	4	1	6	3	8
Z24	8	auto mic		7	6	5	4	3	2	1	8
Z25	8	auto tilt		3	6	5	4	3	2	5	8
Z26	8	tilt	K14	5	2	3	4	5	6	3	8

Table B.4. Knapsack equality facets $\sum_{i=1}^{n} \rho_i x_i \geq \rho_n$

Id #	n	Class	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9	ρ_n
K1	3	cyc	1	0	1							1
K2	4	1,0,-1	2	1	0	2						2
K3	5	cyc	1	0	1	0	1					1
K4	5	cyc	1	2	0	1	2					2
K5	6	3lin1slp	1	2	1	0	1	2				2
K6	6	1,0,-1	6	2	3	4	0	6				6
K7	7	cyc	1	0	1	0	1	0	1			1
K8	7	cyc	2	1	0	2	1	0	2			2
K9	7	cyc	1	2	3	0	1	2	3			3
K10	7	3lin1slp	2	4	1	3	0	2	4			4
K11	8	cyc	1	2	0	1	2	0	1	2		2
K12	8	3lin1slp	2	4	6	3	0	2	4	6		6
K13	8	1,0,-1	3	1	4	2	0	3	1	4		4
K14	8	1,0,-1	8	2	3	4	5	6	0	8		8
K15	9	cyc	1	0	1	0	1	0	1	0	1	1
K16	9	cyc	3	2	1	0	3	2	1	0	3	3
K17	9	cyc	1	2	3	4	0	1	2	3	4	4
K18	9	cyc	4	3	2	6	0	4	3	2	6	6
K19	9	3lin1slp	1	2	3	1	2	0	1	2	3	3
K20	9	mod1slp	3	6	2	5	1	4	0	3	6	6
K21	9	mod1slp	6	12	4	3	9	8	0	6	12	12
K22	9	1,0,-1	9	2	3	4	5	6	7	0	9	9

Table B.5. Knapsack covering facets $\sum_{i=1}^{n-1} \sigma_i x_i \geq \sigma_n$

Id #	n	Class	Tilt From	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_n
V1	3	allG		1	1						2
V2	3	allG		1	2						3
V3	4	allG		1	1	1					2
V4	4	allG		1	2	3					4
V5	5	allG		1	1	1	1				2
V6	5	allG	K3	1	1	2	2				3
V7	5	allG	K4	1	2	2	3				4
V8	5	allG		1	2	3	4				5
V9	6	allG		1	1	1	1	1			2
V10	6	allG	K5	1	2	2	2	3			4
V11	6	allG		1	2	3	4	5			6
V12	6	allGhalf	K6	2	2	3	4	4			6
V13	7	allG		1	1	1	1	1	1		2
V14	7	allG	K8	1	1	1	2	2	2		3
V15	7	allG	K7	1	1	2	2	3	3		4
V16	7	allG		1	2	2	2	2	3		4
V17	7	allG	K10	1	2	2	3	3	4		5
V18	7	allG	K9	1	2	3	3	4	5		6
V19	7	allG		1	2	3	4	5	6		7
V20	8	allG		1	1	1	1	1	1	1	2
V21	8	allG	K13	1	1	2	2	2	3	3	4
V22	8	allG		1	2	2	2	2	2	3	4
V23	8	allG	K11	1	2	2	3	4	4	5	6
V24	8	allG	K12	1	2	3	3	3	4	5	6
V25	8	allGhalf		2	2	2	3	4	4	4	6
V26	8	allG		1	2	3	4	5	6	7	8
V27	8	tilt	K14	2	2	3	4	5	6	6	8

References

1. Aráoz, J.: Polyhedral Neopolarities. PhD thesis, University of Waterloo, Department of Computer Sciences, 1974
2. Balas, E., Ceria, S., Cornuejols, G., Natraj, N.: Gomory cuts revisited. Operations Research Letters **19**, 1–9 (1996)
3. Balas, E.: Facets of the knapsack polytope. Mathematical Programming **8**(2), 146–164 (1975)
4. Gastou, G.: On Faces of Integer Programming Polyhedra. PhD thesis, Yale University, Administrative Sc. Dept., 1982
5. Gilmore, P.C., Gomory, R.E.: The theory and computation of knapsack functions. Operations Research **14**(6), 1045–1074 (1966)
6. Gomory, R.E.: Some polyhedra related to combinatorial problems. Linear Algebra and Applications **2**, 451–558 (1969)
7. Gomory, R.E., Johnson, E.L.: Some continuous functions related to corner polyhedra, I. Mathematical Programming **3**, 23–85 (1972)
8. Gomory, R.E., Johnson, E.L.: Some continuous functions related to corner polyhedra, II. Mathematical Programming **3**, 359–389 (1972)
9. Gomory, R.E., Johnson, E.L.: T-space and cutting planes. Mathematical Programming the same issue 2003
10. Gomory, R.E., Johnson, E.L., Evans, L.: Corner polyhedra and their connection with cutting planes. Mathematical Programming the same issue 2003
11. Hammer, P.L., Johnson, E.L., Peled, U.N.: Facets of regular 0-1 polytopes. Mathematical Programming **8**(2), 179–206 (1975)
12. Motzkin, T.S., Raiffa, H., Thompson, G.L., Thrall, R.M.: The double description method. In: H.W. Kuhn, A.W. Tucker, editors. Contributions to the Theory of Games, Vol. 2, number 28 in Annals of Mathematics Studies, pages 51–73, Princeton, N.J., 1953. Princeton University.