Osborne Reynolds at the age of ~24 (ca. 1866).

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NOTE ON THE HISTORY OF THE REYNOLDS NUMBER

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1. NAME ORIGINS

In 1908, Arnold Sommerfeld presented a paper on hydrodynamic stability at the 4th International Congress of Mathematicians in Rome (Sommerfeld 1908). In the equation known today as the Orr-Sommerfeld equation, he introduced a number R as "eine reine Zahl, die wir die Reynolds'sche Zahl nennen wollen." (freely translated: "R is a pure number; we will call it the Reynolds number.") The terminology introduced by Sommerfeld has not changed ever since, and the use of the expression "Reynolds number" has spread into all branches of fluid mechanics.

Actually Sommerfeld's work is not foremost in one's mind when one thinks of the direct continuation of the ideas set forth by Osborne Reynolds in 1883 (Reynolds 1883). This is probably the reason that Sommerfeld's use of the expression "Reynolds number" was generally forgotten before von Kármán (1954) drew attention to it in his book *Aerodynamics: Selected Topics in the Light of Their Historical Development*. He referred there to work by Sommerfeld from the year 1908 but did not mention the title and the place of publication. Von Kármán returned to this subject in his paper published in the Albert Betz anniversary issue of the *Zeitschrift für Flugwissenschaften* (von Kármán 1956). Unfortunately the reference there is to Sommerfeld's (1904) paper on the Reynolds theory of lubrication. In that work the inertial terms are neglected, and the notion of the Reynolds number is neither needed nor used. While it is not difficult to reconstruct the facts by the use of von Kármán's book, it may be useful, nevertheless, to have these sources collected and recollected here again.

I also wish to correct and to explain an incorrect statement that I made on this subject in an earlier article in this series (Rott 1985). There I stated

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that the expression "Reynolds number" was introduced by Prandtl. This remark was based on personal recollections of conversations with Jakob Ackeret. Given the fallibility of memories that are not supported by tangible evidence, this remark should never have been made. Thanks are due to numerous colleagues and friends who drew my attention to the facts uncovered by von Kármán. However, I have decided to investigate, after having made the allegations, the actual role of Prandtl in the history of the Reynolds number. This has led also to studies of the early history of the notion and notation before Sommerfeld and Prandtl. A summary of my findings follows.

Many people in fluid mechanics, if asked to guess, would be inclined to attribute the expression "Reynolds number" to Blasius. His work is the first that is fully devoted to the extraction from experiments of a function that, according to Reynolds' similarity law, depends only on the Reynolds number. The reader consulting the originals will find that Blasius actually uses the expression "Reynolds number," but only twice in the first publication of his results (Blasius 1912, pp. 640, 642). In the subsequent extended VDI report, he uses the expressions again in the corresponding passages (Blasius 1913, pp. 7, 9).

Prandtl's role in the history of the Reynolds number, as revealed in his collected works, begins with his early paper on the Reynolds analogy (Prandtl 1910). There he introduces "die in der Hydrodynamik bekannte Reynolds'sche Zahl" but calls it later simply ξ. The words "known in hydrodynamics" refer to the general field. Thus it cannot be ascertained whether Prandtl has chosen his words following Sommerfeld or some other source known to him, or whether he made this choice independently. On the other hand, Prandtl's influence on Blasius is highly probable.

In his encyclopedia article on "Flüssigkeitsbewegung," Prandtl (1913) writes more deliberately: "Die vorstehende Grösse, eine dimensionslose Zahl, wird nach dem Entdecker dieses Ähnlichkeitsgesetzes, Osborne Reynolds, die Reynolds'sche Zahl genannt." ("The forementioned quantity, a nondimensional number, is named after the discoverer of this similarity, Osborne Reynolds, [and is called] the Reynolds number".) I am indebted to Professor Itiro Tani for pointing out this passage (in Volume 3, p. 1445 of the collected works), as well as for other information used in this note.

A study of Prandtl's collected works also shows the crucial role that he and his coworkers in Göttingen played in the application of hydrodynamic similarity to the drag problem. In the years around 1912, an international dispute erupted over the drag of spheres, with Prandtl and Eiffel (who made the famous drop experiments from his tower) as the main protagonists. In 1913, Lord Rayleigh noted briefly that this discussion should be considered in the light of hydrodynamic similarity (Rayleigh 1913). He credited Stokes before Reynolds with the discovery of this notion. Stokes indeed noted the similarity properties of his basic equations of viscous flow together with their derivation in 1850. The resolution of the drag problem, however, called for the specific similarity law established by Reynolds (1883) for the pressure drop in pipes, which is also applicable—*mutatis mutandis*—to the drag problem. Interestingly, it was Lord Rayleigh who made what is almost certainly the first reference ever to Reynolds' similarity law, in the introduction to his paper "On the question of the stability of the flow of fluids" (Rayleigh 1892). (This is discussed later.)

The resolution of the sphere drag problem was completed by Prandtl in his paper of 1914, where he introduced a new concept that complemented his boundary-layer theory—namely, the idea of the transition of the boundary layer from laminar to turbulent at a *critical* Reynolds number (Prandtl 1914). Prandtl made here no more explanatory comments on the usage of this word. After his paper it became common knowledge that the drag coefficient depends on the Reynolds number, a term that became a household word in aerodynamics and aeronautics. The general acceptance of the term *and* of the notion came much later in hydraulic engineering, in spite of the fact that a textbook on hydraulics by von Mises appeared in 1914 that fully exploited Reynolds similarity (albeit without using the term "Reynolds number"). The memory of this book (von Mises 1914), which is now largely forgotten, is kept alive in Rouse & Ince's *History of Hydraulics* (Rouse & Ince 1957).

After World War I, a treatise on similarity and its use for model experiments appeared in the *Jahrbuch der Schiffbautechnischen Gesellschaft* by Moritz Weber, Professor of Naval Architecture in Berlin (Weber 1919). As noted by Rouse & Ince, Weber not only put the Reynolds number to use but also introduced the Froude number and a new number involving capillarity, which later was named for him. This was apparently the beginning of a new era in the use of names for nondimensional numbers.

Reference has been made here repeatedly to the seminal paper of Reynolds (1883). There he gave, by dimensional analysis, unprecedented applications that led to specific results: He introduced the notion of the critical Reynolds number and established the similarity law for the pressure drop in pipes. The early history of these ideas is discussed in the following sections.

2. REYNOLDS' FLOW-VISUALIZATION EXPERIMENTS

Reynolds gave a visual demonstration of the transition from laminar to turbulent flow in a pipe, using an experimental setup that is still popular today. Figure 1, which is reproduced from Reynolds (1883), shows an artist's concept of the original device. An elevated platform permits the use of a siphon that is high enough to reach the critical velocities; the valve at the exit is manipulated by a lever that extends to the platform. Water is drawn from a glass-walled box into a glass tube, together with a filament of dye. The tube has a trumpet-shaped inlet. Readings of the water level, accurate to 1/100 of an inch, were used to determine the velocity.

The dye filaments at transition are reproduced in many texts, using Reynolds' original drawings; but as his apparatus is still in existence at the University of Manchester, modern photographs can also be obtained. A series is shown in Van Dyke's (1982) *Album of Fluid Motion*, with a telling comment: "Modern traffic in the streets of Manchester made the critical Reynolds number lower than the value 13,000 found by Reynolds."

Cautioning remarks were already voiced by Reynolds himself concerning the importance attached to the actual value of this critical number.

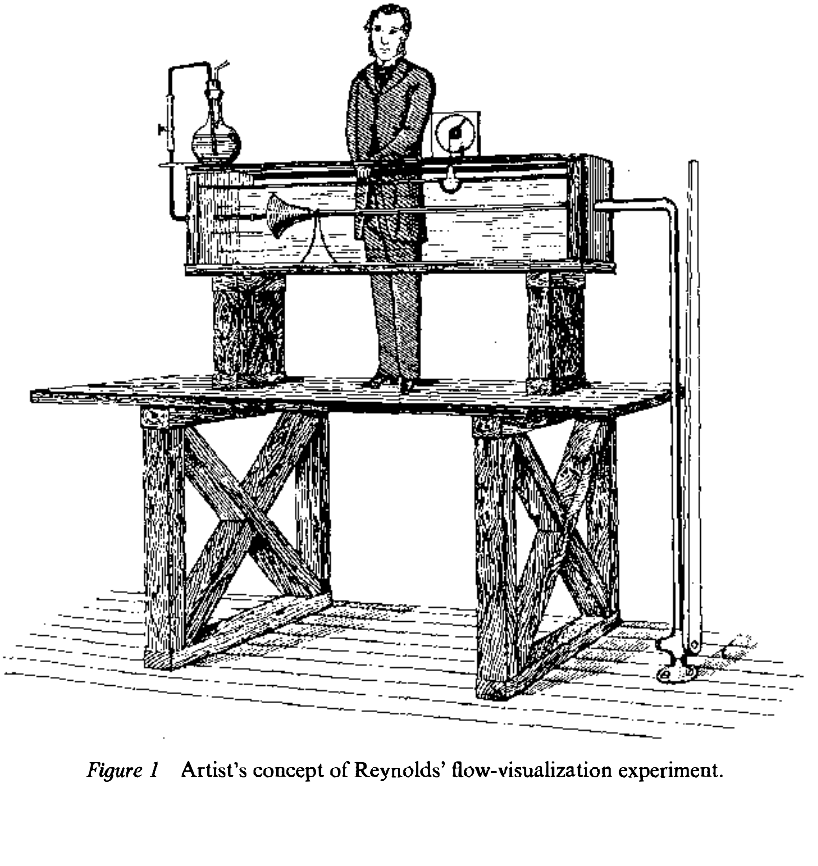

Figure 1 Artist's concept of Reynolds' flow-visualization experiment.

It was clear to him that he needed a carefully shaped inlet to avoid eddies created at the entrance of his tube. He also wrote (Reynolds 1883, p. 955): "The fact . . . that this relation has only been obtained by the utmost care to reduce the external disturbances in the water to a minimum must not be lost sight of." This is a reminder repeated from p. 943, where he notes (we revert to present-day terminology) that turbulent flow has been observed at much lower Reynolds numbers; he continues: "This showed that the steady motion was unstable for large disturbances long before the critical velocity was reached, a fact which agreed with the full-blown manner in which the eddies appeared."

This discussion admits the interpretation that, by more careful experiments, an upper critical Reynolds number could be found that is the stability limit for small disturbances. Reynolds' experiments were repeated a quarter-century later by V. Walfrid Ekman, who visited Manchester to use Reynolds' original equipment. By smoothing the wooden trumpet-shaped inlet, he was able to reach critical Reynolds numbers up to 44,000 (and later more) but found strong scatter approaching these high values. He conjectured (Ekman 1910) that the flow is stable for small disturbances. In a footnote on the subject of the existence of the upper critical Reynolds number, Ekman notes: "It appears that Reynolds himself was somewhat doubtful on this point. It is not easy to find out from his paper what his final opinion really was."

The modern point of view, as formulated by Drazin & Reid (1981, p. 219), is that investigations in this century "have led to the belief that Poiseuille flow in a circular pipe is stable with respect to axisymmetric disturbances. There is also increasing evidence that it is stable with respect to non-axisymmetric disturbances. . . ." An infinite upper critical Reynolds number explains the experimental results. However, a full explanation has also to account for the stability in the varying environment of an inlet, where the Poiseuille profile is not yet fully developed.

In summary, the simplest and most successful visualization of flow transition devised by Reynolds is not, at the same time, the simplest demonstration from the theoretical point of view.

Reynolds first presented his experimental results in a form that did not take full advantage of the clarification of the concepts that he had obtained by dimensional analysis. (A close account is given in Lamb 1932.) For instance, Reynolds wrote for his upper critical velocity the expression $U = P/BD$, where D is the diameter, $P = \nu/\nu_0$ is the kinematic viscosity ν of water divided by its value ν_0 ($=0.01779$ cm^2 s^{-1}) at 0°C, and P is a function of the temperature as measured by Poiseuille. Finally, B is the parameter for which the critical value is determined by experiment. Reynolds found that B is about 43.79 in seconds per square meter units.

The Reynolds number is the reciprocal of the product Bv_0; this gives 12,830.

In early classic (pre-Reynolds) hydraulics, measurements of the temperature dependence of the viscosity in water were often combined with studies of the flow resistance in pipes. Reynolds actually used the function P to extend the scope of his experiments: Critical velocities obtained for 5°C and 22°C were compared and found to have the ratio of about 1.4, in accordance with Poiseuille's formula for P. This complemented the experiments with different tube diameters of 1, 1/2, and 1/4 inch.

3. THE CRITICAL REYNOLDS NUMBER

In his original investigations Reynolds (1883, p. 946) came to the conclusion that "there must be another critical velocity, at which previously existing eddies would die out, and the motion become steady as the water proceeded along the tube. This conclusion has been verified."

Reynolds determined this critical velocity by measuring the pressure drop in a 5-ft section at the end of a 16-ft pipe. He found the pressure loss to be proportional to the first power of the velocity for low speed but varying with a higher power beyond a "lower critical velocity," which occurred for the value $B = 278 \text{ s m}^{-2}$ of his parameter defined above. This parameter value corresponds to a "lower critical" Reynolds number of 2020. (Later, this was simply called *the* critical Reynolds number.) The supercritical pressure-loss dependence on the velocity was found to follow a power law with the exponent 1.723.

Reynolds returned to the discussion of these results in 1895, when he first presented critical values by using explicitly the Reynolds number proper (Reynolds 1895). He called it K and based it (as an engineer always would, reading from a drawing or a caliper) on the pipe diameter. He quoted for the critical K of transition a value between 1900 and 2000, based on his own experience.

Reynolds' purpose in his 1895 paper was to calculate K, or at least to find a lower bound for K, by identifying dissipation with what is today called turbulent production. He thus created the foundation of the main branch of modern turbulence theory by writing down the "Reynolds-averaged" equations of motion. The history of these ideas is beyond the scope of this note.

Reynolds had already observed that plugs of laminar and turbulent flow alternate in a pipe near the critical Reynolds number, causing what is today called "intermittency" of the flow. For this reason Reynolds, as well as experimenters after him, preferred to give an interval for the critical Reynolds number instead of an "exact" value; it was found practically impossible to make an inlet sufficiently "rough" so that a weakly supercritical flow would start without any laminar spots.

Intermittency is easily demonstrated by letting a horizontal jet emerge from a near-critical pipe flow. The jet oscillates—i.e. it reaches different distances as laminar and turbulent flow alternate at the exit. As explained by Julius Rotta (1956), this occurs because for the same mass flow, laminar and turbulent parts do not have the same impulse. The laminar part has the higher impulse and thus moves farther horizontally when it emerges as a jet.

Oscillations are strongly accentuated if they are coupled to changes of the mass flow. This always happens when the flow resistance in the near-critical pipe, where laminar and turbulent flow plugs alternate, is a significant part of the overall pressure drop in the system. Experiments in such systems are not suitable for the determination of the critical Reynolds number. To assure the constancy of the mass flux, Rotta experimented with a (low-pressure) airflow regulated by a sonic throat. Velocity and intermittency were determined by hot-wire measurements. Rotta's observations, supported by theoretical arguments, showed that the supercritical turbulent plugs grew mostly at their front end. He proposed to define the critical Reynolds number by the constancy of the intermittency factor. However, as the growth speed became very slow approaching the critical Reynolds number, reliable measurements would have required a tube of excessive length. Rotta measured a 2% excess of the plug growth speed over the mean speed at a Reynolds number of 2300, from which he extrapolated to a putative critical Reynolds number of 2000. (This happens to be the value first proposed by Reynolds.)

For pipes without a streamlined inlet, the critical value quoted in most contemporary textbooks is 2300.

4. THE POWER LAW

Reynolds determined the critical Reynolds number by the change of the dependence of the pressure loss on velocity. As already noted, he found for turbulent flow a velocity power law with the exponent 1.723, valid for his experiments over a range of 1 to 50 (Reynolds 1883, p. 975). He actually wrote down (on the same page) the similarity law for the pressure drop in its full generality, namely (in present-day notation)

$$p(L)-p(0) = \frac{1}{2}\rho U^2 \frac{L}{D} f(R),$$

$$R = \frac{\rho U D}{\mu} \equiv \frac{UD}{\nu},$$

and then considered the special function $f(R) = cR^{-n}$.

Representation of experimental results by a power law has led, during its long history, to the discovery of analytic relations at best and to useful local approximations at least. Dimensional analysis extends the scope and power of this tool immensely. The experimentalist then is guided by knowing that the exponents obtained for the different physical variables that enter the similarity parameter have to fulfill certain relations.

Reynolds was the first to use the power law in this sense. Many laws with odd exponents were established before him without consideration of dimensionality or similarity. We can understand why Reynolds did not see the necessity of quoting them.

Lord Rayleigh (1892) pointed to the two limits of Reynolds similarity where exact explicit laws follow: The classical slow-flow result ($n = 1$) is independent of density, while the limiting velocity-square law ($n = 0$) for turbulent flow at high speeds is independent of viscosity. Rayleigh considered the latter case as a further illustration of his principle that inviscid solutions and solutions obtained in the limit of vanishing viscosity can be fundamentally different. He proceeded to show that inviscid stability calculations for simple channel flows do not lead to unstable solutions.

The next discussion of Reynolds' paper of 1883 (but with no mention of his work of 1895) appeared in 1897. That it was "next" is conjectured because the author, G. H. Knibbs of the University of Sydney, apparently was very conscientious in searching the literature, and he quotes only Lord Rayleigh's paper dealing with Reynolds' work. Knibbs (1897) could not accept Rayleigh's point of view and, moreover, he rejected Reynolds' similarity principle. To analyze such errors is futile. With some good will, one can say that Knibbs attempted (following older ideas) to establish formulas both for open channels and for pipes, using a common principle. When both viscosity and gravity play a role, then two empirical exponents are required for the most general power law, and the guidance given by Reynolds' similarity is lost. Knibbs described the history of the many contributions to the power law and took issue with Reynolds for ignoring them.

We have already sided with Reynolds and adopt now an idea of Blasius for the selection of references: Only papers are quoted in which the power-law dependence is stated both for the velocity U and for the diameter D, in a way that is compatible with Reynolds similarity. According to the formula that states the similarity law (see above), the difference of the exponents of U and of D has to be 3. Then, only one author is left on Knibbs' reference list, namely the German hydraulic engineer Gotthilf Heinrich Ludwig Hagen (1797–1884), of Hagen-Poiseuille fame.

Knibbs was aware of the compatibility of Hagen's empirical results with Reynolds' theory, but this was for him only an isolated case. Hagen obtained his results without knowing about similarity; he found that the best fit to the averaged results of his measurements with three different tube diameters was obtained with the exponent $n = 1/4$, the same as today's accepted value. Actually, Hagen gave credit to the German engineer Reinhard Woltman (1746–1822) for first proposing the exponent 1.75 for the velocity, in work dating back to 1790. The fact that a power less than 2 describes pressure losses in pipes was already observed earlier, by Pierre Louis Georges Du Buat (1734–1809), one of the great hydraulicists of eighteenth-century France. For more historical material, the reader may want to consult the eminently readable book of Rouse & Ince (1957).

Hagen's measurements were published in 1854 (Hagen 1854); in the same paper he made an observation for which he is probably best known. Knibbs quoted the whole passage in its original form: It is the first description of the transition between laminar and turbulent flow in a pipe. Hagen used sawdust ("Sägespähne") as a means of flow visualization; later (Hagen 1869), he recommended the use of filings of dark amber. Without the help of similarity laws, however, no general conclusions could be drawn.

After the 1897 paper of Knibbs, it took 15 more years before Blasius reintroduced Reynolds similarity to the power law for pipes. In the meantime, however, important experiments were conducted with new means, and Hagen's feat was repeated on a grand scale. Pressure drop in (smooth) pipes was measured by two graduate students at Cornell University, August V. Saph and Ernest W. Schoder, Jun., working at the Hydraulic Laboratory (founded in 1899). Their results were published in the *Transactions of the American Society of Civil Engineers* (Saph & Schoder 1903). They fitted their data, both for velocity and diameter, with $n = 1/4$ but did not consider the dependence on viscosity. Blasius only had to introduce the kinematic viscosity of water to obtain the nondimensional constant that multiplies the power law. (Actually he also made experiments of his own and surveyed data from many other sources. Reynolds' own experiments did not turn out to be useful for the nondimensional constant.)

The paper of Saph & Schoder (1903) was published together with a written discussion of their work by leading hydraulicists. One of them, A. Flamant of France, pointed out that he had already obtained the formula of Saph & Schoder in 1892. This lead proved to be interesting. Flamant's influential textbook *Hydraulique* first appeared in 1891. In the second edition (Flamant 1900), he gave a "new formula" that he obtained by inspecting existing results for accuracy and convenience; it agreed with the results of Saph & Schoder. Flamant also mentioned Reynolds' results in his book: He quoted Reynolds' power-law exponent 1.723 but not the law of similarity.

Finally, around 1910, the acceptance of Reynolds similarity began to spread. In 1911, von Kármán exhorted certain authors who were measuring pressure loss in pipes for different fluids to be mindful of Reynolds similarity (von Kármán 1911). Blasius wrote in the extended version of his paper (Blasius 1913, p. 5) that the Reynolds law "has not penetrated, as of today, into the pertinent fields of engineering" (in the original: "ist in die einschlägigen Gebiete der Ingenieurwissenschaften bis heute noch nicht eingedrungen"). His paper became influential in leading to a change, and it was widely used in aeronautical and mechanical engineering.

The combination of the power law with similarity considerations proved to be a valuable tool for further developments in the early stages of turbulence theory and has led to results of lasting importance for engineering applications. The connection between the wall stress, the dynamic pressure, and the Reynolds number, as given by Blasius' formula for pipe flow, is directly applicable to the turbulent boundary layer on a flat plate and can be extended to a multitude of interesting cases.

A higher level of sophistication in the application of these tools was reached with the derivation of the 1/7-power law for the turbulent velocity profile. However, this also signaled the demise of the power-law era: Both discoverers of the 1/7-power law, Prandtl and von Kármán, moved on to the logarithmic velocity distribution. (According to a personal communication from Ackeret, Prandtl never believed in the deep physical significance of simple fractions as exponents: He *wanted* to find a logarithm.)

Actually, neither Hagen nor Reynolds felt particularly committed to the use of the power law. Hagen, who was also a practical engineer and builder of public works, discussed in 1869 other types of formulas (Hagen 1869). For Reynolds, the power law was mostly a convenient tool for the determination of transition. He had continued interest in the critical Reynolds number and inspired other researchers to measure it. He remained active until 1905, when he retired from the position to which he was appointed in 1868: Professor of Engineering at the University of Manchester.

5. EPILOGUE

Reynolds begins his 1883 paper by stating: "The results of this investigation have both a practical and a philosophical aspect." This could be a useful quotation for instructors who teach a first course in fluid mechanics to juniors in an engineering college. There is a yearly battle going on for students' minds; history might help to convince them that the use of the Reynolds number as an independent variable is an application of a basic truth, and not just a useful convention for a handy diagram.

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