REGULAR HONEYCOMBS IN HYPERBOLIC SPACE

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1. *Introduction.* Schlegel (1883, pp. 444, 454) made a study of honeycombs whose cells are equal regular polytopes in spaces of positive, zero, and negative curvature. The spherical and Euclidean honeycombs had already been described by Schläfli (1855), but the only earlier mention of the hyperbolic honeycombs was when Stringham (1880, pp. 7, 12, and *errata*) discarded them as "imaginary figures", or, for the two-dimensional case, when Klein (1879) used them in his work on automorphic functions. Interest in them was revived by Sommerville (1923), who investigated their metrical properties.

The honeycombs considered by the above authors have finite cells and finite vertex figures. It seems desirable to make a slight extension so as to allow infinite cells, and infinitely many cells at a vertex, because of applications to indefinite quadratic forms (Coxeter and Whitrow 1950, pp. 424, 428) and to the close packing of spheres (Fejes Tóth 1953, p. 159). However, we shall restrict consideration to cases where the fundamental region of the symmetry group has a finite content, like that of a space group in crystallography. This extension increases the number of three-dimensional honeycombs from four to fifteen, the number of four-dimensional honeycombs from five to seven, and the number of five-dimensional honeycombs from zero to five.

A further extension allows the cell or vertex figure to be a *star*-polytope, so that the honeycomb covers the space several times. Some progress in this direction was made in two earlier papers: one (Coxeter 1933), not insisting on finite fundamental regions, was somewhat lacking in rigour; the other (Coxeter 1946) was restricted to two dimensions. The present treatment is analogous to § 14.8 of *Regular Polytopes* (Coxeter 1948, p. 283). We shall find that there are four regular star-honeycombs in hyperbolic 4-space, as well as two infinite families of them in the hyperbolic plane.

2. *Two-dimensional honeycombs.* In the Euclidean plane, the angle of a regular p-gon, $\{p\}$, is $(1 - 2/p)\pi$. In the hyperbolic plane it is smaller, gradually decreasing to zero when the side increases from 0 to ∞. Hence, if p and q are positive integers satsifying

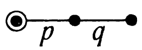
2.1 $$(p-2)(q-2) > 4,$$

we can adjust the size of the polygon so as to make the angle $2\pi/q$. Then q

such $\{p\}$'s will fit together round a common vertex, and we can add further $\{p\}$'s indefinitely. In this manner we construct a two-dimensional honeycomb or *tessellation* $\{p, q\}$, which is an infinite collection of regular p-gons, q at each vertex, filling the whole hyperbolic plane just once (Schlegel 1883, p. 360). We call $\{p\}$ the *face*, $\{q\}$ the *vertex figure*. The centres of the faces of $\{p, q\}$ are the vertices of the *reciprocal* (or dual) tesselation $\{q, p\}$, whose edges cross those of $\{p, q\}$. The simplest instances, $\{7, 3\}$ and $\{3, 7\}$, are shown conformally in Figs. 1 and 2.

As limiting cases, we admit $\{\infty, p\}$, whose faces $\{\infty\}$ are inscribed in horocycles instead of finite circles, and its reciprocal $\{p, \infty\}$, whose vertices are all at infinity (i.e., on the absolute conic).

The lines of symmetry, in which $\{p, q\}$ reflects into itself, are its edges (produced) and the lines of symmetry of its faces. They form a network of congruent triangles whose angles are π/q (at a vertex of $\{p, q\}$), $\pi/2$ (at the mid-point of an edge), and π/p (at the centre of a face). The symmetry group is generated by reflections in the sides of such a *characteristic* triangle $P_0P_1P_2$. Alternatively, we may begin with the triangle and derive $\{p, q\}$ by Wythoff's construction (Coxeter 1948, p. 87) as indicated by the symbol



This means that the vertices of $\{p, q\}$ are the images of the vertex P_0 (where the angle is π/q) in the kaleidoscope formed by mirrors along the three sides of the triangle. (The nodes of the graph represent mirrors, those not directly joined being at right angles.)

For some instances of the network of characteristic triangles, see Klein (1879, p. 448), Fricke (1892, p. 458), and Coxeter (1939, pp. 126, 127).

The classical formulae for a right-angled triangle (Coxeter 1947, p. 238) enable us to compute the sides

$$\varphi = P_0P_1, \qquad \chi = P_0P_2, \qquad \psi = P_1P_2$$

in the form

$$\cosh\varphi = \cos\frac{\pi}{p} \Big/ \sin\frac{\pi}{q}, \quad \cosh\chi = \cot\frac{\pi}{p}\cot\frac{\pi}{q}, \quad \cosh\psi = \cos\frac{\pi}{q} \Big/ \sin\frac{\pi}{p}$$

(Sommerville 1923, p. 86; cf. Coxeter 1948, pp. 21, 64). Then we may describe $\{p, q\}$ as a tesselation of edge 2φ, whose faces are p-gons of circum-radius χ and in-radius ψ.

3. *Three-dimensional honeycombs*. When

$$(p-2)(q-2) < 4,$$

the symbol $\{p, q\}$ denotes a Platonic solid (e.g., $\{4, 3\}$ is a regular hexahedron) which exists not only in Euclidean space but equally well in hyperbolic space. When the edge increases from 0 to ∞, the dihedral angle decreases from its Euclidean value

$$2 \operatorname{arc} \sin \left(\cos \frac{\pi}{q} / \sin \frac{\pi}{p}\right)$$

to $(1 - 2/q)\pi$. For, the solid angle at a vertex resembles a Euclidean solid angle in that its section by a sphere is a spherical q-gon, whose angle must exceed $(1 - 2/q)\pi$.

Thus the dihedral angle can take the value $2\pi/r$ whenever p, q, r are integers, greater than 2, satisfying

3.1 $$\sin \frac{\pi}{p} \sin \frac{\pi}{r} < \cos \frac{\pi}{q}$$

and $(q - 2)(r - 2) < 4$. Then r such $\{p, q\}$'s will fit together round a common edge, and we can add further $\{p, q\}$'s indefinitely. In this manner we construct a three-dimensional honeycomb $\{p, q, r\}$, which is an infinite collection of $\{p, q\}$'s, r at each edge, filling the whole hyperbolic space just once. The arrangement of cells round a vertex is like the arrangement of faces of the polyhedron $\{q, r\}$, which is called the *vertex figure*. The centres of the cells of $\{p, q, r\}$ are the vertices of the *reciprocal* honeycomb $\{r, q, p\}$, whose edges cross the $\{p\}$'s of $\{p, q, r\}$.

The actual instances are

$$\{3, 5, 3\}, \qquad \{4, 3, 5\}, \qquad \{5, 3, 4\}, \qquad \{5, 3, 5\},$$

in which the cells are respectively: an icosahedron of angle $2\pi/3$, a hexahedron of angle $2\pi/5$, and dodecahedra of angles $\pi/2$ and $2\pi/5$ (Schlegel 1883, p. 444). The existence of the "hyperbolic dodecahedron space" (Weber and Seifert 1933, pp. 241—243) reveals an interesting property of the self-reciprocal honeycomb $\{5, 3, 5\}$: its symmetry group has a subgroup (of index 120) which is transitive on the vertices and has the whole cell for a fundamental region. This resembles the translation group of the Euclidean honeycomb of cubes, $\{4, 3, 4\}$, only now instead of translations we have screws.

Returning to the general discussion, we allow the dihedral angle of the Platonic solid $\{p, q\}$ to take its minimum value $(1 - 2/q)\pi$, so as to obtain the further honeycombs

$$\{3, 4, 4\}, \qquad \{3, 3, 6\}, \qquad \{4, 3, 6\}, \qquad \{5, 3, 6\},$$

whose vertices are all at infinity (i.e., on the absolute quadric). We naturally consider also the respective reciprocals

$$\{4, 4, 3\}, \qquad \{6, 3, 3\}, \qquad \{6, 3, 4\}, \qquad \{6, 3, 5\},$$

whose cells are inscribed in horospheres instead of finite spheres (Coxeter and Whitrow 1950, p. 426), as well as the three self-reciprocal honeycombs

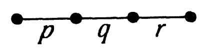
$$\{6, 3, 6\}, \qquad \{4, 4, 4\}, \qquad \{3, 6, 3\},$$

which suffer from both peculiarities at once.

In other words, necessary and sufficient conditions for the existence of a hyperbolic honeycomb $\{p, q, r\}$ are 3.1 and

3.2 $$(p-2)(q-2) \leqq 4, \qquad (q-2)(r-2) \leqq 4.$$

The honeycomb can be derived by Wythoff's construction from its characteristic simplex, which is the quadrirectangular hyperbolic tetrahedron



(Coxeter 1948, p. 139). The inequalities 3.2 ensure that this tetrahedron has a finite volume, being entirely accessible except that it may have one or two vertices at infinity. From relations between the edges and angles of the tetrahedron, we find the edge-length of $\{p, q, r\}$ to be 2φ while its cell $\{p, q\}$ has circum-radius χ and in-radius ψ, where

$$\cosh \varphi = \cos \frac{\pi}{p} \sin \frac{\pi}{r} \Big/ \sin \frac{\pi}{h_{q,r}}, \qquad \cosh \psi = \sin \frac{\pi}{p} \cos \frac{\pi}{r} \Big/ \sin \frac{\pi}{h_{p,q}},$$

$$\cosh \chi = \cos \frac{\pi}{p} \cos \frac{\pi}{q} \cos \frac{\pi}{r} \Big/ \sin \frac{\pi}{h_{p,q}} \sin \frac{\pi}{h_{q,r}},$$

$h_{p,q}$ being given by

$$\cos^2 \frac{\pi}{h_{p,q}} = \cos^2 \frac{\pi}{p} + \cos^2 \frac{\pi}{q}$$

(Coxeter 1948, p. 19). For the particular cases, see Table III, the first part of which was given earlier by Sommerville (1923, p. 96), whose e, R, r are our $2\varphi, \chi, \psi$.

4. *Four-dimensional honeycombs*. When

4.1 $$\sin \frac{\pi}{p} \sin \frac{\pi}{r} > \cos \frac{\pi}{q},$$

the symbol $\{p, q, r\}$ denotes a regular four-dimensional polytope (Coxeter 1948, p. 135), which has smaller angles in hyperbolic space than in Euclidean. A discussion analogous to that of § 3 shows that infinitely many such cells $\{p, q, r\}$ can be fitted together, s round each plane face, to make a four-dimensional honeycomb

$$\{p, q, r, s\},$$

whenever both $\{p, q, r\}$ and $\{q, r, s\}$ are finite polytopes or Euclidean honey-combs such that

$$\frac{\cos^2 \pi/q}{\sin^2 \pi/p} + \frac{\cos^2 \pi/r}{\sin^2 \pi/s} > 1 \tag{4.2}$$

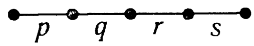
(cf. Coxeter 1948, p. 136). In this manner, the six polytopes

$$\{3, 3, 3\},\quad \{3, 3, 4\},\quad \{4, 3, 3\},\quad \{3, 4, 3\},\quad \{3, 3, 5\},\quad (5, 3, 3\},$$

and the cubic honeycomb $\{4, 3, 4\}$, yield the seven hyperbolic honeycombs

$$\{3, 3, 3, 5\},\quad \{4, 3, 3, 5\},\quad \{5, 3, 3, 5\},\quad \{5, 3, 3, 4\},\quad \{5, 3, 3, 3\},$$
$$\{3, 4, 3, 4\},\qquad \{4, 3, 4, 3\}.$$

Each of these can be derived from the appropriate characteristic simplex



by Wythoff's construction (Coxeter 1948, p. 199).

The edge-length 2φ, and the circum- and in-radii of a cell, χ and ψ, are given by the formulae

$$\operatorname{sech}^2 \varphi = \frac{(-1, 1)(0, 5)}{(1, 5)}, \qquad \operatorname{sech}^2 \psi = \frac{(-1, 4)(3, 5)}{(-1, 3)},$$
$$\operatorname{sech}^2 \chi = (-1, 4)(0, 5)$$

(Coxeter 1948, p. 161), where the symbols (j, k) are derived by the recurrence formula

$$(j, k) = \frac{(j, k-1)(j+1, k) - 1}{(j+1, k-1)}, \quad (j, j+1) = 1,$$

from any convenient sequence of numbers

$$(-1, 1), \qquad (0, 2), \qquad (1, 3), \qquad (2, 4), \qquad (3, 5)$$

satisfying

$$(-1, 1)(0, 2) = \sec^2 \frac{\pi}{p}, \quad (0, 2)(1, 3) = \sec^2 \frac{\pi}{q},$$
$$(1, 3)(2, 4) = \sec^2 \frac{\pi}{r}, \quad (2, 4)\,(3, 5) = \sec^2 \frac{\pi}{s}.$$

In any particular case, the values are most easily computed by arranging the (j, k)'s in a triangular table:

$$\begin{array}{ccccccccc}
(-1, 1) & & (0, 2) & & (1, 3) & & (2, 4) & & (3, 5) \\
& (-1, 2) & & (0, 3) & & (1, 4) & & (2, 5) & \\
& & (-1, 3) & & (0, 4) & & (1, 5) & & \\
& & & (-1, 4) & & (0, 5) & & & \\
& & & & (-1, 5) & & & &
\end{array}$$

e.g., the respective tables for $\{5, 3, 3, 5\}$ and $\{4, 3, 4, 3\}$ are

$$\begin{matrix}
2\tau^{-2} & & 2 & & 2 & & 2 & & 2\tau^{-2} \\
& \sqrt{5}\tau^{-3} & & 3 & & 3 & & \sqrt{5}\tau^{-3} & \\
& & 2\tau^{-4} & & 4 & & 2\tau^{-4} & & \\
& & & \tau^{-6} & & \tau^{-6} & & & \\
& & & & -2\sqrt{5}\tau^{-6} & & & &
\end{matrix}
\qquad\qquad
\begin{matrix}
1 & & 2 & & 2 & & 1 & & 4 \\
& 1 & & 3 & & 1 & & 3 & \\
& & 1 & & 1 & & 2 & & \\
& & & 0 & & 1 & & & \\
& & & & -1 & & & &
\end{matrix}$$

where $\tau = \frac{1}{2}(\sqrt{5} + 1)$, so that $\tau^{-1} = \frac{1}{2}(\sqrt{5} - 1)$. The negative value of $(-1, 5)$ provides a verification that the honeycomb is hyperbolic, and the zero for $(-1, 4)$ indicates that the cell of $\{4, 3, 4, 3\}$ is infinite. (For the results of this computation, see Table IV.)

5. *Five-dimensional honeycombs.* Similarly, the cell $\{p, q, r, s\}$ and vertex figure $\{q, r, s, t\}$ of a five-dimensional honeycomb $\{p, q, r, s, t\}$ must occur among the finite polytopes

$$\{3, 3, 3, 3\}, \qquad \{3, 3, 3, 4\}, \qquad \{4, 3, 3, 3\}$$

or among the Euclidean honeycombs

$$\{3, 3, 4, 3\}, \qquad \{3, 4, 3, 3\}, \qquad \{4, 3, 3, 4\}$$

(Coxeter 1948, p. 136). Since

$$\{3, 3, 3, 3, 3\}, \qquad \{3, 3, 3, 3, 4\}, \qquad \{4, 3, 3, 3, 3\}$$

are finite, while $\{4, 3, 3, 3, 4\}$ is Euclidean, the only hyperbolic honeycombs $\{p, q, r, s, t\}$ are

$$\{3, 3, 3, 4, 3\}, \quad \{4, 3, 3, 4, 3\}, \quad \{3, 3, 4, 3, 3\}, \quad \{3, 4, 3, 3, 4\} \quad \{3, 4, 3, 3, 3\}$$

all of which have either infinite cells or all their vertices at infinity.

The edge-length and radii are now given by the formulae

$$\operatorname{sech}^2 \varphi = \frac{(-1, 1)(0, 6)}{(1, 6)}, \ \operatorname{sech}^2 \psi = \frac{(-1, 5)(4, 6)}{(-1, 4)},$$

$$\operatorname{sech}^2 \chi = (-1, 5)(0, 6),$$

with $(3, 5)(4, 6) = \sec^2 \pi/t$. The triangular tables for $\{3, 3, 3, 4, 3\}$ and $\{3, 3, 4, 3, 3\}$ are

$$\begin{matrix}
2 & & 2 & & 2 & & 2 & & 1 & & 4 \\
& 3 & & 3 & & 3 & & 1 & & 3 & \\
& & 4 & & 4 & & 1 & & 2 & & \\
& & & 5 & & 1 & & 1 & & & \\
& & & & 1 & & 0 & & & & \\
& & & & & -1 & & & & &
\end{matrix}
\qquad\qquad
\begin{matrix}
2 & & 2 & & 2 & & 1 & & 4 & & 1 \\
& 3 & & 3 & & 1 & & 3 & & 3 & \\
& & 4 & & 1 & & 2 & & 2 & & \\
& & & 1 & & 1 & & 1 & & & \\
& & & & 0 & & 0 & & & & \\
& & & & & -1 & & & & &
\end{matrix}$$

yielding $\varphi = \chi = \infty$, $\psi = \log \tau$ for the former, and $\varphi = \chi = \psi = \infty$ for the latter. (See Table V.)

This is the end of the story, so far as honeycombs of density 1 are concerned. For, if $n > 5$, the only finite polytopes and Euclidean honeycombs that might serve as cells and vertex figures are

$$\alpha_n = \{3, 3, \ldots, 3, 3\}, \qquad \beta_n = \{3, 3, \ldots, 3, 4\},$$
$$\gamma_n = \{4, 3, \ldots, 3, 3\}, \qquad \delta_n = \{4, 3, \ldots, 3, 4\};$$

and these yield only α_{n+1}, β_{n+1}, γ_{n+1}, and δ_{n+1}. Hence (Schlegel 1883, p. 455) *There are no regular honeycombs in hyperbolic space of six or more dimensions.*

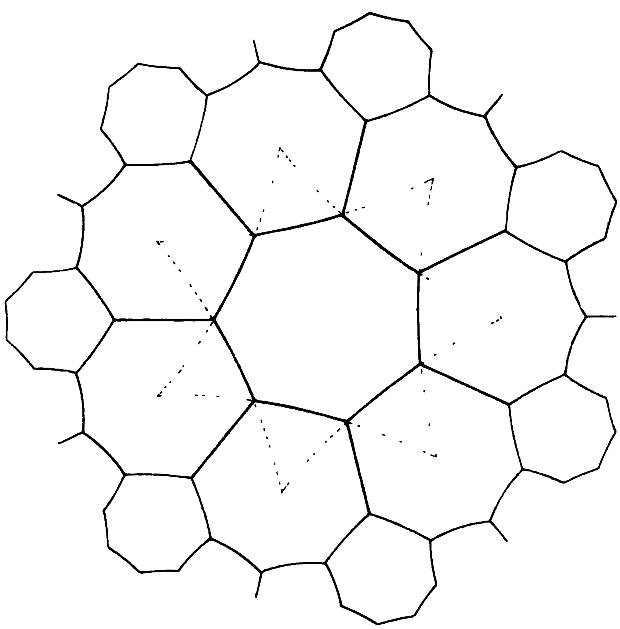


Figure 1

6. *Two-dimensional star-honeycombs.* If n/d is a fraction in its lowest terms, whose value is greater than 2, the symbol $\{n/d\}$ denotes a regular star-polygon whose n sides surround its centre d times: briefly, a regular n-gon of density d, such as the pentagram $\{\frac{5}{2}\}$. It is natural to ask whether the symbol $\{p, q\}$ for a hyperbolic tessellation remains valid when p or q is fractional. We can obviously *begin* to construct such a star-tessellation whenever 2.1 is satisfied. The question is whether it will cover the plane a finite number of times, i.e., whether its *density* is finite.

The symmetry group of such a tessellation $\{p, q\}$ is still generated by reflections in the sides of its characteristic triangle, whose angles are π/q, $\pi/2$, and π/p (Coxeter 1948, p. 109). If p or q is fractional, this triangle is dissected into smaller triangles by "virtual mirrors" (Coxeter 1948, pp. 75, 76), and the process of subdivision will continue until we come to a triangle all of whose angles are submultiples of π (or possibly zero). There might conceivably be

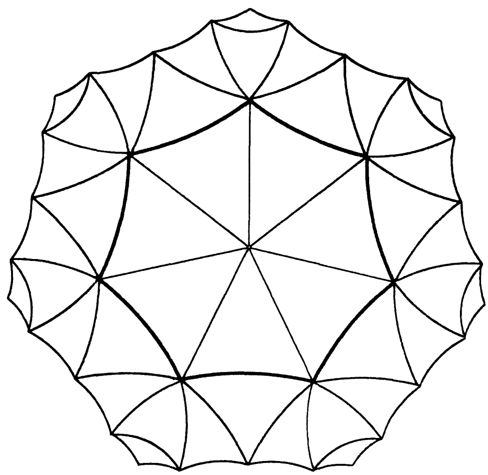


Figure 2

several different triangles of this kind, but the smallest of them will serve as a fundamental region for the group. We denote this smallest triangle by $(l\ m\ n)$ to indicate that its angles are π/l, π/m, π/n. The number of repetitions of it that fill the characteristic triangle $(2\ p\ q)$ is an integer $D > 1$, which is equal to the density of the tessellation (Coxeter 1948, p. 110).

Since each angle of $(2\ p\ q)$ must be a multiple of one of the angles of $(l\ m\ n)$, a zero angle of the former implies a zero angle of the latter. But since the subdivision of an asymptotic triangle $(2\ p\ \infty)$ yields at least one finite piece, this is impossible; both p and q must be finite.

Since the triangle $(2\ p\ q)$ is filled with D triangles $(l\ m\ n)$, of area $(1 - l^{-1} - m^{-1} - n^{-1})\pi$, we have

$$\tfrac{1}{2} - p^{-1} - q^{-1} = D(1 - l^{-1} - m^{-1} - n^{-1}),$$

where the numerators of the rational numbers p and q are divisors of one or two of the integers l, m, n.

The numerators of p and q cannot divide two *different* integers among l, m, n, say m and n; for then we would have

$$\tfrac{1}{2} - p^{-1} - q^{-1} \leqq \tfrac{1}{2} - m^{-1} - n^{-1} \leqq 1 - l^{-1} - m^{-1} - n^{-1},$$

implying $D \leqq 1$. Thus we may assume that these numerators both divide m, so that $p = m/x$ and $q = m/y$, where x and y are positive integers. Since p and q are not both integers,

$$x + y \geqq 3.$$

Moreover, $D \geqq 3$, since the triangle $(2\ p\ q)$ obviously cannot be *bisected* by a line through one of its acute vertices. Hence

$$\begin{aligned} \frac{1}{2} &= \frac{x}{m} + \frac{y}{m} + D\left(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\right) \\ &\geqq \frac{3}{m} + 3\left(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\right) = 3\left(1 - \frac{1}{l} - \frac{1}{n}\right), \end{aligned}$$

i.e.,

$$\frac{1}{l} + \frac{1}{n} \geqq \frac{5}{6}.$$

Since also

$$\frac{1}{l} + \frac{1}{n} < \frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1,$$

the integers l and n must be 2 and 3, and

$$x + y = D = 3.$$

Thus the only possibility is the triangle $(2\ m\ m/2)$ dissected into three triangles $(2\ m\ 3)$. In other words,

The only regular star-tessellations in the hyperbolic plane are

$$\left\{\frac{m}{2}, m\right\} \text{ and } \left\{m, \frac{m}{2}\right\},$$

of density 3, *where* m *is any odd number greater than* 5.

Such star-tessellations exist also when $m = 5$, but then they are not hyperbolic but spherical. In fact, they are essentially the small stellated dodecahedron of Kepler and the great dodecahedron of Poinsot (Coxeter 1948, p. 95). But the other two Kepler-Poinsot polyhedra have no hyperbolic analogues.

We may describe the faces of $\{m/2, m\}$ as the stellated faces of $\{m, 3\}$. (One face of $\{\frac{7}{2}, 7\}$ is indicated by broken lines in Fig. 1.) Dually $\{m, m/2\}$ is derived from $\{3, m\}$ by regarding the same vertices and edges as forming m-gons (such as the heptagon of $\{7, \frac{7}{2}\}$ which is emphasized in Fig. 2) instead of triangles. The density 3 can be observed in the fact that each triangle of $\{3, m\}$ lies within three m-gons of $\{m, m/2\}$. In ascribing the same density to $(m/2, m\}$, we regard the middle part of each $\{m/2\}$ as being covered twice over. The relationship of $(m/2, m\}$ and $\{m, m/2\}$ is such that any face of either has the same vertices as a corresponding face of the other. Both have the same vertices as $\{3, m\}$, and the same face-centres as $\{m, 3\}$. Thus

$$\chi(m, m/2) = 2\varphi(m, m/2) = 2\varphi(3, m),$$

in agreement with Table II.

7. *Four-dimensional star-honeycombs.* One might expect to find a three-dimensional star-honeycomb $\{p, q, r\}$ whose cell $\{p, q\}$ or vertex figure $(q, r\}$ is one of the Kepler-Poinsot polyhedra

$$\{\tfrac{5}{2}, 5\}, \qquad \{5, \tfrac{5}{2}\}, \qquad \{\tfrac{5}{2}, 3\}, \qquad \{3, \tfrac{5}{2}\}.$$

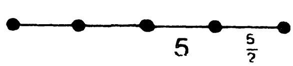
However, in every instance (Coxeter 1948, p. 264) the values of p, q, r satisfy 4.1, not 3.1. Hence

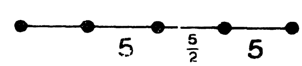
There are no regular star-honeycombs in hyperbolic 3-*space.*

Hoping for a more positive result in four dimensions, we seek a honeycomb $\{p, q, r, s\}$ in which both $\{p, q, r\}$ and $\{q, r, s\}$ occur among the sixteen regular polytopes (Coxeter 1948, pp. 293—294) while at least one of $p, q, r. s$ has the fractional value $\frac{5}{2}$. A list of the twenty-three possibilities reveals that only four satisfy 4.2:

$$\{3, 3, 5, \tfrac{5}{2}\}, \qquad \{3, 5, \tfrac{5}{2}, 5\}, \qquad \{5, \tfrac{5}{2}, 5, 3\}, \qquad \{\tfrac{5}{2}, 5, 3, 3\}$$

(Coxeter 1948, p. 264: 14.15). That these four are genuine hyperbolic honeycombs, of finite density, may be seen by verifying that the reflections in the bounding hyperplanes of their characteristic simplexes





generate discrete groups. In Table I (cf. Coxeter 1948, p. 283) these simplexes appear as X and Z, and we see how they are dissected, by virtual mirrors (bisecting their dihedral angles $2\pi/5$, which are indicated by the mark $\frac{5}{2}$ in the graphical symbols), into smaller simplexes T and W, W and Y. Similarly W and Y, having angles $2\pi/3$, are further subdivided, until the process ends with the "quantum" T, which is thus seen to be the fundamental region for both these groups, as it is also for the symmetry group of the ordinary honeycombs $\{3, 3, 3, 5\}$ and $\{5, 3, 3, 3\}$.

The accuracy of Table I can be checked by observing that, in each graph, the branch with a fractional mark forms, with any third node, the symbol for a spherical triangle whose dissection is obvious; e.g., the dissection $Y = 2V$ embodies

$$(5\ 5\ \tfrac{3}{2}) = (5\ 2\ 3) + (2\ 5\ 3) \text{ and } (3\ 2\ \tfrac{3}{2}) = (3\ \tfrac{3}{2}\ 3) + (3\ 2\ 3)$$

(Coxeter 1948, p. 113). Since $X = 5T$ and $Z = 10T$, the densities of the star-honeycombs are 5 and 10.

Thus the six hyperbolic honeycombs

$$\{5, 3, 3, 3\}, \quad \{\tfrac{5}{2}, 5, 3, 3\}, \quad \{5, \tfrac{5}{2}, 5, 3\}, \quad \{3, 5, \tfrac{5}{2}, 5\}, \quad \{3, 3, 5, \tfrac{5}{2}\}, \quad \{3, 3, 3, 5\}$$

all have the same symmetry group, and their densities are the binomial coefficients

$$1, \quad 5, \quad 10, \quad 10, \quad 5, \quad 1.$$

It is interesting to compare them with the five spherical honeycombs (or Euclidean polytopes)

$$\{5, 3, 3\}, \quad \{\tfrac{5}{2}, 5, 3\}, \quad \{5, \tfrac{5}{2}, 5\}, \quad \{3, 5, \tfrac{5}{2}\}, \quad \{3, 3, 5\},$$

whose densitities are 1, 4, 6, 4, 1, and with the four spherical tessellations (or Euclidean polyhedra)

$$\{5, 3\}, \quad \{\tfrac{5}{2}, 5\}, \quad \{5, \tfrac{5}{2}\}, \quad \{3, 5\},$$

whose densities are 1, 3, 3, 1. We can summarize these results by saying that, when the Schläfli symbol for an n-dimensional honeycomb has $\frac{5}{2}$ in the r^{th} place with 5 before and after and 3 everywhere else, the density is $\binom{n}{r}$.

The dissection of characteristic simplexes can be translated into a direct derivation of the honeycombs from one another. The first stage is to derive $\{\frac{5}{2}, 5, 3, 3\}$ from $\{5, 3, 3, 3\}$ by stellating each cell $\{5, 3, 3\}$ to form a $\{\frac{5}{2}, 5, 3\}$ (Coxeter 1948, p. 264). The second stage is to replace each $\{\frac{5}{2}\}$ of $\{\frac{5}{2}, 5, 3, 3\}$ by the pentagon that has the same vertices, and consequently each $\{\frac{5}{2}, 5\}$ by a $\{5, \frac{5}{2}\}$ and each $\{\frac{5}{2}, 5, 3\}$ by a $\{5, \frac{5}{2}, 5\}$. In this process the vertex figure gets stellated from $\{5, 3, 3\}$ to $\{\frac{5}{2}, 5, 3\}$, and the result is $\{5, \frac{5}{2}, 5, 3\}$. The third stage is to replace each $\{5, \frac{5}{2}\}$ by the icosahedron that has the same edges, and consequently each $\{5, \frac{5}{2}, 5\}$ by a $\{3, 5, \frac{5}{2}\}$. The vertex figure $\{\frac{5}{2}, 5, 3\}$ is changed into $\{5, \frac{5}{2}, 5\}$, and the result is $\{3, 5, \frac{5}{2}, 5\}$. The fourth stage is to replace each $\{3, 5, \frac{5}{2}\}$ by the $\{3, 3, 5\}$ that has the same faces, so that the vertex figure $\{5, \frac{5}{2}, 5\}$ is changed into $\{3, 5, \frac{5}{2}\}$ (which has the same edges), and we obtain $\{3, 3, 5, \frac{5}{2}\}$. The fifth and last stage is to replace the cell $\{3, 3, 5\}$ by a cluster of 600 regular simplexes $\{3, 3, 3\}$ surrounding a common vertex, so as to obtain $\{3, 3, 3, 5\}$. Each simplex belongs to five of the clusters, as we would expect from the fact that $\{3, 3, 5, \frac{5}{2}\}$ has density 5.

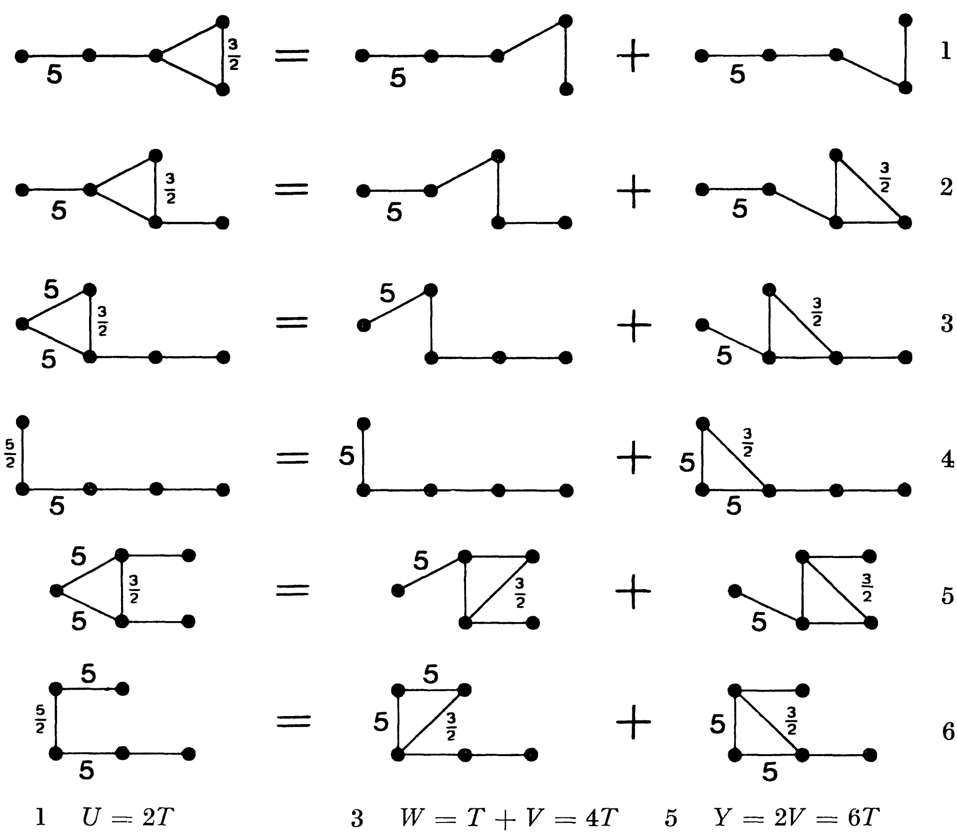
Taking the same stages in the reverse order, we may say that the vertices, edges, faces and cells of $\{3, 3, 3, 5\}$ belong also to $\{3, 3, 5, \frac{5}{2}\}$; the vertices, edges and faces belong also to $\{3, 5, \frac{5}{2}, 5\}$; the vertices and edges belong also to $\{5, \frac{5}{2}, 5, 3\}$; and the vertices belong also to $\{\frac{5}{2}, 5, 3, 3\}$, which is the "stellated" $\{5, 3, 3, 3\}$.

The formulae for φ, χ, ψ (§ 4) apply to star-honeycombs without any alteration. We see from Table IV that, since $\tau^3 = 2\tau^2 - 1$, the value of χ for each of them is equal to the value of 2φ for $\{3, 3, 3, 5\}$.

Finally, since there are no regular star-polytopes in five or more dimensions (Coxeter 1948, p. 278) to serve as cell or vertex figure,

There are no regular star-honeycombs in hyperbolic space of five or more dimensions.

Table I. The dissection of characteristic simplexes in hyperbolic 4-space.



1 $U = 2T$

2 $V = T + U = 3T$

3 $W = T + V = 4T$

4 $X = T + W = 5T$

5 $Y = 2V = 6T$

6 $Z = W + Y = 10T$

Table II. Regular honeycombs in the hyperbolic plane

Tessellation	Density	$\cosh \varphi$	$\cosh \chi$	$\cosh \psi$
$\{p, q\}$ (integers)	1	$\cos \frac{\pi}{p} \operatorname{cosec} \frac{\pi}{q}$	$\cot \frac{\pi}{p} \cot \frac{\pi}{q}$	$\operatorname{cosec} \frac{\pi}{p} \cos \frac{\pi}{q}$
$\left\{\frac{m}{2}, m\right\}$ (m odd)	3	$\cos \frac{2\pi}{m} \operatorname{cosec} \frac{\pi}{m}$	$\cot \frac{\pi}{m} \cot \frac{2\pi}{m}$	$\frac{1}{2} \operatorname{cosec} \frac{\pi}{m}$
$\left\{m, \frac{m}{2}\right\}$		$\frac{1}{2} \operatorname{cosec} \frac{\pi}{m}$		$\cos \frac{2\pi}{m} \operatorname{cosec} \frac{\pi}{m}$

Table III. The fifteen regular honeycombs in hyperbolic 3-space

eycomb	Density	cosh $^2\varphi$	$\cosh^2 \chi$	$\cosh^2 \psi$
5, 3}	1	$\frac{3}{4}\tau^2$	$\frac{1}{4}\tau^6$	$\frac{3}{4}\tau^2$
3, 5}	1	$\frac{1}{2}\sqrt{5}\,\tau$	$\frac{1}{2}\tau^4$	$\frac{1}{2}\tau^2$
3, 4}		$\frac{1}{2}\tau^2$		$\frac{1}{2}\sqrt{5}\,\tau$
3, 5}	1	$\frac{1}{4}\sqrt{5}\,\tau^3$	$\frac{1}{4}\tau^8$	$\frac{1}{4}\sqrt{5}\,\tau^3$
3, 6}	1	∞	∞	$\frac{9}{8}$
3, 3}		$\frac{9}{8}$		∞
4, 4}	1	∞	∞	$\frac{3}{2}$
4, 3}		$\frac{3}{2}$		∞

Honeycomb	Density	$\cosh^2 \varphi$	$\cosh^2 \chi$	$\cosh^2 \psi$
$\{3, 6, 3\}$	1	∞	∞	∞
$\{4, 3, 6\}$	1	∞	∞	$\frac{3}{2}$
$\{6, 3, 4\}$		$\frac{3}{2}$		∞
$\{4, 4, 4\}$	1	∞	∞	∞
$\{5, 3, 6\}$	1	∞	∞	$\frac{3}{4}\sqrt{5}\,\tau$
$\{6, 3, 5\}$		$\frac{3}{4}\sqrt{5}\,\tau$		∞
$\{6, 3, 6\}$	1	∞	∞	∞

$$\left(\tau = \frac{\sqrt{5}+1}{2}\right)$$

Table IV. The eleven regular honeycombs in hyperbolic 4-space

Honeycomb	Density	cosh φ	cosh χ	cosh ψ
$\{3, 3, 3, 5\}$	1	τ	$\sqrt{\frac{1}{5}}\,\tau^3$	$\sqrt{\frac{2}{5}}\,\tau$
$\{5, 3, 3, 3\}$		$\sqrt{\frac{2}{5}}\,\tau$		τ
$\{4, 3, 3, 5\}$	1	$\sqrt{2}\,\tau$	τ^3	$\sqrt{\frac{1}{2}}\,\tau$
$\{5, 3, 3, 4\}$		$\sqrt{\frac{1}{2}}\,\tau$		$\sqrt{2}\,\tau$
$\{5, 3, 3, 5\}$	1	τ^2	τ^6	τ^2

Honeycomb	Density	cosh φ	cosh χ	cosh
$\{3, 4, 3, 4\}$	1	∞	∞	[illegible]
$\{4, 3, 4, 3\}$		$\sqrt{2}$		∞
$\{\frac{5}{2}, 5, 3, 3\}$	5	$\sqrt{2}\,\tau$	τ^3	τ
$\{3, 3, 5, \frac{5}{2}\}$		τ		$\sqrt{2}$
$\{3, 5, \frac{5}{2}, 5\}$	10	τ	τ^3	τ
$\{5, \frac{5}{2}, 5, 3\}$				

Table V. The five regular honeycombs in hyperbolic 5-space

Honeycomb	Density	φ	χ	ψ
$\{3, 3, 3, 4, 3\}$	1	∞	∞	$\log\tau$
$\{3, 4, 3, 3, 3\}$		$\log\tau$		∞
$\{3, 3, 4, 3, 3\}$	1	∞	∞	∞
$\{3, 4, 3, 3, 4\}$	1	∞	∞	∞
$\{4, 3, 3, 4, 3\}$				

References

H. S. M. Coxeter 1933. The densities of the regular polytopes, Part 3, Proc. Camb. Phil. Soc. **29**, 1—22.

H. S. M. Coxeter 1939. The abstract groups $G^{m,n,p}$, Trans. Amer. Math. Soc. **45**, 73—150.

H. S. M. Coxeter 1946. The nine regular solids, Proc. 1st Canadian Math. Congress, 252—264.

H. S. M. Coxeter 1947. Non-Euclidean geometry, Toronto (University Press).

H. S. M. Coxeter 1948. Regular polytopes, London (Methuen).

H. S. M. Coxeter and G. J. Whitrow 1950. World-structure and non-Euclidean honey combs, Proc. Roy. Soc. A, **201**, 417—437.

L. FEJES TÓTH 1953. On close-packings of spheres in spaces of constant curvature, Publ. Math. Debrecen, **3**, 158—167.

R. FRICKE 1892. Ueber den arithmetischen Charakter der zu den Verzweigungen (2, 3, 7) und (2, 4, 7) gehörenden Dreiecksfunctionen, Math. Ann. **41**, 443—468.

F. KLEIN 1879. Ges. Math. Abh. **3** (Berlin, 1923), 90—136.

L. SCHLÄFLI 1855. Ges. Math. Abh. **2** (Basel, 1953), 164—190.

V. SCHLEGEL 1883. Theorie der homogen zusammengesetzten Raumgebilde, Nova Acta Leop. Carol. **44**, 343—459.

D. M. Y. SOMMERVILLE 1923. Division of space by congruent triangles and tetrahedra, Proc. Roy. Soc. Edinburgh, **43**, 85—116.

W. I. STRINGHAM 1880. Regular figures in n-dimensional space, Amer. J. Math. **3**, 1—14.

C. WEBER and H. SEIFERT 1933. Die beiden Dodekaederräume, Math. Z, **37**, 237—253.