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Almost All Generalized Extraspecial p-Groups Are Resistant

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A p-group P is called resistant if, for any finite group G having P as a Sylow p-subgroup, the normalizer $N_G(P)$ controls p-fusion in G. The aim of this paper is to prove that any generalized extraspecial p-group P is resistant, excepting the case when $P = E \times A$, where A is elementary abelian and E is dihedral of order 8 (when $p = 2$) or extraspecial of order p^3 and exponent p (when p is odd). This generalizes a result of Green and Minh.

1. INTRODUCTION

Let G be a finite group and let H be a subgroup of G. Two elements of H are said to be *fused* in G if they are conjugate in G but not in H. We are interested in p-groups P such that, for any finite group G having P as a Sylow p-subgroup, the p-fusion is controlled only by the normalizer $N_G(P)$ of P (that is, any two elements of P which are fused in G are fused in $N_G(P)$). This is equivalent to the requirement that any such group G does not contain *essential* p-subgroups (Definition 2.2). Following the terminology suggested by Jesper Grodal, we will call such a group *resistant*.

In fact, by a theorem of Mislin [Mi], the notion of *resistant group* is equivalent to what Martino and Priddy [MP] call *Swan group*. We recall that P is a Swan group if, for any G as before, the mod-p cohomology ring $H^*(G)$ is isomorphic to the mod-p cohomology ring $H^*(N_G(P))$.

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In a recent paper [GM], Green and Minh proved that almost all extraspecial p-groups are Swan groups. In our paper, we find the same result for *generalized extraspecial* p-groups (Definition 3.1) and give a proof avoiding cohomological methods.

2. ESSENTIAL GROUPS

Let $\mathcal{F}_p(G)$ be the Frobenius category of a finite group G. We recall that the objects in this category are the nontrivial p-subgroups of G and the morphisms are the group homomorphisms given by the conjugation by elements of G. For a subgroup H of G, we denote by $\mathcal{F}_p(G)_{\leq H}$ the full subcategory of $\mathcal{F}_p(G)$ containing the nontrivial p-subgroups of H.

A natural question is: What is the minimal information needed to completely characterize these morphisms? For a Sylow p-subgroup P of G, Alperin showed in [Al] that these morphisms are locally controlled, i.e., by normalizers $N_G(Q)$ for Q a subgroup of P. Nine years later, Puig [Pu1] refined this and required Q to be an *essential* p-subgroup of G. In what follows, we will give the definition and some basic properties of essential p-subgroups of G.

DEFINITION 2.1. We say that Q is p-**centric** if Q is a Sylow p-subgroup of $QC_G(Q)$ or, equivalently, $Z(Q)$ is a Sylow p-subgroup of $C_G(Q)$.

In the literature [Th, p. 324], a p-centric subgroup is also called p-self-centralizing. Note that if Q is p-centric, then $C_P(Q) = Z(Q)$ for any Sylow p-subgroup P of G containing Q.

Consider now the Quillen complex $\mathcal{S}_p(H)$ of a finite group H whose vertices are the objects in $\mathcal{F}_p(H)$ and whose simplices are given by chains of groups ordered by inclusion.

DEFINITION 2.2. We say that Q is an **essential** subgroup of G if the Quillen complex $\mathcal{S}_p(N_G(Q)/Q)$ is disconnected and $C_G(Q)$ does not act transitively on the connected components.

One can find in [Th, Theorem 48.8] that

PROPOSITION 2.3. *Q is an essential p-subgroup of G if and only if Q is p-centric and $\mathcal{S}_p(N_G(Q)/QC_G(Q))$ is disconnected.*

The proof has been done in a more general case. In the terminology and notation of [Th, Theorem 48.8], it suffices to replace *local pointed groups* by *p-subgroups*, $\mathcal{N}_{>Q}$ by $\mathcal{S}_p(N_G(Q))_{>Q}$, and $\mathcal{O}G$ by G. In most of the proofs of this paper, we will use this proposition as an alternative definition of essential subgroups. For $g \in G$, we denote by ${}^gQ$ the conjugate by g of Q.

DEFINITION 2.4. We say that a subgroup H of a group G **controls** p-**fusion** in G if $(|G : H|, p) = 1$ and for any $g \in G$ and any Q, such that Q and ${}^gQ$ are contained in H, there exists $h \in H$ and $c \in C_G(Q)$ such that $g = hc$, or, equivalently, if the inclusion $H \hookrightarrow G$ induces an equivalence of categories $\mathcal{F}_p(H) \simeq \mathcal{F}_p(G)$.

The notions of control of fusion and essential p-subgroups are strongly linked. The next proposition shows one of the aspects of this link.

PROPOSITION 2.5 [Pu1, Ch. IV, Prop. 2]. *The normalizer $N_G(P)$ controls p-fusion in G if and only if there are no essential p-subgroups in G.*

The proof is based on the variant of Alperin's theorem using essential p-subgroups (see, for instance, [Th, Theorem 48.3]) and on the fact that the essential p-subgroups are preserved by any equivalence of categories.

3. GENERALIZED EXTRASPECIAL GROUPS

From now on, C_n will denote the cyclic group of order n.

DEFINITION 3.1. A p-group P is called **generalized extraspecial** if its Frattini subgroup, $\Phi(P)$, has order p, $\Phi(P) = [P, P] \simeq C_p$, and $Z(P) \geq \Phi(P)$. If, moreover, $Z(P) = \Phi(P)$, P is called **extraspecial**.

LEMMA 3.2. *Let P be a generalized extraspecial p-group. Then either $Z(P)$ is isomorphic to $\Phi(P) \times A$ and P is isomorphic to $E \times A$, or $Z(P)$ is isomorphic to $C_{p^2} \times A$ and E is isomorphic to $(E * C_{p^2}) \times A$, where E is an extraspecial p-group, A is an elementary abelian group, and $*$ means central product.*

Proof. As $\Phi(P)$ is a cyclic subgroup of order p, the center $Z(P)$ does not admit more than one factor isomorphic to C_{p^2} in its decomposition in cyclic subgroups, and if this factor exists, it contains $\Phi(P)$. Let A be an elementary abelian subgroup of $Z(P)$ such that $Z(P) \simeq \Phi(P) \times A$, when there is no C_{p^2} factor in $Z(P)$, and $Z(P) \simeq C_{p^2} \times A$, otherwise. We have, in both cases, $[P, P] \cap A = 1$ and $[P, A] = 1$, so A is a direct factor of P. It is then straightforward that the complement of A in P is isomorphic either to E or to $E * C_{p^2}$.

Recall that for $|P| = p^3$, we have that P is isomorphic either to $(C_p \times C_p) \rtimes C_p$ (in this case we say that P is of order p^3 and exponent p) or to $C_{p^2} \rtimes C_p$, for p odd, and either to the dihedral group D_8 or the quaternion group Q_8, for $p = 2$.

Let β: $P/Z(P) \times P/Z(P) \to \Phi(P)$ defined by $\beta(\bar{x}, \bar{y}) = [x, y]$. It is a bilinear nondegenerate symplectic form on $U := P/Z(P)$ viewed as a vector space over $\mathbf{F}_p$. We recall that an isotropic vector subspace of U with respect to β is a subspace on which β is identically zero. A maximal isotropic subspace of U has dimension equal to half of the dimension of U.

LEMMA 3.3. *Let Q be a p-centric subgroup of P. Then Q contains $Z(P)$ and $Q/Z(P)$ contains a maximal isotropic subspace of $P/Z(P)$.*

Proof. A p-centric subgroup of P clearly contains the center $Z := Z(P)$ of P. Suppose that $V := Q/Z(P)$, considered as vector space, does not contain a maximal isotropic subspace of $U := P/Z(P)$ with respect to β. This means that there exists $u \in U \backslash V$ with $\beta(u, x) = 0$, $\forall\, x \in V$. By taking a representative e of u in P, we have $e \in P \backslash Q$ and e commutes with all the elements of Q. So $e \in C_P(Q) \backslash Z(Q)$, which is a contradiction to the fact that Q is p-centric.

4. RESISTANT GROUPS

DEFINITION 4.1. A p-group P is called **resistant** if, for any finite group G such that P is a Sylow p-subgroup of G, the normalizer $N_G(P)$ controls p-fusion in G.

Here is now the main result of this paper.

THEOREM 4.2. *Let P be a generalized extraspecial p-group. Then P is resistant excepting the case when $P = E \times A$, where A is elementary abelian and E is dihedral of order* 8 (*when $p = 2$*) *or extraspecial of order p^3 and exponent p* (*when p is odd*).

COROLLARY 4.3. *If P satisfies the conditions of the theorem, then P is a Swan group.*

Proof of Theorem 4.2. We will prove that the only cases where G contains essential p-subgroups are the exceptions of our theorem. Let Q be a proper p-centric subgroup of P. This forces Q to contain $Z(P)$ and hence also $\Phi := \Phi(P)$. Denote by R the subgroup of $N := (N_G(Q) \cap N_G(\Phi))/C_G(Q)$ acting trivially on Φ and Q/Φ. We have that R centralizes the quotients of the central series $1 \lhd \Phi \lhd Q$, so it is a normal p-subgroup [Gor, Theorem 5.3.2] of N. Now R contains $P/Z(Q)$ as P acts trivially on Φ and Q/Φ. As P is a Sylow p-subgroup of G, this forces $R = P/Z(Q)$, and thus R is the unique Sylow p-subgroup of N, and thus $S_p(N)$ is connected.

Assume that Q is essential. Then $S_p(N_G(Q)/QC_G(Q))$ is disconnected and therefore $N_G(Q) \neq N_G(Q) \cap N_G(\Phi)$. As the $\Phi(Q)$ is characteristic in Q and is contained in Φ, we have that $\Phi(Q)$ is a proper subgroup of Φ, hence trivial; this gives that Q is elementary abelian. Take $x \in N_G(Q) \backslash N_G(\Phi)$. Now $R = P/Q$ is not contained in $(N_G(Q) \cap N_G({}^x\Phi))/C_G(Q)$; otherwise

$N/C_G(Q)$ and $(N_G(Q) \cap N_G({}^x\Phi))/C_G(Q)$ would have the same Sylow p-subgroup R, implying that $P/Q = {}^x(P/Q)$ and thus that x normalizes P. It follows that $\Phi = {}^x\Phi$, which is in contradiction with the choice of x. As ${}^x\Phi$ is a subgroup of P of order p, the vector subspace ${}^x\Phi/(Z(P) \cap {}^x\Phi)$ of $P/Z(P)$ admits an orthogonal complement with respect to β which is either all $P/Z(P)$ or a hyperplane. This gives that $|P : C_P({}^x\Phi)| = 1$ or p. If Q is a proper subgroup of $C_P({}^x\Phi)$, then $C_P({}^x\Phi)$ is non-abelian, and therefore $\Phi = \Phi(C_P({}^x\Phi))$. Moreover, ${}^{x^{-1}}(C_P({}^x\Phi)/Q) \subset (C_{N_G(Q)}(\Phi)/Q)$ so, by Sylow's theorem, there exists $c \in (C_{N_G(Q)}(\Phi)/Q)$ such that ${}^{cx^{-1}}(C_P({}^x\Phi)/Q) \subset (C_P(\Phi)/Q)$. This implies that ${}^{cx^{-1}}\Phi = \Phi$, which is equivalent to $\Phi = {}^x\Phi$, and we obtain once again a contradiction. Hence $Q = C_P({}^x\Phi)$ and $|P : Q| = p$. We also have that $Q/Z(P)$ is a maximal isotropic subspace of $P/Z(P)$; it follows that $|P : Z(P)| = p^2$. Moreover, $C_P({}^x\Phi)$ is a proper subgroup of P, so ${}^x\Phi$ is not contained in $Z(P)$, implying that $Z(P) \neq {}^xZ(P)$. By the same type of arguments, taking x^{-1} instead of x, we can also prove that Φ is not contained in ${}^xZ(P)$.

Finally, take $A := Z(P) \cap {}^xZ(P)$. As $|Q : Z(P)| = |Q : {}^xZ(P)| = p$ and $Z(P) \neq {}^xZ(P)$, we obtain that $|Z(P) : A| = p$, so Q/A is isomorphic to $C_p \times C_p$. Moreover, A does not contain Φ so, by Lemma 3.2, $Z(P) \simeq \Phi \times A$ and $P \simeq E \times A$, where E is an extraspecial group of order p^3. First, as Q/A is isomorphic to $C_p \times C_p$, E cannot be isomorphic to the quaternion group. Second, we will prove that the case where E is isomorphic to $C_{p^2} \rtimes C_p$ also yields to a contradiction. The result is due to Glauberman [MP], but the proof we give, which is more elegant, is due to Jacques Thévenaz.

Let $K := \langle P/Q, {}^x(P/Q) \rangle$, which is isomorphic to a subgroup of $\mathrm{Aut}(Q/A)$ viewed as a subgroup of $\mathrm{GL}(2, \mathbf{F}_p)$. As $P/Q \neq {}^x(P/Q)$, they generate all $\mathrm{SL}(2, \mathbf{F}_p)$, so $\mathrm{SL}(2, \mathbf{F}_p)$ is a subgroup of K containing P/Q. Now P/Q is a Sylow p-subgroup of K and we will prove that the exact sequence $1 \to Q/A \to E \to P/Q \to 1$ can be extended to an exact sequence $1 \to Q/A \to L \to K \to 1$ and hence to an exact sequence $1 \to Q/A \to L' \to \mathrm{SL}(2, \mathbf{F}_p) \to 1$. To have this, it suffices to verify [Br, pp. 84–85] that the class $h(E)$ determined by E in $H^2(P/Q, Q/A)$ is K-stable; that is, for any $k \in K$, we have

$$\mathrm{res}^{P/Q}_{P/Q \cap {}^k(P/Q)} h(E) = \mathrm{res}^{{}^k(P/Q)}_{P/Q \cap {}^k(P/Q)} \mathrm{conj}_k(h(E)). \qquad (*)$$

Here res is the restriction in cohomology and conj_k is the morphism induced by the conjugation by k in cohomology. If $P/Q \neq {}^k(P/Q)$, then $P/Q \cap {}^k(P/Q) = 1$ and the relation $(*)$ is trivially satisfied. Suppose that $P/Q = {}^k(P/Q)$. Take $\tilde{k}$ to be a representative of k in $N_G(Q)$ that normalizes P. We have that $\tilde{k}$ induces the conjugation by k on Q and P/Q. So the conjugation by $\tilde{k}$ induces conj_k on $H^2(P/Q, Q/A)$. Thus

$h(E) = \mathrm{conj}_k(h(E))$ and $(*)$ is again satisfied. Now, for $E \simeq C_{p^2} \rtimes C_p$, $h(E)$ is not trivial.

The contradiction comes from the fact that $H^2(\mathrm{SL}(2, \mathbf{F}_p), Q/A) = 0$, so the cohomology class $h(E)$ induced by E in $H^2(P/Q, Q/A)$ would be trivial. Indeed let $U := \{\left(\begin{smallmatrix}1 & * \\ 0 & 1\end{smallmatrix}\right)\}$ be a Sylow p-subgroup of $\mathrm{SL}(2, \mathbf{F}_p)$. Write $S := \mathrm{SL}(2, \mathbf{F}_p)$ and $N(U) := N_{\mathrm{SL}(2,\mathbf{F}_p)}(U)$. The restriction to U in cohomology induces a monomorphism $\mathrm{res}_U^S \colon H^2(S, Q) \to H^2(U, Q)^{N(U)}$, where $H^2(U, Q)^{N(U)}$ are the fixed points under the natural action of $N(U)$. Now $U = \langle u \rangle$ is a cyclic group, so [Be, p. 60] its cohomology is

$$H^2(U, Q) = Q^U \Big/ \left\{ \left(\sum_{i=0}^{p-1} u^i \right) v \,\middle|\, v \in Q \right\}.$$

By a simple computation, we obtain $Q^U = \langle z \rangle$, where z is a generator of $\Phi(P)$ and $\{(\sum_{i=0}^{p-1} u^i)v | v \in Q\} = 0$, so $H^2(U, Q) = \langle z \rangle$. As z is not fixed by $N(U)$, we have $H^2(U, Q)^{N(U)} = 0$, and therefore $H^2(S, Q) = 0$.

We prove now that the remaining case, $P = E \times A$ with E either dihedral of order 8 (when $p = 2$) or extraspecial of order p^3 and exponent p (when p is odd), is indeed an exception to Theorem 4.2. Let us start with a property of resistant groups:

Proposition 4.4. *Let P be a p-group and let B be a finite abelian p-group. If P is not resistant, then the direct product $P \times B$ is not resistant.*

Proof. Let G be a finite group with P as Sylow p-subgroup and let Q be an essential p-subgroup of G embedded in P. Such a G exists because we suppose that P is not resistant. In this case, $\widetilde{P} := P \times B$ is a Sylow p-subgroup of $\widetilde{G} := G \times B$. As Q is p-centric in P, so is $\widetilde{Q} := Q \times B$ in $\widetilde{P}$. Moreover, $N_{\widetilde{G}}(\widetilde{Q})/\widetilde{Q}C_{\widetilde{G}}(\widetilde{Q}) \simeq N_G(Q)/QC_G(Q)$. This means that, as $\mathcal{S}_p(N_G(Q)/QC_G(Q))$ is disconnected, so is $\mathcal{S}_p(N_{\widetilde{G}}(\widetilde{Q})/\widetilde{Q}C_{\widetilde{G}}(\widetilde{Q}))$. Then $\widetilde{Q}$ is an essential p-subgroup of $\widetilde{G}$. This proves that $\widetilde{P}$ is not resistant.

Proposition 4.5. *Let $P = E \times A$, where A is elementary abelian and E is dihedral of order* 8 (*when* $p = 2$) *or of order p^3 and exponent p* (*when p is odd*). *Then P is not resistant.*

Proof. We can realize E as a Sylow p-subgroup of $\mathrm{GL}(3, \mathbf{F}_p)$. One can verify that

$$Q_1 = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad Q_2 = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

are essential subgroups of G. So E is not resistant. As P is isomorphic to $E \times A$, where A is elementary abelian, by Proposition 4.4, P is not resistant.

In a very recent paper [Pu2], Puig introduced the notion of "full Frobenius system," which is a category over a finite p-group P whose objects are the subgroups of P and whose morphisms are a set of injective morphisms between the subgroups of P containing the conjugation by the elements of P. The morphisms satisfy some natural axioms which are inspired by the local properties of P when P is a Sylow p-subgroup of a finite group or a defect group of a block in a group algebra. Puig defined in this context the concept of "essential group" and proved that, on a full Frobenius system, the analog of Alperin's Fusion Theorem holds. Full Frobenius systems are the generalization of the Frobenius category of a group, and of the Brauer and Puig categories of a block.

The theorem in this paper remains true and all the arguments were chosen to remain valid in a full Frobenius system over P. This permits us to generalize the results to Brauer pairs and pointed groups.

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