

Extended-order algebras and fuzzy implicators

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Abstract

In this work we reconsider the notion of implicator in a complete lattice L and discuss its properties, taking into account the viewpoint of the implication operation of classes of *(weak) extended-order algebras*, introduced by C. Guido and P. Toto and included in the class of implicative algebras considered by E. Rasiowa. In fact, such an implication, that is an extension of an order relation, can be viewed as an implicator in L , whose properties depend on those characterizing the structure of the algebra.

We also propose in a *(weak) right-distributive complete extended-order algebra* (L, \rightarrow, \top) with adjoint product \otimes a *relative implication* as a further implicator beyond \rightarrow .

The relative implication allows an extension of the inclusion relation between L -sets A and B , different from the subethood degree, that consists in seeing to which extent A is included in its conjunction with B .

Moreover, we introduce in (L, \rightarrow, \top) a further binary operation that we call *conditional conjunction* that can be read as “ a ” and “ b , given a ”, which motivates the term we have chosen to denote it.

This operation, strictly related to the divisibility condition of BL algebras, satisfies most conditions usually asked to a conjunction and it is well behaved with the adjoint product \otimes and the meet operation \wedge .

Keywords Lattice-ordered structures, Residuated lattices, Extended-order algebras, Implicators, Conjunctors.

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1 Introduction

The wide development of lattice-valued mathematics in the last decades has stimulated the study of several kinds of lattice-ordered algebras whose operations and properties reflect the main characters of the underlying non-classical logics; fundamental monographs are [4, 9, 13]. These kinds of algebras fall within the context of residuated structures (see also [3, 14]).

Recently, a quite general approach, with the same starting point, e. g. implicative algebras, as in the Rasiowa's monograph [15], has been introduced in [12] and developed in [6], where *extended-order algebras* have been considered and studied. The structure of these algebras is based on a very general implication \rightarrow that is a many-valued extension of the order relation of a poset (L, \leq, \top) with a greatest element \top , in the sense that $a \rightarrow b = \top$ if and only if $a \leq b$ (see Definition 3.1 below). It is shown in [6, 12] that the algebraic structure of extended-order algebras, that include most residuated lattices, is completely determined by the implication (a sort of internal many-valued order) just like the structure of any lattice is determined by the underlying order. In fact, one can say that [6, 12] present an approach to residuated lattices that is more order-theoretical than algebraic, for instance paying more attention to the completeness of the underlying order than to the monoidal character of the structure. Details on this topic that are needed in the present paper are explained in Section 3.

Some applications of extended-order algebras can be found in [8], where fuzzy Galois connections determined by many-valued binary relations are considered and in [11] where a lattice-valued approach to category theory is proposed. This paper deals with further applications of extended-order algebras in connection with the fundamental concepts of implicators and conjunctors largely used and studied in many contexts (see, for instance, [1, 2, 7, 16, 17]).

In Section 4 the notion of implicator will be reconsidered in the light of the conditions that the implication of an extended-order algebra satisfies, and the notion of relative implication is also introduced.

In Section 5 the notion of conditional conjunction is introduced and studied, which is strictly related to the divisibility condition of BL-algebras.

2 Preliminaries

In this Section we recall some standard notions and notations on order and lattice structures according to [4, 10, 13].

If (L, \leq) is a poset, i.e. a partially ordered set, and $S \subseteq L$, we denote by $Ub(S)$ the set of all upper bounds of S and by $Lb(S)$ the set of all lower bounds of S . We use, moreover, the following notation: $Ub(a) = Ub(\{a\}) = \uparrow a$ and $Lb(a) = Lb(\{a\}) = \downarrow a$, for all $a \in L$. $\bigvee S$ denotes the least upper bound of S and $\bigwedge S$ the greatest lower bound of S , if these exist and we call $\bigvee S$ ($\bigwedge S$, respectively) the *supremum* or *join* (*infimum* or *meet*, respectively) of S . (L, \leq) , with $|L| \geq 2$, is a *lattice* if, for all finite subsets $F \subseteq L$, $\bigvee F$ and $\bigwedge F$ exist in L . (L, \leq) , with $|L| \geq 2$, is a *complete lattice* if, for all arbitrary subsets $F \subseteq L$, $\bigvee F$ and $\bigwedge F$ exist in L . An *adjunction* also called *isotonic Galois connection* between partially ordered sets L and M , denoted

by $f \dashv g : (L, \leq) \rightarrow (M, \leq)$ or simply by $f \dashv g$, is a pair of maps $f : L \rightarrow M$ and $g : M \rightarrow L$ satisfying the condition

$$\forall x \in L, y \in M: x \leq g(y) \Leftrightarrow f(x) \leq y.$$

The map f is called *left adjoint* of g and g *right adjoint* of f .

In the following we denote $f(A) = \{f(a) \mid a \in A\}$ and $f_-(B) = \{a \in L \mid f(a) \in B\}$, for all $f : L \rightarrow M$, $A \subseteq L$, $B \subseteq M$.

Lemma 2.1. *Let (L, \leq) and (M, \leq) be posets and $f : L \rightarrow M$ and $g : M \rightarrow L$ be maps.*

- *If $f \dashv g$, then f preserves existing joins and g preserves existing meets. Moreover one has*
 1. *for every $x \in L$, $f(x) = \bigwedge \{y \in M \mid x \leq g(y)\}$ and f is the unique left adjoint of g ;*
 2. *for every $y \in M$, $g(y) = \bigvee \{x \in L \mid f(x) \leq y\}$ and g is the unique right adjoint of f .*
- *If (L, \leq) is a complete lattice and $f : L \rightarrow M$ preserves \bigvee , then the function*

$$g : M \rightarrow L, y \mapsto g(y) = \bigvee \{x \in L \mid f(x) \leq y\}$$

is the unique right adjoint of f .

- *If (M, \leq) is a complete lattice and $g : M \rightarrow L$ preserves \bigwedge , then the function*

$$f : L \rightarrow M, x \mapsto f(x) = \bigwedge \{y \in M \mid x \leq g(y)\}$$

is the unique left adjoint of g .

A *Galois connection* also called *antitonic Galois connection* between two posets (L, \leq) and (M, \leq) , denoted by $[f, g] : (L, \leq) \leftrightarrow (M, \leq)$ or simply by $[f, g]$, is a pair of maps $f : L \rightarrow M$ and $g : M \rightarrow L$ satisfying the condition

$$\forall x \in L, y \in M: x \leq g(y) \Leftrightarrow y \leq f(x).$$

Of course $[g, f]$ is a Galois connection if and only if $[f, g]$ is.

Lemma 2.2. • *If $[f, g] : (L, \leq) \leftrightarrow (M, \leq)$ is a Galois connection then f maps existing joins into corresponding meets and for every $y \in M$, $g(y) = \bigvee \{x \in L \mid y \leq f(x)\}$ and g is the unique function such that $[f, g]$ is a Galois connection.*

- *If (L, \leq) is a complete lattice, (M, \leq) a poset and $f : L \rightarrow M$ a function such that $f(\bigvee A) = \bigwedge f(A)$, for every $A \subseteq L$, then the function $g : M \rightarrow L$ defined by*

$$g(y) = \bigvee \{x \in L \mid y \leq f(x)\}, \text{ for every } y \in M,$$

is the unique function such that $[f, g]$ is a Galois connection.

3 Extended-order algebras

All the definitions (explicitly referenced) and results recalled in this Section come from [6] and [12], where one can find more details on motivations, examples and technical aspects of (weak) extended-order algebras and their relationship with other lattice-ordered structures.

Definition 3.1. [12] Let L be a non-empty set, $\rightarrow: L \times L \rightarrow L$ a binary operation and \top a fixed element of L . The triple (L, \rightarrow, \top) is a weak extended-order algebra, shortly w-eo algebra, if for all $a, b, c \in L$ the following conditions are satisfied

- (o₁) $a \rightarrow \top = \top$ (upper bound condition);
- (o₂) $a \rightarrow a = \top$ (reflexivity condition);
- (o₃) $a \rightarrow b = \top$ and $b \rightarrow a = \top \Rightarrow a = b$ (antisymmetry condition);
- (o₄) $a \rightarrow b = \top$ and $b \rightarrow c = \top \Rightarrow a \rightarrow c = \top$ (weak transitivity condition).

Remark 3.2. w-eo algebras are exactly the implicative algebras defined in [15]; however, the subclasses of w-eo algebras considered in [6, 12] and in this paper, as well, are rather different from the subclasses of implicative algebras treated in [15].

Proposition 3.3. For every w-eo algebra (L, \rightarrow, \top) the relation determined by the operation \rightarrow , by means of the equivalence

$$a \leq b \text{ if and only if } a \rightarrow b = \top, \text{ for all } a, b \in L$$

is an order relation called natural ordering in L with greatest element \top . This order relation is called the natural ordering in L .

Conversely, if (L, \leq) is a poset with greatest element \top and $\rightarrow: L \times L \rightarrow L$ extends \leq , i.e. $a \rightarrow b = \top \Leftrightarrow a \leq b$, for all $a, b \in L$, then (L, \rightarrow, \top) is a w-eo algebra.

Definition 3.4. [12] (L, \rightarrow, \top) is a right w-eo algebra if it satisfies the axioms (o₁), (o₂), (o₃) and (o₅) $a \rightarrow b = \top \Rightarrow (c \rightarrow a) \rightarrow (c \rightarrow b) = \top$ (weak isotonic condition in the second variable).

(L, \rightarrow, \top) is a left w-eo algebra if it satisfies the axioms (o₁), (o₂), (o₃) and

(o₅') $a \rightarrow b = \top \Rightarrow (b \rightarrow c) \rightarrow (a \rightarrow c) = \top$ (weak antitonic condition in the first variable).

(L, \rightarrow, \top) is an extended-order algebra, shortly eo algebra, if it is a right and a left w-eo algebra.

All the above defined and the subsequently considered algebras (L, \rightarrow, \top) are said to be *complete* if L with the natural ordering \leq induced by \rightarrow is a complete lattice; in this case \top is the greatest element, of course, and the least element is denoted by \perp . So we shall consider, with a short obvious notation, (left), (right) w-ceo algebras and ceo algebras.

In this paper we consider only complete structures; we note that the completeness condition is restrictive for w-eo algebras, but it is not restrictive for eo algebras, which in fact can be embedded in their MacNeille completion. For details see [6, 12].

In the following we denote $P \rightarrow Q = \{p \rightarrow q \mid p \in P, q \in Q\}$, for any $P, Q \subseteq L$.

Definition 3.5. [12] Let (L, \rightarrow, \top) be a w-ceo algebra.

(L, \rightarrow, \top) is right-distributive if it satisfies the right distributivity condition

(d_r) for any $a \in L, B \subseteq L$: $a \rightarrow \bigwedge B = \bigwedge (a \rightarrow B)$.

(L, \rightarrow, \top) is left-distributive if it satisfies the left distributivity condition

(d_l) for any $b \in L, A \subseteq L \Rightarrow (\bigvee A) \rightarrow b = \bigwedge (A \rightarrow b)$.

(L, \rightarrow, \top) is distributive if it satisfies the distributivity condition

(d) for any $A, B \subseteq L: \bigvee A \rightarrow \bigwedge B = \bigwedge (A \rightarrow B)$.

Of course (L, \rightarrow, \top) is distributive if and only if it is left-distributive and right-distributive. We also say that the operation \rightarrow is (right-)(left-)distributive if $((d_r))((d_l))(d)$ is satisfied. We adopt the short notation *cdeo* for distributive complete extended-order algebras.

Note that every distributive complete *w-eo* algebra (L, \rightarrow, \top) is a complete *eo* algebra.

Most of our constructions and results only need to assume (L, \rightarrow, \top) to be a right-distributive *w-ceo* algebra, which implies that the condition (o_5) is satisfied; stronger assumptions will be specified when they will be necessary.

If (L, \rightarrow, \top) is a right-distributive *w-ceo* algebra, then the adjoint product $\otimes : L \times L \rightarrow L$ is defined by

$$a \otimes x = \bigwedge \{t \in L \mid x \leq a \rightarrow t\}.$$

Then \otimes and \rightarrow form an adjoint pair, i.e. for all $x, y, z \in L$:

$$x \otimes y \leq z \Leftrightarrow y \leq x \rightarrow z.$$

Proposition 3.6. *Let (L, \rightarrow, \top) be a right-distributive *w-ceo* algebra and let \otimes be the adjoint product. The following properties hold, for all $a, b, c \in L, A, B \subseteq L$.*

1. $a \otimes b \leq a$;
2. $a \otimes \top = a$;
3. $a \otimes \perp = \perp \otimes a = \perp$;
4. $\top \otimes a = a$ if and only if $(\forall x \in L : a \leq \top \rightarrow x \Leftrightarrow a \leq x)$;
5. $a \otimes (\bigvee B) = \bigvee (a \otimes B)$;
6. if $b \leq c$, then $a \otimes b \leq a \otimes c$;
7. $a \otimes (a \rightarrow b) \leq b \leq a \rightarrow (a \otimes b)$;
8. if, moreover, L is a *ceo* algebra and $a \leq b$, then $a \otimes c \leq b \otimes c$;
9. if L is a *cdeo* algebra, then $(\bigvee A) \otimes b = \bigvee (A \otimes b)$.

Definition 3.7. [6] A *w-ceo* algebra (L, \rightarrow, \top) is called symmetrical if there exists a binary operation $\rightsquigarrow : L \times L \rightarrow L$ such that $(L, \rightsquigarrow, \top)$ is a *w-eo* algebra, \rightarrow and \rightsquigarrow have the same natural ordering and $y \leq x \rightsquigarrow b \Leftrightarrow x \leq y \rightarrow b$, for all $b, x, y \in L$. The *w-ceo* algebras (L, \rightarrow, \top) , $(L, \rightsquigarrow, \top)$ and their implications $\rightarrow, \rightsquigarrow$ are said to be dual to each other and $[\rightarrow, \rightsquigarrow]$ is the related Galois pair.

The above Definition has a symmetrical character, so, with the above notation, $(L, \rightsquigarrow, \top)$ is symmetrical if and only if (L, \rightarrow, \top) is symmetrical.

Lemma 3.8. *If (L, \rightarrow, \top) is a symmetrical w-ceo algebra and \rightsquigarrow is the dual implication, then for all $a, b \in L$ one has*

$$a \leq (a \rightsquigarrow b) \rightarrow b \text{ and } a \leq (a \rightarrow b) \rightsquigarrow b.$$

Moreover, since \rightarrow and \rightsquigarrow form a Galois pair, each of them is uniquely determined by the other.

The following Theorem gives an internal characterization of symmetrical w-ceo algebras.

Theorem 3.9. *Let (L, \rightarrow, \top) be a w-ceo algebra. L is symmetrical if and only if it is left-distributive and the equivalence $a \leq \top \rightarrow x \Leftrightarrow a \leq x$ holds, for all $a, x \in L$.*

Proposition 3.10. *If (L, \rightarrow, \top) is a symmetrical right-distributive w-ceo algebra and \rightsquigarrow the dual implication, then both (L, \rightarrow, \top) and $(L, \rightsquigarrow, \top)$ satisfy (d_l) and the one is a cdeo algebra if and only if the other is.*

The following Proposition characterizes the symmetrical cdeo algebras in terms of the adjoint product.

Proposition 3.11. *Let (L, \rightarrow, \top) be a right-distributive w-ceo algebra and let \otimes be the adjoint product. Then the following are equivalent:*

1. (L, \rightarrow, \top) is symmetrical.
2. If $a \in L$ and $B \subseteq L$ then $\top \otimes a = a$ and $(\bigvee B) \otimes a = \bigvee (B \otimes a)$.

Proposition 3.12. *Let (L, \rightarrow, \top) be a symmetrical cdeo algebra, \rightsquigarrow the dual implication and \otimes the adjoint product. Then, for all $a, b, c \in L$, the following properties hold.*

1. $a \leq b \rightsquigarrow c \Leftrightarrow a \otimes b \leq c$;
2. $a \otimes b \leq a \wedge b$;
3. $(a \rightsquigarrow b) \otimes a \leq b \leq a \rightsquigarrow (b \otimes a)$;
4. $b \leq a \rightarrow b, b \leq a \rightsquigarrow b$;
5. $\top \rightarrow a = a, \top \rightsquigarrow a = a$.

It is a standard way to define a negation \neg in a bounded implicative structure (L, \rightarrow, \top) with minimum \perp by setting $\neg a = a \rightarrow \perp$.

Definition 3.13. [6] *In a w-ceo algebra (L, \rightarrow, \top) the negation is defined by*

$$[\cdot]^- : L \rightarrow L, x \mapsto x^- = x \rightarrow \perp.$$

If (L, \rightarrow, \top) is symmetrical, then a dual negation is defined by

$$[\cdot]^\sim : L \rightarrow L, x \mapsto x^\sim = x \rightsquigarrow \perp$$

The negation $[\cdot]^-$ ($[\cdot]^\sim$, respectively) and the corresponding algebra is called involutive if $x^{--} = x$ ($x^{\sim\sim} = x$, respectively), for every $x \in L$.

The following Proposition states basic properties of the negation in a quite general context; all the properties we shall state for a negation also hold for the dual negation under the assumption of symmetry.

Proposition 3.14. *Let (L, \rightarrow, \top) be a left w-ceo algebra. Then the following hold, for all $x, y \in L, \{x_i\}_{i \in I} \subseteq L$.*

1. $\perp^- = \top$;
2. $x \leq y \Rightarrow y^- \leq x^-$.
3. $(\bigvee_{i \in I} x_i)^- \leq \bigwedge_{i \in I} x_i^-$;
4. $(\bigwedge_{i \in I} x_i)^- \geq \bigvee_{i \in I} x_i^-$;

If, moreover, the negation $[\cdot]^-$ is involutive, then

5. $\top^- = \perp$;
6. $(\bigvee_{i \in I} x_i)^- = \bigwedge_{i \in I} x_i^-$;
7. $(\bigwedge_{i \in I} x_i)^- = \bigvee_{i \in I} x_i^-$.

Proposition 3.15. *Let (L, \rightarrow, \top) be a w-ceo algebra, $x, y \in L$ and $\{x_i\}_{i \in I} \subseteq L$; then under (d_r) one has*

1. $x \otimes x^- = \perp$;
2. $x \leq y^- \Leftrightarrow y \otimes x = \perp$;
while (d_l) implies
3. $(\bigvee_{i \in I} x_i)^- = \bigwedge_{i \in I} x_i^-$.

Eventually, we list further properties involving the pair of dual negations of a symmetrical cdeo algebra.

Proposition 3.16. *Let (L, \rightarrow, \top) be a symmetrical cdeo algebra. Then the following hold, for all $x, y \in L$.*

1. $\top^- = \perp, \top^\sim = \perp$;
2. $x^\sim \otimes x = \perp$;
3. $x \leq x^{-\sim}, x \leq x^{\sim-}$;
4. $x \leq y^\sim \Leftrightarrow x \otimes y = \perp$;

5. $x \leq y^\sim \Leftrightarrow y \leq x^-$;
6. $x \leq x^- \rightsquigarrow y, x \leq x^\sim \rightarrow y$;
7. $x^{--} = x^-, x^{\sim\sim} = x^\sim$.

Definition 3.17. [12] A *w-ceo algebra* (L, \rightarrow, \top) is commutative if and only if it satisfies the weak exchange condition

(c) $a \rightarrow (b \rightarrow c) = \top \Leftrightarrow b \rightarrow (a \rightarrow c) = \top$, for all $a, b, c \in L$.

Proposition 3.18. For a right-distributive *w-ceo algebra* the following are equivalent:

1. (L, \rightarrow, \top) is commutative;
2. the adjoint product \otimes is commutative;
3. (L, \rightarrow, \top) is symmetrical and \rightsquigarrow coincides with \rightarrow .

Definition 3.19. [12] A *w-ceo algebra* (L, \rightarrow, \top) is associative if and only if it satisfies the following condition

(a) $a \rightarrow (b \rightarrow c) = (\bigwedge \{x \mid a \rightarrow (b \rightarrow x) = \top\}) \rightarrow c$, for all $a, b, c \in L$.

Remark 3.20. Let (L, \rightarrow, \top) be a right-distributive *w-ceo algebra*. If \otimes denotes the adjoint product, then the associativity condition (a) is equivalent to

(a') $a \rightarrow (b \rightarrow c) = (b \otimes a) \rightarrow c$, for all $a, b, c \in L$.

Proposition 3.21. A right-distributive *w-ceo algebra* (L, \rightarrow, \top) is associative if and only if its adjoint product \otimes is associative.

Corollary 3.22. Let (L, \rightarrow, \top) be a symmetrical *cdeo algebra* and \rightsquigarrow the dual implication. Then (L, \rightarrow, \top) is associative if and only if $(L, \rightsquigarrow, \top)$ is associative.

Remark 3.23. Associative symmetrical (commutative) *cdeo algebras* are exactly the implicative fragment of complete integral (commutative) residuated lattices. The possible lack of monoidal structure is the main character of extended-order algebras: even when the adjoint product exists, i.e. under (d_r) , it need not be associative and if the algebra is not symmetrical \top need not be a left unit. The strength of the associativity condition is explained in [6] and it allows the following results.

Proposition 3.24. Let (L, \rightarrow, \top) be a *cdeo algebra* and let \otimes be its adjoint product. If L is associative, then, for all $a, b, c \in L$:

1. $(a \rightarrow b) \otimes (b \rightarrow c) \leq a \rightarrow c$;
2. $(b \rightarrow c) \leq (a \rightarrow b) \rightarrow (a \rightarrow c)$;
3. $a \rightarrow b^- = (b \otimes a)^-$.

Proposition 3.25. Let (L, \rightarrow, \top) be a symmetrical *cdeo algebra*. If it is associative, then the following properties hold, for all $a, b, c \in L$.

1. $a \rightsquigarrow b \leq b^{\sim} \rightarrow a^{\sim}, a \rightarrow b \leq b^{-} \rightsquigarrow a^{-};$
2. $a \rightsquigarrow b^{-} = b \rightarrow a^{\sim}, a \rightarrow b^{\sim} = b \rightsquigarrow a^{-};$
3. $a \rightsquigarrow a^{-} = a \rightarrow a^{\sim};$
4. $b^{\sim} \rightarrow a^{\sim} = a^{\sim-} \rightsquigarrow b^{\sim-} = a \rightsquigarrow b^{\sim-}, b^{-} \rightsquigarrow a^{-} = a^{-\sim} \rightarrow b^{-\sim} = a \rightarrow b^{-\sim};$
5. $a \rightsquigarrow (b \rightarrow c) = b \rightarrow (a \rightsquigarrow c).$

4 Fuzzy implicators and the conditional implication

In the literature the term fuzzy implication operator, shortly fuzzy implicator, usually is meant as an extension of the implication defined in classical logic; in fact in the most general sense it is defined as a map $\mathcal{I} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ that satisfies the boundary conditions

$$(b) \mathcal{I}(0, 0) = \mathcal{I}(0, 1) = \mathcal{I}(1, 1) = 1 \text{ and } \mathcal{I}(1, 0) = 0.$$

Further properties are considered and assumed on \mathcal{I} in different theoretical approaches and applications; a detailed list of the most important (some of which imply the equalities in the boundary condition (b)) is given in [17] as follows:

- (fi_1) $a \leq b \Rightarrow \mathcal{I}(b, c) \leq \mathcal{I}(a, c)$, for all $a, b, c \in [0, 1]$;
- (fi_2) $a \leq b \Rightarrow \mathcal{I}(c, a) \leq \mathcal{I}(c, b)$, for all $a, b, c \in [0, 1]$;
- (fi_3) $\mathcal{I}(0, b) = 1$, for every $b \in [0, 1]$;
- (fi_4) $\mathcal{I}(a, 1) = 1$, for every $a \in [0, 1]$;
- (fi_5) $\mathcal{I}(1, 0) = 0$;
- (fi_6) $\mathcal{I}(1, b) = b$, for every $b \in [0, 1]$;
- (fi_7) $\mathcal{I}(a, \mathcal{I}(b, c)) = \mathcal{I}(b, \mathcal{I}(a, c))$, for all $a, b, c \in [0, 1]$;
- (fi_8) $\mathcal{I}(a, b) = 1 \Leftrightarrow a \leq b$, for all $a, b \in [0, 1]$;
- (fi_9) the map N' , defined as $N'(a) = \mathcal{I}(a, 0)$, for every $a \in [0, 1]$, is an involutive fuzzy negation;
- (fi_{10}) $\mathcal{I}(a, b) \geq b$, for all $a, b \in [0, 1]$;
- (fi_{11}) $\mathcal{I}(a, a) = 1$, for every $a \in [0, 1]$;
- (fi_{12}) $\mathcal{I}(a, b) = \mathcal{I}(N(b), N(a))$, for every $a \in [0, 1]$, where N is an involutive fuzzy negation;
- (fi_{13}) \mathcal{I} is a continuous mapping.

Evidently there are several interrelationships among these axioms. In [17] there is a complete view of these: there, taking (fi_1)-(fi_5) as the basic system of axioms for a fuzzy implicator the authors investigate the dependence and independence of the other axioms, given the first five that, in their turn, implies the boundary condition (b).

In this work we reconsider the above conditions and we give a definition of an internal implicator in a complete lattice L , taking into account the point of view of the implication operation of extended-order algebras.

This leads to a different arrangement and grouping of the basic requirements a fuzzy implicator should satisfy, according to the interpretation of the implication operation in the semantic of many-valued logic that motivated the introduction and development of extended-order algebras [6, 12],

strictly related to implicative algebras considered in [15].

We recalled in Section 3 that the implication of a *w-eo* algebra (i.e. implicative algebra) is any internal extension of the order relation of any poset with a greatest element (true value \top), as stated in Proposition 3.3. This criterion states exactly in which cases the implication $a \rightarrow b$ is true saying nothing, in general, on when the implication is false, even if the existence of the value "false" (the least element \perp) in the algebra is assumed.

So our feeling is that we should agree with the requirement $\mathcal{I}(0, 0) = \mathcal{I}(0, 1) = \mathcal{I}(1, 1) = 1$ without asking to $\mathcal{I}(1, 0)$ nothing but to be different from \top , in general, which makes the implication of a *w-ceo* algebra a bounded impicator, as we shall see. Nevertheless, we shall weaken further the requirements of an *impicator*, making it more general than the implication operation of a *w-ceo* algebra.

Definition 4.1. Let (L, \leq) be a complete lattice, with greatest element \top and least element \perp . A map $\mathcal{I} : L \times L \rightarrow L$ is an **impicator** in L if it satisfies the following axiom

(i) $\mathcal{I}(a, b) = \top \Rightarrow a \leq b \Rightarrow \mathcal{I}(b, a) \leq \mathcal{I}(a, b)$, for all $a, b \in L$.

The impicator \mathcal{I} is **bounded** if it satisfies the following axioms

(i₁) $\mathcal{I}(\perp, b) = \top$, for every $b \in L$;

(i₂) $\mathcal{I}(a, \top) = \top$, for every $a \in L$.

The impicator \mathcal{I} is **weak-ordered** if it satisfies the following axiom

(i₃) $\mathcal{I}(a, b) = \top \Leftrightarrow a \leq b$, for all $a, b \in L$.

The impicator \mathcal{I} is **isotonic** if it satisfies the following axioms

(i₄) $a \leq b \Rightarrow \mathcal{I}(c, a) \leq \mathcal{I}(c, b)$, for all $a, b, c \in L$;

(i₅) $a \leq b \Rightarrow \mathcal{I}(b, c) \leq \mathcal{I}(a, c)$, for all $a, b, c \in L$.

The impicator \mathcal{I} is **crisp-bounded** if it satisfies the following axiom

(b) $\mathcal{I}(\perp, \perp) = \mathcal{I}(\perp, \top) = \mathcal{I}(\top, \top) = \top$ and $\mathcal{I}(\top, \perp) = \perp$.

The impicator \mathcal{I} is **ordered** if it satisfies the axioms (i₃), (i₄) and (i₅).

The impicator \mathcal{I} is **distributive** if it satisfies the following axioms

(i₆) $\mathcal{I}(a, \bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (\mathcal{I}(a, b_i))$, for all $a \in L, \{b_i\}_{i \in I} \subseteq L$;

(i'₆) $\mathcal{I}(\bigvee_{i \in I} a_i, b) = \bigwedge_{i \in I} (\mathcal{I}(a_i, b))$, for all $b \in L, \{a_i\}_{i \in I} \subseteq L$.

The impicator \mathcal{I} is **continuous** if it satisfies (i₆), (i'₆) and the following axioms

(i₇) $\mathcal{I}(a, \bigvee_{i \in I} b_i) = \bigvee_{i \in I} (\mathcal{I}(a, b_i))$, for all $a \in L, \{b_i\}_{i \in I} \subseteq L$;

(i'₇) $\mathcal{I}(\bigwedge_{i \in I} a_i, b) = \bigvee_{i \in I} (\mathcal{I}(a_i, b))$, for all $b \in L, \{a_i\}_{i \in I} \subseteq L$.

The impicator \mathcal{I} is **commutative** or **interchanging** if it satisfies the axiom

(i₈) $\mathcal{I}(a, \mathcal{I}(b, c)) = \top \Leftrightarrow \mathcal{I}(b, \mathcal{I}(a, c)) = \top$, for all $a, b, c \in L$.

The impicator \mathcal{I} is **involutive** if it satisfies the axiom

(i₉) the map N , defined by $N(a) = \mathcal{I}(a, \perp)$, for every $a \in L$, is an order reversing involution.

The impicator \mathcal{I} is **contrapositive** if it satisfies the axiom

(i₁₀) $\mathcal{I}(a, b) = \mathcal{I}(N(b), N(a))$, for all $a, b \in L$.

Eventually we list the other axioms for a fuzzy impicator considered in [17].

(i₁₁) $\mathcal{I}(\top, b) \leq b$, for every $b \in L$;

(i₁₂) $\mathcal{I}(\top, b) \geq b$, for every $b \in L$;

$(i_{13}) \mathcal{I}(a, b) \geq b$, for all $a, b \in L$.

The following Proposition shows in the item (1) that the implication of every one of the complete algebras we have considered in Section 3 is a weak-ordered, hence bounded, implicator; the subsequent statements (2)-(11) only state the further properties it satisfies depending on which kind of *w-ceo* algebra is considered.

Proposition 4.2. 1. *If (L, \rightarrow, \top) is a *w-ceo* algebra, then the operation \rightarrow is a bounded weak-ordered implicator.*

2. *If (L, \rightarrow, \top) is a right *w-ceo* algebra, then the implicator \rightarrow satisfies the axiom (i_4) , too.*
3. *If (L, \rightarrow, \top) is a left *w-ceo* algebra, then the implicator \rightarrow satisfies the axiom (i_5) , too.*
4. *If (L, \rightarrow, \top) is a *ceo* algebra, then \rightarrow is an ordered implicator, too.*
5. *If (L, \rightarrow, \top) is a right-distributive *w-ceo* algebra, then the implicator \rightarrow satisfies the axioms (i_4) and (i_6) , too.*
6. *If (L, \rightarrow, \top) is a left-distributive *w-ceo* algebra, then the implicator \rightarrow satisfies the axioms (i_5) and (i'_6) , too.*
7. *If (L, \rightarrow, \top) is a *cdeo* algebra, then \rightarrow is an ordered distributive implicator, too.*
8. *If (L, \rightarrow, \top) is a symmetrical *cdeo* algebra, then both \rightarrow and \rightsquigarrow are ordered, distributive and crisp-bounded implicators that satisfy the axioms (i_{11}) , (i_{12}) and (i_{13}) .*
9. *If (L, \rightarrow, \top) is an involutive left *w-ceo* algebra, then \rightarrow is an involutive implicator, too.*
10. *If (L, \rightarrow, \top) is a commutative *cdeo* algebra, then \rightarrow is an ordered, distributive, crisp-bounded and commutative implicator that satisfies the axioms (i_{11}) , (i_{12}) and (i_{13}) .*
11. *If (L, \rightarrow, \top) is a commutative, associative and involutive *cdeo* algebra, then \rightarrow is an ordered, distributive, crisp-bounded, commutative and contrapositive implicator that satisfies (i_{11}) , (i_{12}) and (i_{13}) .*

Proof. 1. From Proposition 3.3 \rightarrow satisfies the following equivalence $a \rightarrow b = \top \Leftrightarrow a \leq b$ and, in particular, it follows trivially that $a \rightarrow b = \top \Leftrightarrow a \leq b \Rightarrow b \rightarrow a \leq a \rightarrow b$, for all $a, b \in L$. Hence it satisfies the axioms (i) and (i_3) .

Moreover the following equivalences are true $\perp \leq b \Leftrightarrow \perp \rightarrow b = \top$, $a \leq \top \Leftrightarrow a \rightarrow \top = \top$, for all $a, b \in L$. Therefore, \rightarrow satisfies (i_1) and (i_2) .

2. From (o_5) it follows that $a \leq b \Leftrightarrow a \rightarrow b = \top \Rightarrow c \rightarrow a \leq c \rightarrow b$, for all $a, b, c \in L$. Then \rightarrow satisfies (i_4) .
3. From (o'_5) it follows that $a \leq b \Leftrightarrow a \rightarrow b = \top \Rightarrow b \rightarrow c \leq a \rightarrow c$, for all $a, b, c \in L$. Then \rightarrow satisfies (i_5) .
4. It follows easily from (2) and (3).

5. Since (d_r) implies (o_5) , it follows that (L, \rightarrow, \top) is right w -ceo algebra. Then from (2) one has that \rightarrow satisfies (i_4) . From (d_r) it follows that $a \rightarrow (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \rightarrow b_i)$; hence \rightarrow satisfies (i_6) .
6. Since (d_l) implies (o'_5) it follows that (L, \rightarrow, \top) is left w -ceo algebra. Then from (3) one has that \rightarrow satisfies (i_5) .
From (d_l) it follows that $(\bigvee_{i \in I} b_i) \rightarrow a = \bigwedge_{i \in I} (a \rightarrow b_i)$; hence \rightarrow satisfies (i'_6) .
7. Since (d) implies (o_5) and (o'_5) , it follows that (L, \rightarrow, \top) is a ceo algebra. The the statement follows from (4), (5) and (6).
8. By assumption and from Proposition 3.10 it follows that (L, \rightarrow, \top) and $(L, \rightsquigarrow, \top)$ are cdeo algebras. Hence, from (7) \rightarrow and \rightsquigarrow are ordered and distributive implicators.
From Proposition 3.12 (5) it follows that \rightarrow and \rightsquigarrow satisfy (i_{11}) , (i_{12}) and (b) .
Moreover, from Proposition 3.12 (4) it follows that \rightarrow and \rightsquigarrow satisfy (i_{13}) .
9. It follows trivially from Definitions 3.13, 4.1 and Proposition 3.14 (2).
10. From Proposition 3.18 it follows that (L, \rightarrow, \top) is a symmetrical cdeo algebra. Then from (8) \rightarrow is an ordered, distributive and crisp-bounded impicator that satisfies (i_{11}) , (i_{12}) and (i_{13}) . Moreover it is commutative by Definition 4.1.
11. From (10) it follows that \rightarrow is an ordered, distributive, commutative and crisp-bounded impicator that satisfies (i_{11}) , (i_{12}) and (i_{13}) . Moreover, by assumption and from Propositions 3.18 and 3.25 (4) it follows that \rightarrow is a contrapositive impicator.

□

The above Proposition explains and suggests the possibility to think of an impicator as nothing but the implication operation of some w -ceo algebra. In particular, if L is a complete lattice and $\mathcal{I} : L \times L \rightarrow L$, then

- \mathcal{I} is a weak-ordered impicator if and only if (L, \mathcal{I}, \top) is a w -ceo algebra;
- \mathcal{I} is an ordered impicator if and only if (L, \mathcal{I}, \top) is a ceo algebra.

Fuzzy implicators have been mostly considered, since [2], to express and to evaluate the inclusion between fuzzy sets. Classically, crisp inclusion between fuzzy (or lattice-valued) sets has been considered to be the order relation point-wisely induced by the lattice-order of L in the L -powerset L^X of any set X .

In order to determine the inclusion degree between two L -sets A, B with respect to an impicator \rightarrow in L it seems reasonable, of course, to evaluate the extended-order relationship between A and B ; this is usually done by means of the so called subethood degree $S(A, B) = \bigwedge_x A(x) \rightarrow B(x)$. Here we further propose the possibility to evaluate the inclusion of A in B by seeing to which extent A is included in the conjunction of A and B (or how much of A is in B).

In terms of the right-distributive (w -)ceo algebra structure this purpose can be realized using the relative implication given by the following Definition.

Definition 4.3. Let (L, \rightarrow, \top) be a right-distributive w-ceo algebra. The **relative implication** is the binary operation

$$\begin{aligned} \rightarrow_{\otimes}: L \times L &\rightarrow L \\ (a, b) &\longmapsto a \rightarrow_{\otimes} b = a \rightarrow (a \otimes b), \end{aligned}$$

where \otimes is the adjoint product of L .

Lemma 4.4. If (L, \rightarrow, \top) is a right-distributive w-ceo algebra, then the relative implication satisfies the axioms (i_1) , (i_2) , (i_4) and (i_{12}) .

Proof. From (o_2) and Proposition 3.6 (3) it follows that $\perp \rightarrow_{\otimes} b = \perp \rightarrow (\perp \otimes b) = \perp \rightarrow \perp = \top$. From (o_2) and Proposition 3.6 (2) one has that $a \rightarrow_{\otimes} \top = a \rightarrow (a \otimes \top) = a \rightarrow a = \top$. From (o_5) and Proposition 3.6 (6) it follows that $a \leq b \Rightarrow c \otimes a \leq c \otimes b \Rightarrow c \rightarrow (c \otimes a) \leq c \rightarrow (c \otimes b)$; hence $c \rightarrow_{\otimes} a \leq c \rightarrow_{\otimes} b$. From Proposition 3.6 (7) one has that $\top \rightarrow_{\otimes} b = \top \rightarrow (\top \otimes b) \geq b$. Hence \rightarrow_{\otimes} satisfies the axioms (i_1) , (i_2) , (i_4) and (i_{12}) . \square

Proposition 4.5. If (L, \rightarrow, \top) is a symmetrical cdeo algebra that satisfies the condition $a \leq b \Rightarrow b \otimes a \leq a \otimes b$, then the relative implications \rightarrow_{\otimes} and \leadsto_{\otimes} are bounded and crisp-bounded implicators that satisfy the axioms (i_4) , (i_{11}) (i_{12}) .

Proof. From Proposition 3.12 (2), (o_5) and the assumption it follows that $a \rightarrow_{\otimes} b = \top \Leftrightarrow a \leq a \otimes b \leq b \Rightarrow a \leq b \Rightarrow b \rightarrow_{\otimes} a \leq a \rightarrow_{\otimes} b$. Hence \rightarrow_{\otimes} is an impicator; trivially, by the above Lemma \rightarrow_{\otimes} is bounded and satisfies (i_4) and (i_{12}) . Moreover, from Propositions 3.11 and 3.12 (5) it follows that $\top \rightarrow_{\otimes} b = \top \rightarrow (\top \otimes b) = \top \rightarrow b = b$, for every $b \in L$ and, in particular, $\top \rightarrow_{\otimes} \perp = \top \rightarrow (\top \otimes \perp) = \top \rightarrow \perp = \perp$. Hence \rightarrow_{\otimes} satisfies the axioms (i_{11}) and (b) . Similarly for \leadsto_{\otimes} . \square

Clearly a commutative cdeo algebra satisfies the assumption of Proposition 4.5; but even the symmetry condition, and hence the commutativity are not necessary for \rightarrow_{\otimes} to be an impicator, as the following Example shows.

Example 4.6. Let $L = \{\perp, a, b, \top\}$, $a \leq b$, and \rightarrow the binary operation in L described in the table below.

\rightarrow	\perp	a	b	\top
\perp	\top	\top	\top	\top
a	a	\top	\top	\top
b	a	b	\top	\top
\top	\perp	b	b	\top

It is easily seen that (L, \rightarrow, \top) is a non-symmetrical cdeo algebra, with \otimes and \rightarrow_{\otimes} described below

It is easily seen that \rightarrow_{\otimes} is an impicator.

\otimes	\perp	a	b	\top
\perp	\perp	\perp	\perp	\perp
a	\perp	\perp	a	a
b	\perp	\perp	a	b
\top	\perp	a	a	\top

\rightarrow_{\otimes}	\perp	a	b	\top
\perp	\top	\top	\top	\top
a	a	a	\top	\top
b	a	a	b	\top
\top	\perp	b	b	\top

Proposition 4.7. *Let (L, \rightarrow, \top) be a cdeo algebra. Then \rightarrow_{\otimes} is a weak-ordered implicator if and only if the equivalence*

$$a \leq b \Leftrightarrow a = a \otimes b \text{ holds, for all } a, b \in L.$$

Proof. Assume \rightarrow_{\otimes} to be a weak-ordered implicator. Then it follows from $a \leq b$ that $a \rightarrow (a \otimes b) = \top$, hence $a \leq a \otimes b \leq a$. Conversely, from $a = a \otimes b$ one has $a \rightarrow_{\otimes} b = \top$, hence $a \leq b$. Now assume the stated equivalence. Then the equivalence $a \rightarrow_{\otimes} b = \top \Leftrightarrow a \leq a \otimes b \leq a \Leftrightarrow a \leq b$ proves that \rightarrow_{\otimes} is a weak-ordered implicator. \square

Corollary 4.8. *If (L, \rightarrow, \top) is a symmetrical cdeo algebra, then the dual relative implication $\rightsquigarrow_{\otimes}$ defined by*

$$a \rightsquigarrow_{\otimes} b = a \rightsquigarrow (b \otimes a), \text{ for all } a, b \in L$$

is a weak-ordered implicator if and only if the equivalence $a \leq b \Leftrightarrow a = b \otimes a$ holds, for all $a, b \in L$.

Proof. It follows trivially by the above Proposition since, under the assumption, $(L, \rightsquigarrow, \top)$ is a cdeo algebra with adjoint product $\tilde{\otimes}$ defined by $a \tilde{\otimes} b = b \otimes a$. \square

Corollary 4.9. *If (L, \rightarrow, \top) is a symmetrical cdeo algebra and \otimes is idempotent, then \rightarrow_{\otimes} and $\rightsquigarrow_{\otimes}$ are weak-ordered implicators.*

Proof. Under the assumption, for all $a \leq b$ in L one has

$$a = a \otimes a \leq a \otimes b \leq a \text{ and } a = a \otimes a \leq b \otimes a \leq a. \text{ Conversely, it follows from } a = a \otimes b \text{ that } a = a \otimes b \leq b \text{ and from } a = b \otimes a \text{ that } a = b \otimes a \leq b.$$

Now the statement follows from Proposition 4.7 and Corollary 4.8. \square

Remark 4.10. The idempotency condition is necessary for each of \rightarrow_{\otimes} and $\rightsquigarrow_{\otimes}$ to be a weak-ordered implicator. In fact, if this is the case it follows from either Proposition 4.7 or Corollary 4.8 and from $a \leq a$ that $a = a \otimes a$, for every $a \in L$.

However idempotency is not necessary for \rightarrow_{\otimes} to be an implicator (see Example 4.6) and the symmetry condition is not necessary for \rightarrow_{\otimes} to be a weak-ordered implicator, as the following Example shows.

Example 4.11. Let $L = \{\perp, a, b, c, \top\}$, $a \leq c, b \leq c$ and \rightarrow the binary operation in L described in the table below.

It is easily seen that (L, \rightarrow, \top) is a non-symmetrical cdeo algebra, with \otimes and \rightarrow_{\otimes} described below

Also, it can be seen that the condition of Proposition 4.7 is satisfied, hence \rightarrow_{\otimes} is a weak-ordered implicator.

\rightarrow	\perp	a	b	c	\top
\perp	\top	\top	\top	\top	\top
a	b	\top	b	\top	\top
b	a	a	\top	\top	\top
c	\perp	a	b	\top	\top
\top	\perp	\perp	\perp	c	\top

\otimes	\perp	a	b	c	\top
\perp	\perp	\perp	\perp	\perp	\perp
a	\perp	a	\perp	a	a
b	\perp	\perp	b	b	b
c	\perp	a	b	c	c
\top	\perp	c	c	c	\top

\rightarrow_{\otimes}	\perp	a	b	c	\top
\perp	\top	\top	\top	\top	\top
a	b	\top	b	\top	\top
b	a	a	\top	\top	\top
c	\perp	a	b	\top	\top
\top	\perp	c	c	c	\top

5 The conditional conjunction

In this last Section we consider a new binary operator that can be read as: " a and b , given a " which motivates the term we have chosen to denote it.

Definition 5.1. Let (L, \rightarrow, \top) be a right-distributive w-ceo algebra. The related **conditional conjunction** is the binary operation

$$\begin{aligned} \otimes_{\rightarrow} : L \times L &\rightarrow L \\ (a, b) &\mapsto a \otimes_{\rightarrow} b = a \otimes (a \rightarrow b), \end{aligned}$$

where \otimes is the adjoint product of L .

Proposition 5.2. Let (L, \rightarrow, \top) be a right-distributive w-ceo algebra. The conditional conjunction has the following properties, for all $a, b, b_i \in L$

1. $a \leq b \Rightarrow x \otimes_{\rightarrow} a \leq x \otimes_{\rightarrow} b, \forall x \in L;$
2. $\perp \otimes_{\rightarrow} b = \perp;$
3. $b \otimes_{\rightarrow} \perp = \perp;$
4. $a \otimes_{\rightarrow} \top = a;$
5. $a \otimes_{\rightarrow} x \leq b \otimes_{\rightarrow} x, \forall x \in L \Rightarrow a \leq b;$
6. $\top \otimes_{\rightarrow} b \leq b;$
7. $a \otimes_{\rightarrow} a = a;$
8. $a \otimes_{\rightarrow} b \leq a \wedge b;$
9. $a \leq b \Leftrightarrow a \otimes_{\rightarrow} b = a;$

10. $a \otimes_{\rightarrow} x \leq b \otimes_{\rightarrow} x, \forall x \in L \Rightarrow a \otimes_{\rightarrow} b = b \otimes_{\rightarrow} a = a;$

11. $a \otimes_{\rightarrow} b = \top \Leftrightarrow a = b = \top;$

12. $(a \otimes b) \otimes_{\rightarrow} a = a \otimes b;$

13. $a \otimes_{\rightarrow} (a \otimes b) = a \otimes b;$

14. $a \otimes_{\rightarrow} (a \otimes_{\rightarrow} b) = a \otimes_{\rightarrow} b;$

15. $(a \otimes_{\rightarrow} b) \otimes_{\rightarrow} b = a \otimes_{\rightarrow} b.$

If moreover (L, \rightarrow, \top) is a right-distributive ceo algebra, then

16. $\top \rightarrow x \geq x, \forall x \in L \Rightarrow a \otimes b \leq a \otimes_{\rightarrow} b;$

17. if \otimes is distributive over \bigwedge on the right side, then $a \otimes_{\rightarrow} (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \otimes_{\rightarrow} b_i).$

If (L, \rightarrow, \top) is a symmetrical cdeo algebra, then

18. $\top \otimes_{\rightarrow} b = b;$

19. $a \otimes b \leq a \otimes_{\rightarrow} b;$

20. $a \leq b \Leftrightarrow x \otimes_{\rightarrow} a \leq x \otimes_{\rightarrow} b, \forall x \in L.$

If \rightarrow satisfies the axiom (i_7) , then

21. $a \otimes_{\rightarrow} (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \otimes_{\rightarrow} b_i).$

Proof. 1. Let $a \leq b$; from (o_5) $x \rightarrow a \leq x \rightarrow b$. Then from Proposition 3.6 (6) one has $x \otimes_{\rightarrow} a = x \otimes (x \rightarrow a) \leq x \otimes (x \rightarrow b) = x \otimes_{\rightarrow} b.$

2. From Proposition 3.6 (3) it follows that $\perp \otimes_{\rightarrow} b = \perp \otimes (\perp \rightarrow b) = \perp.$

3. From Proposition 3.6 (7) one has that $b \otimes_{\rightarrow} \perp = b \otimes (b \rightarrow \perp) \leq \perp.$

4. From (o_1) and Proposition 3.6 (2) it follows that $a \otimes_{\rightarrow} \top = a \otimes (a \rightarrow \top) = a \otimes \top = a.$

5. Taking $x = \top$, the assumption gives $a \leq b$ by (4).

6. From Proposition 3.6 (7) one has that $\top \otimes_{\rightarrow} b = \top \otimes (\top \rightarrow b) \leq b.$

7. From Proposition 3.6 (2) and (o_2) it follows that $a \otimes_{\rightarrow} a = a \otimes (a \rightarrow a) = a \otimes \top = a.$

8. From Proposition 3.6 (1) $a \otimes_{\rightarrow} b \leq a$; moreover, from Proposition 3.6 (9) $a \otimes_{\rightarrow} b \leq b$. Hence $a \otimes_{\rightarrow} b \leq a \wedge b.$

9. Let $a \leq b$. Then $a \rightarrow b = \top$, hence, from Proposition 3.6 (2) $a \otimes_{\rightarrow} b = a \otimes (a \rightarrow b) = a \otimes \top = a.$
Conversely, from Proposition 3.6 (7) $a = a \otimes_{\rightarrow} b = a \otimes (a \rightarrow b) \leq b.$

10. The assumption gives $a \leq b$ by (5); then $a \otimes_{\rightarrow} b = a$ by (9).

Moreover, taking $x = a$, by the assumption, (7) and (8) one has $a \leq b \otimes_{\rightarrow} a \leq a.$

11. Assume $a \otimes_{\rightarrow} b = \top$; from (8) it follows that $\top \leq a \wedge b$ and hence $a = b = \top$.
Conversely, it is clear that $\top \otimes_{\rightarrow} \top = \top$.
12. From Proposition 3.6 (1) (2) one has that $(a \otimes b) \otimes_{\rightarrow} a = (a \otimes b) \otimes [(a \otimes b) \rightarrow a] = (a \otimes b) \otimes \top = (a \otimes b)$.
13. From Proposition 3.6 (7) $a \rightarrow (a \otimes b) \geq b$; hence, from Proposition 3.6 (6) $a \otimes_{\rightarrow} (a \otimes b) = a \otimes [a \rightarrow (a \otimes b)] \geq a \otimes b$. The converse inequality follows from (8), so $a \otimes_{\rightarrow} (a \otimes b) = a \otimes b$.
14. From Proposition 3.6 (6)(7) one has that $a \otimes_{\rightarrow} (a \otimes_{\rightarrow} b) = a \otimes [a \rightarrow (a \otimes (a \rightarrow b))] \geq a \otimes (a \rightarrow b) = a \otimes_{\rightarrow} b$. The converse inequality follows from (8).
15. By (8) $a \otimes_{\rightarrow} b \leq b$, hence by (9) $(a \otimes_{\rightarrow} b) \otimes_{\rightarrow} b = a \otimes_{\rightarrow} b$.
16. From (o'_5) and Proposition 3.6 (6) and by assumption the following inequality holds $a \otimes_{\rightarrow} b = a \otimes (a \rightarrow b) \geq a \otimes (\top \rightarrow b) \geq a \otimes b$.
17. By assumption and condition (d_r) it follows that $a \otimes_{\rightarrow} (\bigwedge_{i \in I} b_i) = a \otimes [a \rightarrow (\bigwedge_{i \in I} b_i)] = a \otimes [\bigwedge_{i \in I} (a \rightarrow b_i)] = \bigwedge_{i \in I} [a \otimes (a \rightarrow b_i)] = \bigwedge_{i \in I} (a \otimes_{\rightarrow} b_i)$.
18. From Propositions 3.11 and 3.12 (5) it follows that $\top \otimes_{\rightarrow} b = \top \otimes (\top \rightarrow b) = \top \otimes b = b$.
19. By assumption the equality $\top \rightarrow x = x, \forall x \in L$, follows from Proposition 3.12 (5), hence the statement follows by (16).
20. The implication " \Rightarrow " comes from (1); the converse implication follows by (7) and (8), in fact by the assumption for $x = a$ one has $a = a \otimes_{\rightarrow} a \leq a \otimes_{\rightarrow} b \leq b$.
21. By assumption and Proposition 3.6 (5) it follows that $a \otimes_{\rightarrow} (\bigvee_{i \in I} b_i) = a \otimes [a \rightarrow (\bigvee_{i \in I} b_i)] = a \otimes [\bigvee_{i \in I} (a \rightarrow b_i)] = \bigvee_{i \in I} (a \otimes_{\rightarrow} b_i)$.

□

The properties listed in the above Proposition show that \otimes_{\rightarrow} satisfies most conditions usually asked to a conjunction operator and moreover it is well related to the operator \otimes , the main conjunction operator of a right-distributive *w-ceo* algebra, and to the classical conjunction operator \wedge ; in fact $x \otimes_{\rightarrow} y \leq x \wedge y$ in any case and if (L, \rightarrow, \top) is a symmetrical *cdeo* algebra then

$$x \otimes y \leq x \otimes_{\rightarrow} y \leq x \wedge y, \text{ for all } x, y \in L.$$

We also recall that the equality $x \wedge y = x \otimes_{\rightarrow} y$, for all $x, y \in L$ is just the so called divisibility condition that is assumed in BL-algebras and in MV-algebras, as well (see for instance [13]). We can show that whenever L is a chain the divisibility condition is necessary for the conditional conjunction \otimes_{\rightarrow} to be isotonic in the first argument; more precisely the only conditional conjunction in a right-distributive *w-ceo* chain that is isotonic in the first argument is the meet operation. In fact, the following holds.

Proposition 5.3. *Let (L, \rightarrow, \top) be a right-distributive *w-ceo* algebra and let L be a chain in the natural ordering. If the related conditional conjunction is isotonic in the first argument, then $a \otimes_{\rightarrow} b = a \wedge b$, for all $a, b \in L$.*

Proof. Consider $a, b \in L$ and assume $a \wedge b = a$. Then $a \leq b$ and by the assumption the inequality $a \otimes_{\rightarrow} x \leq b \otimes_{\rightarrow} x$, for every $x \in L$, holds.

Then from Proposition 5.2 (10) $a \otimes_{\rightarrow} b = b \otimes_{\rightarrow} a = a = a \wedge b$. \square

Corollary 5.4. *A right-distributive w-ceo chain is a BL-algebra if and only if the related conditional conjunction is isotonic in the first argument. \square*

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