On the Clique Operator

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Abstract. The clique operator K maps a graph G into its *clique graph*, which is the intersection graph of the (maximal) cliques of G. Among all the better studied graph operators, K seems to be the richest one and many questions regarding it remain open. In particular, it is not known whether recognizing a clique graph is in P. In this note we describe our progress toward answering this question. We obtain a necessary condition for a graph to be in the image of K in terms of the presence of certain subgraphs A and B. We show that being a clique graph is *not* a property that is maintained by addition of twins. We present a result involving distances that reduces the recognition problem to graphs of diameter at most two. We also give a constructive characterization of $K^{-1}(G)$ for a fixed but generic G.

1 Introduction

The *clique operator* K transforms a graph G into a graph $K(G)$ having as vertices all the cliques of G, with two cliques being adjacent when they intersect. The graph $K(G)$ is called the *clique graph* of G. (This and other definitions can be found in Sect. 2.)

In this note we will be interested in the image $K(\mathcal{G})$ of the operator K, where $\mathcal{G}$ is the class of all graphs. We are particularly interested in the complexity of the recognition problem for $K(\mathcal{G})$, which is still open. Roberts and Spencer [5], building upon ideas from Hamelink [3], gave a characterization of $K(\mathcal{G})$, but direct application of these results leads to an exponential time algorithm. While trying to find alternative characterizations that could possible shed more light into the problem, we came across an interesting question: is $K(\mathcal{G})$ the same as $K^2(\mathcal{G})$?

This question turns out to be very difficult, and the present paper represents an effort toward its solution. Our contribution can be summarized as follows. In Sect. 3 we investigate the structure of graphs in $K(\mathcal{G})$ that are not Helly graphs, finding certain subgraphs that have to be present in this situation. Because Helly graphs are all in $K^2(\mathcal{G})$, non-Helly graphs are the only candidates to separate $K^2(\mathcal{G})$ from the rest of $K(\mathcal{G})$. We show that one of these subgraphs belongs to $K^2(\mathcal{G})$. We do not know if the other one does.

In Sect. 4 we show that H ia a graph in $K^{-1}(G)$ if and only if it is the intersection graph of an ERS family of G (please see definition in Sect. 2). We also study several properties of both RS and ERS families of G.

In Sect. 5 we obtain results that show that it is enough to study the recognition problem for $K(\mathcal{G})$ in graphs with diameter at most two.

The study of $K^2(\mathcal{G})$ is further complicated by the fact that being in $K(\mathcal{G})$ is *not* a property inherited by reduced graphs, as we show in Sect. 6. In fact, when $H \notin K(\mathcal{G})$ it is possible to get a graph in $K(\mathcal{G})$ by adding twin vertices to H. Of course, this addition will not modify $K(H)$. Finally, Sect. 7 contains our concluding remarks.

Some proofs are omitted for space limitations. All proofs appear in full in the extended version of this paper.

2 Definitions

In this note all graphs are simple, i.e., without loops or multiple edges. Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. A set C of vertices of G is *complete* when any two vertices of C are adjacent. A maximal complete subset of $V(G)$ is called a *clique*. We denote by $\mathcal{C}(G)$ the clique family of G.

Let $\mathcal{F} = (F_i)_{i \in I}$ be a finite family of finite sets. Its *dual family* $\mathcal{F}^*$ is the family $(F(x))_{x \in X}$ where $X = \bigcup_{i \in I} F_i$ and $F(x) = \{i \in I, x \in F_i\}$. We denote by $\Omega\mathcal{F}$ the *intersection graph* of $\mathcal{F}$, i.e., $V(\Omega\mathcal{F}) = I$ and two vertices i and j are adjacent if and only if $F_i \cap F_j \neq \emptyset$. We also say that $\mathcal{F}$ *represents* $\Omega\mathcal{F}$.

The *2-section* of $\mathcal{F}$, denoted by $\mathcal{F}_2$, is the graph with $V(\mathcal{F}_2) = \bigcup_{i \in I} F_i$ and two vertices x and y are adjacent if and only if there exists $i \in I$ such that $x, y \in F_i$.

It is easy to see that $\Omega\mathcal{F} = \mathcal{F}_2^*$ [1].

A family $\mathcal{F}$ of arbitrary sets satisfies the *Helly property*, or is *Helly*, when for every subfamily $J \subseteq \mathcal{F}$ such that any two sets $A, B \in J$ intersect, we have $\bigcap_{A \in J} A \neq \emptyset$. A graph is *Helly* when the family of its cliques is Helly. We denote by $\mathcal{H}$ the class of Helly graphs. A family $\mathcal{F}$ is *conformal* when the cliques of $\mathcal{F}_2$ are all members of $\mathcal{F}$. This amounts to saying that its dual family $\mathcal{F}^*$ is Helly [1]. A family $\mathcal{F}$ is *reduced* when none of its members is contained in another one.

As we said earlier, the *clique operator* K transforms a graph G into a graph $K(G)$ having as vertices all the cliques of G, with two cliques being adjacent when they intersect. Thus, $K(G)$ is nothing else than the intersection graph of the family of all cliques of G. The graph $K(G)$ is called the *clique graph* of G.

In this note we will be interested in the image $K(\mathcal{G})$ of the operator K, where $\mathcal{G}$ is the class of all graphs. In particular, we would like to determine the complexity of recognizing whether a graph is a clique graph, that is, is in $K(\mathcal{G})$.

There are only two general results about $K(\mathcal{G})$ in the literature. The first result, due to Hamelink [3], says that $\mathcal{H}$ is properly contained in $K(\mathcal{G})$. In the second result, based on the previous one, Roberts and Spencer [5] find the following characterization of $K(\mathcal{G})$: