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On the complexity of deciding whether the distinguishing chromatic number of a graph is at most two

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ABSTRACT

In an article Cheng (2009) [3] published recently in this journal, it was shown that when $k \geq 3$, the problem of deciding whether the distinguishing chromatic number of a graph is at most k is NP-hard. We consider the problem when $k = 2$. In regards to the issue of solvability in polynomial time, we show that the problem is at least as hard as graph automorphism, but no harder than graph isomorphism.

1. Introduction

We consider simple undirected graphs. A *nontrivial* automorphism of a graph is an automorphism that is not the identity mapping. We use the abbreviation NTA for nontrivial automorphism. A graph that has no nontrivial automorphism is said to be *asymmetric*. A vertex k-coloring of graph $G = (V, E)$ is a mapping $V \to \{1, 2, \ldots, k\}$. A vertex k-coloring of graph G is *proper* if no two adjacent vertices of G map to the same color. $\chi(G)$ is the chromatic number of graph G, namely, the smallest positive integer k such that G admits a proper vertex k-coloring.

In some of our results, we use the polynomial-time many-one reducibility (also called Karp reducibility) that is usually used in the definition of NP-completeness (denoted $\leq_m$) and we also use polynomial-time Turing reductions (denoted $\leq_T$) in some cases. For two problems A and B, if $A \leq_m B$ and $B \leq_m A$ (respectively, $A \leq_T B$ and $B \leq_T A$), then A and B are said to be *polynomial-time many-one equivalent* (respectively, *polynomial-time Turing equivalent*) and we write $A \equiv_m B$ (respectively, $A \equiv_T B$). Many-one reducibility is a special case of Turing reducibility. Furthermore, there are problems between which a polynomial-time Turing reduction exists and there is no polynomial-time many-one reduction (see [8]); however, the only problems known to have this property are outside NP $\cup$ co-NP. Since every decision problem can be trivially reduced to its complement via a polynomial-time Turing reduction, the equivalence of the two types of reductions within NP $\cup$ co-NP would imply NP $=$ co-NP. As a result, Turing reducibility is believed to be more general than many-one reducibility within NP $\cup$ co-NP, though no proof of this fact is known.

A vertex k-coloring of graph G is *distinguishing* if the only automorphism of G that preserves the coloring is the identity automorphism; that is, there is no NTA that maps each vertex to a vertex of the same color. The *distinguishing number of graph G*, denoted $D(G)$, is the smallest positive integer k such that G admits a k-coloring (not necessarily proper) that is

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distinguishing. Similarly, the *distinguishing chromatic number of graph G*, denoted $\chi_D(G)$, is the smallest positive integer k such that G admits a *proper* k-coloring that is distinguishing. The concept of the distinguishing number of a graph was introduced by Albertson and Collins in [1]. Later, Collins and Trenk [4] introduced the notion of distinguishing chromatic numbers of graphs.

The computational complexities of the problems of computing $D(G)$ and $\chi_D(G)$ have been investigated in the recent past. It was shown by Russell and Sundaram [7] that given a graph G and an integer k, deciding whether $D(G) \leq k$ belongs to AM, the set of languages for which there exist Arthur and Merlin games. In a more recent paper, Arvind, Cheng, and Devanur [2] showed that given a planar graph G and an integer k, whether $D(G) \leq k$ can be decided in polynomial time. Cheng [3] has shown that given an interval graph G and an integer k, whether $\chi_D(G) \leq k$ can be decided in polynomial time. In contrast to this, Cheng [3] also established that given an arbitrary graph G and an integer k, where $k \geq 3$, deciding whether $\chi_D(G) \leq k$ is NP-hard. Further, the problem remains NP-hard when $k = 3$ and the input graph is planar with maximum degree at most five [3]. In regards to the problem of deciding whether $\chi_D(G) \leq 2$, given a graph G, Cheng remarked in [3] that "it will be interesting to consider what the corresponding results are" for deciding whether $\chi_D(G) \leq 2$.

We show that given a *connected* graph G, deciding whether $\chi_D(G) \leq 2$ is polynomial-time Turing equivalent to the problem of deciding whether a given graph H has a NTA. Thus, given an *arbitrary* graph G, deciding whether $\chi_D(G) \leq 2$ is at least as hard as deciding whether a graph H has any NTA. We then show that given an arbitrary graph G, the problem of deciding whether $\chi_D(G) \leq 2$ is no harder than deciding whether given graphs G_1 and G_2 are isomorphic.

Next, we introduce the definitions of some needed problems. Then, we present our main results. Finally, we conclude with some discussion.

2. Graph automorphism and graph isomorphism

Consider the following decision problems each of which is known to be in NP, but neither of which is known to be in P or NP-complete. Graph isomorphism has long been considered a candidate to be in NP but neither in P nor NP-complete (Ladner [6] proved that such problems exist if P $\neq$ NP).

GRAPH AUTOMORPHISM (**GA**)
Instance: Graph G.
Question: Does G have a nontrivial automorphism?

GRAPH ISOMORPHISM (**GI**)
Instance: Graphs G_1 and G_2.
Question: Is $G_1 \cong G_2$?

It is known that **GA** $\leq_m$ **GI** (see [5]); however, as stated in [5], "**GI** does not seem to be reducible to **GA**". Thus, it is possible that **GA** is easier to compute than **GI**.

3. Results

It can be observed based on the definitions that $\chi(G) \leq \chi_D(G)$ and that $D(G) \leq \chi_D(G)$. If G is asymmetric, then $\chi_D(G) = \chi(G)$. Clearly, $D(G) = 1$ if and only if G is asymmetric. Therefore, given graph G, deciding whether $D(G) = 1$ is polynomial-time Turing equivalent to **GA**. In contrast, given graph G, deciding whether $\chi_D(G) = 1$ is trivial; $G = K_1$ is the only graph with $\chi_D(G) = 1$. When $\chi_D(G) = 2$, G is necessarily bipartite. In the remainder of the paper, we use 2-coloring to refer to a proper 2-coloring.

Our focus is on the following problem:

DISTINGUISHING 2-COLORABILITY (**D2C**)
Instance: Graph G.
Question: Is $\chi_D(G) \leq 2$?

We first consider the problem **D2C** restricted to connected graphs. As previously observed, all YES instances of **D2C** must be bipartite graphs. Note that a connected bipartite graph G has a unique bipartition and therefore (up to renaming the colors) a unique 2-coloring; thus either every 2-coloring of G is distinguishing or none of them are. Furthermore, every automorphism of a connected bipartite graph $G = (X, Y, E)$ maps all vertices of X to Y and vice versa, or maps X to X and Y to Y. If $|X| \neq |Y|$ then only the latter type of automorphism is possible.

Consequently, **D2C** for connected graphs is polynomial-time many-one equivalent to the problem: Given a connected bipartite graph G and a 2-coloring c of G, is c a distinguishing coloring? Thus, since a given coloring c of graph G is *not* distinguishing if and only if there is a NTA of G that preserves c, the complement of **D2C** restricted to connected graphs seems closely related to **GA**. The next theorem shows that those two problems are in fact polynomial-time many-one equivalent.

In the remainder of the paper, we refer to the *complement* of **D2C** restricted to *connected* graphs as **CC**:

COMPLEMENT OF **D2C** ON CONNECTED GRAPHS (**CC**)
Instance: *Connected* graph G.
Question: Is $\chi_D(G) \not\leq 2$?

Theorem 1. *Problems* **CC** *and* **GA** *are polynomial-time many-one equivalent.*

Proof. First, we show that **GA** $\leq_m$ **CC**.

Since a graph has a NTA if and only if its complement has a NTA, and the complement of a disconnected graph is connected, we may assume that the given instance $G = (V, E)$ of **GA** is connected.

If $G = K_1$, G is a NO instance of **GA** and we can easily construct a NO instance G' of **CC**. Otherwise, let $G' = (V' = V \cup E, E')$ be the graph obtained from G by subdividing each edge of G once. We note that for an edge xy of G, we use xy to refer to the edge of G as well as the vertex of G' that subdivides the edge xy of G. Clearly, G' is connected and bipartite, and every vertex in E has degree 2. In order to complete the reduction from **GA** to **CC**, we prove that G has a NTA if and only if $\chi_D(G') \not\leq 2$.

If all the vertices of V' have degree 2 then G is a chordless cycle of length ≥ 3 and therefore has a NTA. In this case, for every 2-coloring of G', the vertices in V are mapped to one color, the vertices in E are mapped to the other color, and also there is a NTA of G' that preserves the coloring. Therefore, G has a NTA and $\chi_D(G') \not\leq 2$.

In the remaining case, one color class of G' consists entirely of degree 2 vertices (vertices in E) and the other color class (vertices in V) contains a vertex of degree $\neq 2$. Thus, every NTA of G' must map V to V and E to E.

First, we show that if $f : V \mapsto V$ is a NTA of G, then for every 2-coloring of G' there exists a NTA $f' : V' \mapsto V'$ that preserves the coloring of G' (and hence $\chi_D(G') \not\leq 2$). Note that as G' is connected, it is enough to consider a particular 2-coloring of G'.

Suppose $f : V \mapsto V$ is a NTA of G. Define $f' : V' \mapsto V'$ where

$$f'(x) = \begin{cases} f(x) & \text{if } x \in V \\ f(y)f(z) & \text{if } x = yz \in E. \end{cases}$$

We now show that f' is an NTA that preserves every 2-coloring of G'. Since f is an automorphism, f' is a bijection. To see that f' is an automorphism, observe that:

$$\begin{aligned} uv \in E' &\Leftrightarrow u \in V \text{ and } v \in E \text{ and } v = uw \text{ for some } w \in V \text{ (or vice versa)} \\ &\Leftrightarrow f'(u) = f(u) \in V \text{ and } f'(v) = f(u)f(w) \\ &\Leftrightarrow f'(u)f'(v) \in E' \text{ since } f'(v) \text{ corresponds to an element of } E \text{that is incident with } f'(u) \text{ in } G. \end{aligned}$$

Since G' is a connected bipartite graph, it has a unique 2-coloring, and that 2-coloring is preserved by f' since f' maps V to V and E to E. Finally, as f is a NTA of G, f' is a NTA of G'.

Next, we show that if $\chi_D(G') \not\leq 2$, then G has a NTA. Suppose $\chi_D(G') \not\leq 2$. Let c be the unique 2-coloring of G' and let $f' : V' \mapsto V'$ be a NTA of G' that preserves c. Define $f : V \mapsto V$ such that $f(x) = f'(x)$ for all $x \in V$. Since f' preserves c, it maps V to V and E to E. As f' is a NTA of G', the V to V mapping of f is a NTA of G.

This completes the proof of **GA** $\leq_m$ **CC**.

We now prove the reduction in the other direction, that is, **CC** $\leq_m$ **GA**. Let $G = (X, Y, E)$ be a connected bipartite graph that is not K_1 or K_2. (K_1 and K_2 are NO instances of **CC** and any connected non-bipartite graph G is a YES instance of **CC**. In these cases, we can construct G' accordingly.) Note that G has a unique 2-coloring with color classes X and Y. Define

$$G' = \begin{cases} G = (X, Y, E) & \text{if } |X| \neq |Y| \\ (X', Y', E') & \text{otherwise} \end{cases}$$

where $a, b, c \notin X \cup Y$ and

$$\begin{aligned} X' &= X \cup \{b\} \\ Y' &= Y \cup \{a, c\} \\ E' &= E \cup \{ax \mid x \in X\} \cup \{ab, bc\}. \end{aligned}$$

We prove that $\chi_D(G) \not\leq 2$ if and only if G' has a NTA.

Suppose $\chi_D(G) \not\leq 2$. Then there exists a NTA f of G that preserves the unique 2-coloring of G. In the case that $G' = G$, f is also a NTA of G'. In the case that $G' \neq G$ define the mapping $f' : X' \cup Y' \mapsto X' \cup Y'$ where

$$f'(x) = \begin{cases} f(x) & \text{if } x \in X \cup Y \\ x & \text{if } x \in \{a, b, c\}. \end{cases}$$

It is easily seen that f' is a NTA of G'.

Now suppose f is a NTA of $G' = (X', Y', E')$. Since $|X'| \neq |Y'|$ and G' is connected, f preserves the unique 2-coloring of G'. Further, $f(a) = a$, $f(b) = b$, and $f(c) = c$ by the vertex degrees, the connectedness of G, and the fact that $G \ncong K_2$. Thus, f maps X to X and Y to Y and therefore f restricted to G is a NTA of G that preserves the unique 2-coloring of G. Therefore, $\chi_D(G) \not\leq 2$ and the proof of the theorem is complete. □

The following proposition allows us to analyze the complexity of problem **D2C** for graphs that are not necessarily connected. We again use the fact that a connected bipartite graph has a unique (up to renaming the colors) 2-coloring.

Proposition 1. *Let G be a graph. $\chi_D(G) \leq 2$ if and only if*

- *G is bipartite and*
- *for every component C of G:*
  - *$\chi_D(C) \leq 2$,*
  - *C is isomorphic to at most one other component of G, and*
  - *if C is isomorphic to some other component of G then C is asymmetric.*

Proof. The proposition clearly holds when $G = K_1$. Therefore, we now assume G has at least two vertices.

$\Rightarrow$ Suppose that $\chi_D(G) \leq 2$. Then, G is bipartite by an earlier observation. By the definition of χ_D, there is a 2-coloring c of G that is distinguishing.

Let C be a component of G. It is clear that c restricted to C is a distinguishing coloring or else we contradict the choice of c.

Suppose that $C_1 = (X_1, Y_1, E_1)$, $C_2 = (X_2, Y_2, E_2)$, and $C_3 = (X_3, Y_3, E_3)$ are three distinct isomorphic components of G and that there are isomorphisms mapping X_1 to X_2 and X_2 to X_3. No matter how the vertices of C_1, C_2, and C_3 are 2-colored, two of X_1, X_2, X_3 will be in the same color class. Therefore, for every 2-coloring of G, there is an NTA that preserves the coloring (specifically, an automorphism that maps the two X_i's that are in the same color class to one another), contradicting that $\chi_D(G) \leq 2$. Thus, each component can be isomorphic to at most one other component.

Suppose that $C = (X_C, Y_C, E_C)$ is isomorphic to another component $C' = (X_{C'}, Y_{C'}, E_{C'})$ and that some isomorphism f maps X_C to $X_{C'}$. If C is not asymmetric, then it has a NTA g, and every NTA of C maps X_C to Y_C and vice versa, or else we contradict that c is distinguishing. But, now there are two isomorphisms from C to C', namely, f and $g \circ f$, one of which preserves c, a contradiction.

$\Leftarrow$ Let bipartite graph G, with components $C_1, C_2, \ldots, C_k$, satisfy the conditions. Suppose $\chi_D(G) \nleq 2$. Then, for every 2-coloring of G, there is a NTA that preserves the coloring. Let c be a 2-coloring of G in which isomorphic pairs of components $C_1 = (X_1, Y_1, E_1)$ and $C_2 = (X_2, Y_2, E_2)$ are colored such that if there is an isomorphism mapping X_1 to X_2 then X_1 and X_2 have opposite colors in c. Now, every NTA swaps colors within a single component and/or swaps colors in an isomorphic pair of components but, in any case, c is not preserved, which is a contradiction. $\square$

Corollary 1. **D2C** $\leq_T$ **GI**.

Proof. By Proposition 1, an algorithm for **D2C** can be constructed from algorithms for **CC**, **GI**, and **GA**. Since **CC** $\leq_m$ **GA** (Theorem 1) and **GA** $\leq_m$ **GI**, the result follows. $\square$

Corollary 2. **D2C** $\in$ coNP.

Proof. The fact that **CC** $\in$ NP follows from Theorem 1. This observation, combined with the negation of Proposition 1, shows that the complement of **D2C** has an NP certificate. $\square$

4. Discussion

Combining Theorem 1, Corollary 1, and the observation that **CC** $\leq_T$ **D2C**, we have **GA** $\equiv_m$ **CC** $\leq_T$ **D2C** $\leq_T$ **GI**. That is, **D2C** is at least as hard as **GA** and no harder than **GI**, in terms of Turing reductions.

Our results imply that **CC** $\in$ NP and **D2C** $\in$ co-NP. In addition, a direct consequence of Corollary 1 is that for a graph G belonging to a class $\mathcal{C}$ such that the isomorphism problem can be solved in polynomial time for $\mathcal{C}$, deciding whether $\chi_D(G) \leq 2$ can be done in polynomial time.

A question that arises from Theorem 1 and Corollary 1 is: is problem **D2C** polynomial-time Turing equivalent to **GA** or to **GI**, or does its Turing degree lie in between those of problems **GA** and **GI**?

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References

[1] M. Albertson, K. Collins, Symmetry breaking in graphs, The Electronic Journal of Combinatorics 3 (1996) #R18.
[2] V. Arvind, C. Cheng, N. Devanur, On computing the distinguishing numbers of planar graphs and beyond, SIAM Journal on Discrete Mathematics 22 (2008) 1297–1324.
[3] C. Cheng, On computing the distinguishing and distinguishing chromatic numbers of interval graphs and other results, Discrete Mathematics 309 (2009) 5169–5182.
[4] K. Collins, A. Trenk, The distinguishing chromatic number, The Electronic Journal of Combinatorics 13 (2006) #R16.
[5] J. Köbler, U. Schöning, J. Torán, The Graph Isomorphism Problem: Its Structural Complexity, Birkhäuser, 1993.
[6] R.E. Ladner, On the structure of polynomial-time reducibilities, Journal of the Association for Computing Machinery 22 (1975) 155–171.
[7] A. Russell, R. Sundaram, A note on the asymptotics and computational complexity of graph distinguishability, The Electronic Journal of Combinatorics 5 (1998) #R23.
[8] M. Sipser, Introduction to the Theory of Computation, second ed., Thomson, 2006.