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On a Packing and Covering Problem

VOJTĚCH RÖDL

Let positive integers $r<k<N$ and a family $\mathscr{F}$ of k-element subsets of $\{1,2,\ldots,N\}$ be given. We say that $\mathscr{F}$ is *r-dense* if *any* r-element subset of $\{1,2,\ldots,N\}$ is contained in at least one member of $\mathscr{F}$. On the other hand we say that $\mathscr{F}'$ is *r-sparse* if *any* two members of $\mathscr{F}$ intersect in less than r-elements—i.e. every r-element subset of $\{1,2,\ldots,N\}$ is contained in at most one member of $\mathscr{F}'$. It is well known (see [4]) that

$$|\mathscr{F}|\geqslant\frac{\binom{N}{r}}{\binom{k}{r}}\geqslant|\mathscr{F}'| \tag{1}$$

for any r-dense family $\mathscr{F}$ and r-sparse family $\mathscr{F}'$, $\mathscr{F},\mathscr{F}'\subset[\{1,2,\ldots,N\}]^k$. In [2] Erdös and Spencer denote by $M(N,k,r)$ the minimal number of elements of r-dense family $\mathscr{F}\subset[\{1,2,\ldots,N\}]^k$ and by $m(N,k,r)$ the maximal number of elements of r-sparse family $\mathscr{F}'\subset[\{1,2,\ldots,N\}]^k$. From (1) we get

$$M(N,k,r)\geqslant\frac{\binom{N}{r}}{\binom{k}{r}}\geqslant m(N,k,r).$$

It was shown by P. Erdös and J. Spencer [2] that

$$M(N,k,r)\leqslant\left[\binom{N}{r}\Big/\binom{k}{r}\right]\left[1+\log\binom{k}{r}\right].$$

In 1963 P. Erdös and J. Hanani [1] conjectured that

$$\lim M(N,k,r)\binom{k}{r}\binom{N}{r}^{-1}=\lim_{N\to\infty}m(N,k,r)\binom{k}{r}\binom{N}{r}^{-1}=1, \tag{2}$$

for every fixed r and k, $r<k$. They proved (2) for $r=2$ and all k and for $r=3$, $k=p$ or $p+1$ where p is power of a prime.

The objective of this paper is to prove (2) (cf. [5], where a few related remarks are mentioned).

PRELIMINARIES

We find it convenient to work with the following structures. Let J be a k-element set (k-*set*) of positive integers. A *k-partite r-graph* is a pair

$$G=((V_j)_{j\in J},E)$$

such that $|e\cap V_j|\leqslant 1$ for every $j\in J$ and $e\in E$, moreover $|e|=r$ and $e\subset\bigcup_{j\in J}V_j$ for every $e\in E$.

If $G=((V_j)_{j\in J},E)$ then we set $V(G)=\bigcup_{j\in J}V_j$, $E(G)=E$. For $I\in[J]^r$ ($[J]^r$ denotes the set of all r-element subsets of J) $\rho_I=\rho_I(G)$ denotes the cardinality of $E_I(G)$, where

$$E_I(G)=\{e\in E\,;\,e\cap V_i\neq 0\text{ for every }i\in I\}.$$

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We say that subset $L \subset \bigcup_{j\in J} V_j$, $|L| \geq r$ is *complete set* if $[L]^r \subset E$. Let L be a complete set of cardinality k, we say that the set $K = [L]^r$ forms a *k-gon*. Let L be a complete set in G, then $\sigma^L(G)$ will denote the number of complete k-sets containing L as a subset.

Let $A_1, A_2, \dots, A_p$, $A = \bigcup_{i=1}^{p} A_i$ be pairwise disjoint sets. By the symbol $[\{A_i\}_{i=1}^{p}]^r$ we shall denote the system of all r-subsets of A, intersecting each A_i in at most one element. We shall often use the symbol o(1) to denote a quantity (depending on the natural number n) the value of which tends to zero as n (here always n) tends to infinity. For a function $f(n)$ we put $o(f(n)) = o(1)f(n)$. For two functions $f(n)$, $g(n)$ we will write $f(n) \sim g(n)$ if $f(n) = (1+o(1))g(n)$. We also write $f(n) \sim_\delta g(n)$ if $|f(n) - g(n)| < \delta f(n)$. Finally we will state here the following auxiliary.

CLAIM. *For every pair of positive integers n, m, $n > m$ and positive reals p, q, $p + q = 1$ such that*

$$(2p-1)n < m < 2pn$$

we have

$$\binom{n}{m} p^m q^{n-m} < \exp\left(-\frac{1}{3}\frac{(m-np)^2}{npq}\right). \tag{3}$$

We omit the standard proof based on Stirling formula. (For the proof of very similar statements see e.g. [3]).

RESULTS

The objective of this paper is to prove the following theorem.

THEOREM. *For every pair of positive integers r and k, $r < k$,*

$$\lim_{N\to\infty} M(N, k, r)\binom{k}{r}\binom{N}{r}^{-1} = \lim_{N\to\infty} m(N, k, r)\binom{k}{r}\binom{N}{r}^{-1} = 1$$

holds.

First we show the easy fact that

$$\lim_{N\to\infty} M(N, k, r)\binom{k}{r}\binom{N}{r}^{-1} = 1$$

implies that

$$\lim_{N\to\infty} m(N, k, r)\binom{k}{r}\binom{N}{r}^{-1} = 1$$

and thus that it will be sufficient to show the first statement only:

Let

$$\mathscr{F} \subset [\{1, 2, \dots, N\}]^k, |\mathscr{F}| \leq \binom{N}{r}\binom{k}{r}^{-1}(1+\delta)$$

be an r-dense family. For every $e \in [\{1, 2, \dots, N\}]^r$ let $\nu(e)$ be the number of elements of $\mathscr{F}$ containing e. We have

$$\sum\{\nu(e); e \in [\{1, 2, \dots, N\}]^r\} \leq (1+\delta)\binom{N}{r}$$

and thus

$$\sum\{\nu(e);\ \nu(e)\geq 2\}\leq 2\delta\binom{N}{r}.$$

After deleting (from $\mathscr{F}$) all elements of F containing r-sets e with $\nu(e)\geq 2$ we get at least

$$\binom{N}{r}\binom{k}{r}^{-1}(1+\delta)-2\delta\binom{N}{r}=\binom{N}{r}\binom{k}{r}^{-1}\left(1+\delta-2\delta\binom{k}{r}\right)$$

k-sets, forming an r-sparse family. The last quantity tends to

$$\binom{N}{r}\binom{k}{r}^{-1}\quad\text{as } \delta\to 0;$$

this proves our claim.

The proof of the theorem is divided into three lemmas.

LEMMA 1. *Let $G=((V_i)_{i=1}^k, E)$, $|V_1|=|V_2|=\cdots=|V_k|=n$ be a k-partite r-graph, ρ and $\sigma_l(l=r,\ldots,k-1)$ positive reals smaller than one such that*

(α) *$\sigma^L(G)\sim\sigma_l n^{k-l}$ for any L, complete (in G) $k>|L|=l\geq r$.*
(β) *$\rho_I(G)\sim\rho n^r$ for any $I\in[\{1,2,\ldots,k\}]^r$.*

Then for every $\varepsilon>0$ one can select a system $\mathscr{K}$ of k-gons from G such that if we put

$$G^*=((V_i)_{i=1}^k;\ E-\{e;\ \exists K\in\mathscr{K},\ e\in K\}),$$
$$\rho_I^*=\rho_I(G^*),$$
$$\sigma^{L*}=\sigma^L(G^*),$$

the following hold:

(a) $\rho_I^*\sim(\rho\exp(-\sigma_r\varepsilon))n^r$ *for any $I\in[\{1,2,\ldots,k\}]^r$.*

(b) $\sigma^{L*}\sim\left[\sigma_l\exp\left(-\sigma_\mu\varepsilon\left(\binom{k}{r}-\binom{l}{r}\right)\right)\right]n^{k-l}$ *for any L complete (in G^*), $k>|L|=l\geq r$.*

(c) $|\{\{K^1,K^2\},\ K^1,K^2\in\mathscr{K},\ K^1\cap K^2\neq\varnothing\}|\leq 2\sigma_r\varepsilon|\bigcup_{K\in\mathscr{K}}K|$.

REMARK. Roughly speaking this Lemma asserts that for a given $\varepsilon>0$ there is $n>n_0(\varepsilon)$ such that if $\sigma^L(G)$, $\rho_I(G)$ are close to values described in (α)(β) one can select a system $\mathscr{K}$ with ρ_I^*, σ^{L*} close to values described in (a) and (b) and moreover such that (c) holds.

PROOF OF LEMMA 1. Let $G=((V_i)_{i=1}^k, E)$ be a given k-partite r-graph with the properties (α) and (β) of Lemma 1. Suppose without loss of generality that $n=|V_1|=|V_2|=\cdots=|V_k|$ is large positive integer (this will be specified later). Let $\boldsymbol{\mathscr{K}}$ be a random variable the values of which are subsets of the set $\mathscr{K}(G)$ of all k-gons of the r-graph G. If $K\in\mathscr{K}(G)$ then

$$\text{Prob}[K\in\boldsymbol{\mathscr{K}}]=\frac{\varepsilon}{n^{k-r}}$$

and these probabilities are independent for different $K\in\mathscr{K}(G)$. First consider the edges of G which are not covered by chosen k-gons (more precisely the edges of $\boldsymbol{E}_I=\boldsymbol{E}_I(\boldsymbol{G})$, where $I\in[\{1,2,\ldots,k\}]^r$ and $\boldsymbol{G}=(V(G)\ E(G)-\{e;\ \exists K\in\boldsymbol{\mathscr{K}}\ e\in K\})$. For a fixed edge $e\in E_I$ the probability that $e\in E_I(\boldsymbol{G})$ is

$$p_e=\left(1-\frac{\varepsilon}{n^{k-r}}\right)^{\sigma_r n^{k-r}(1+o(1))}\sim\exp(-\varepsilon\sigma_r).$$

These probabilities are independent for different $e \in E_I$. Thus the probability that there are exactly s edges in E_I which are not covered by any k-gon $K \in \mathscr{K}$ is

$$\sum_{E \in [E_I]^s} \prod_{e \in E} p_e \prod_{e \in E_I - E} (1-p_e) =$$
$$= \binom{\rho n^r(1+o(1))}{s} (\exp(-\varepsilon\sigma_r))^s (1-\exp(-\varepsilon\sigma_r))^{\rho n^r(1+o(1))-s} (1+o(1))^{\rho n^r}.$$

For any $\sigma > o$ let S_δ be the set of all integers such that $0 \leq s \leq \rho n^r(1+o(1)) = |E_I|$ and $|s - (\exp(-\sigma_r\varepsilon))\rho n^r| > \delta\rho n^r \exp(-\sigma_r\varepsilon)$, then by (3) we get

$$\sum_{s \in S\delta} \binom{\rho n^r(1+o(1))}{s} (\exp(-\varepsilon\sigma_r))^s (1-\exp(-\varepsilon\sigma_r))^{\rho n^r(1+o(1))-s} (1+o(1))^{\rho n^r}$$
$$< n^r \exp(-c_1 n^r) < \exp(-c_2 n^r)$$

for some $c_1 > 0$, $c_2 > 0$ and n sufficiently large.

Thus we get

$$\left.\begin{aligned} &\rho_I(G) \underset{\delta}{\sim} \rho n^r \exp(-\sigma_r\varepsilon), \\ &\text{for every } I \in [\{1,2,\ldots,k\}]^k \text{ with probability greater than} \\ &1 - \binom{k}{r} \exp(-c_2 n^r) > 1 - \exp(-c_3 n^r), \qquad c_3 > 0 \end{aligned}\right\} \tag{4}$$

Here we used again that n is sufficiently large. Now we prove the following:

CLAIM. *For every $\delta > 0$, $k > l \geq r$,*

$$\left.\begin{aligned} &\text{Prob}[\sigma^L(G) \underset{\delta}{\sim} \sigma^L(G) \exp\left(-\varepsilon\sigma_r\left(\binom{k}{r} - \binom{l}{r}\right)\right), L \text{ complete}, \\ &|L| = l] > 1 - \exp(-c_l' n) \end{aligned}\right\} \tag{5}$$

(here c_l' is the positive constant depending on the size of L only).

We shall proceed by induction on $k - |L|$. If $|L| = k-1$ and L is complete then there is $k' \in \{1,2,\ldots,k\}$ (say $k' = k$) and $t(L) = \sigma_{k-1} n(1+o(1))$ vertices $v_1, v_2, \ldots, v_{t(L)} \in V_{k'} = V_k$ such that

$$\{v_i\} \cup R \in E$$

for every $i \in \{1,2,\ldots,t(L)\}$ and $R \in [L]^{r-1}$. Let A_L be the event that $[L]^r \subset \bigcup\{E_I, I \in [\{1,2,\ldots,k\}]^r\} = E$. For a vertex $v_i 1 \leq i \leq t(L)$ denote by B_i the event that all edges $\{v_i\} \cup R$, $R \in [L]^{r-1}$ remain in E. It follows from (α) of Lemma 1 that for every i, $1 \leq i \leq t(L)$ the number of complete k-sets L' containing v_i and moreover such that $[L']^r \cap [L \cup \{v_i\}]^r \neq \varnothing$ and $[L']^r \cap [L]^r = 0$ is bounded from above by

$$\binom{k-1}{r-1} \sigma_r n^{k-r}(1+o(1))$$

and from below by

$$\binom{k-1}{r-1} \sigma_r n^{k-r}(1+o(1)) - \binom{k-1}{r} \sigma_{r+1} n^{k-r-1}(1+o(1)) \sim \binom{k-1}{r-1} \sigma_r n^{k-r}.$$

As deletion of one of such k-gons causes that B_i fails to be true provided A_L holds, we have

$$\mathrm{Prob}(B_i|A_L)\sim\left(1-\frac{\varepsilon}{n^{k-r}}\right)^{\sigma_r n^{k-r}\binom{k-1}{r-1}}\sim\exp\left(-\binom{k-1}{r-1}\sigma_r\varepsilon\right).$$

Clearly the events $(B_i|A_L)$ are independent for fixed L and different i, $1\leqslant i\leqslant t(L)$.

Thus similarly as in the preceding case we get, using (3) that

$$\mathrm{Prob}\left[\sigma^L(G)\underset{\delta}{\sim}t(L)\exp\left(-\varepsilon\sigma_r\binom{k-1}{r-1}\right);\quad L\text{ complete},\quad L\subset V,\ |L|=k-1\right]>1-\exp(-c'_{k-1}n),$$

where c'_{k-1} is a positive constant independent on n. Here we have again used the fact that there are only polynomially many choices for L). As $t(L)=\sigma^L(G)$ we are done.

Suppose now that we proved our claim (4) for all L, $k-1\geqslant|L|\geqslant k-j$ where $1\leqslant j\leqslant k-r-1$. Let L' be a fixed complete set $|L'|=k-j-1$, $1\leqslant j\leqslant k-r-1$. Without loss of generality suppose that $L'\cap V_i\neq 0$ for $i=1,2,\ldots,k-j-1$. Let v'_i, $1\leqslant i\leqslant t(L')$ be all vertices of V_{k-j} such that $\{v'_i\}\cup R\in E$ for every i, $1\leqslant i\leqslant t(L')$ and $R\in[L']^{r-1}$. Let again $A_{L'}$ be the event that $[L']^r\subset \boldsymbol{E}$ and for i, $1\leqslant i\leqslant t(L')$ let B'_i be the event that all edges $\{v'_i\}\cup R$, $R\in[L']^{r-1}$ remain in $\boldsymbol{E}$. Then, similarly to the preceding case we have

$$\mathrm{Prob}(B'_i|A_{L'})\sim\exp\left(-\varepsilon\sigma_r\binom{k-j-1}{r-1}\right).$$

The events $(B'_i|A_{L'})$ are independent for fixed L' and different i, $1\leqslant i\leqslant t(L')$. Thus again if $\delta>0$, we have again by (3) that

$$|\{i,\{v_i\}\cup R\in \boldsymbol{E}\text{ for every }R\in[L']^{r-1}\}|\underset{\delta}{\sim}t(L')\exp\left(-\varepsilon\sigma_r\binom{k-j-1}{r-1}\right)\tag{6}$$

with the probability bigger than $1-\exp(-c^{L'}_{k-j-1}n)$ for $c^{L'}_{k-j-1}>0$. By the induction assumption, i.e. (4), we have the probability that

$$\sigma^{L^i}(\boldsymbol{G})\underset{\delta}{\sim}\exp\left(-\varepsilon\sigma_r\left(\binom{k}{r}-\binom{k-j}{r}\right)\right)\sigma^{L^i}(G)\quad\text{for every }i,\ 1\leqslant i\leqslant t(L')\text{ (here }L^i=L'\cup\{v_i\}\text{) is larger than }1-\exp(-c_{k-j}n).\tag{7}$$

As the set of all k-gons containing L' in $\boldsymbol{G}$ is the union of all sets of k-gons containing such L^i for which $L^i\subset \boldsymbol{G}$ holds we get, combining (6) and (7) and using $(1+\delta)^2<1+3\delta$ for $\delta<1$, that

$$\begin{aligned}\sigma^{L'}(\boldsymbol{G})&=\sum_{L^i\subset \boldsymbol{G}}\sigma^{L^i}(\boldsymbol{G})\underset{3\delta}{\sim}\\&\underset{3\delta}{\sim}\exp\left(-\varepsilon\sigma_r\left[\left(\binom{k}{r}-\binom{k-j}{r}\right)+\binom{k-j-1}{r-1}\right]\right)\sum_{1\leqslant i\leqslant t(L)}\sigma^{L^i}\\&\sim\exp\left(-\varepsilon\sigma_r\left[\binom{k}{r}-\binom{k-j-1}{r}\right]\right)\sigma^{L'}(G)\end{aligned}$$

for $\delta<1$ with probability exponentially close to 1. As there are only polynomially many choices for L' we infer that

$$\sigma^{L'}(\boldsymbol{G})\underset{3\delta}{\sim}\sigma^{L'}(G)\exp\left[-\varepsilon\sigma_r\left(\binom{k}{r}-\binom{k-j-1}{r}\right)\right]$$

for every L' holds with probability bigger than $1-\exp(-c'_{k-j-1}n)$. This proves the Claim.

Now we shall examine intersections of chosen k-gons. For a system $\mathcal{K}$ of k-gons let $c(\mathcal{K})$ denote the number of pairs $K^1, K^2 \in \mathcal{K}$ such that $K^1 \cap K^2 \neq 0$. Then the expectation

$$E(c(\boldsymbol{\mathcal{K}})) = (1+o(1))\binom{k}{r}\rho n^r \binom{\sigma_r n^{k-r}}{2}\left(\frac{\varepsilon}{n^{k-r}}\right)^2 \leq \frac{1+o(1)}{2}\binom{k}{r}\rho\sigma_r^2\varepsilon^2 n^r,$$

and thus

$$\mathrm{Prob}\left(c(\boldsymbol{\mathcal{K}}) \leq \frac{3}{4}(\rho\sigma_r^2\varepsilon^2 n^r)\binom{k}{r} \geq \frac{1}{3}(1+o(1))\right).$$

According to (4) we have that

$$\left|\bigcup_{K\in\boldsymbol{\mathcal{K}}} K\right| \geq \binom{k}{r}\rho n^r(1+o(1)-(1+\delta)\exp(-\sigma_r\varepsilon))$$

holds with probability bigger than $1-\exp(-c_3 n^r)$ for $n \geq n(\delta)$. As (4) holds for any $\delta > 0$ and $n \geq n(\delta)$, we can assume that

$$\delta \leq \frac{\sigma_r\varepsilon}{10}\exp(\sigma_r\varepsilon).$$

Thus we have for n sufficiently large

$$\frac{c(\boldsymbol{\mathcal{K}})}{|\bigcup_{k\in\boldsymbol{\mathcal{K}}} K|} \leq \frac{\frac{3}{4}\binom{k}{r}\rho\sigma_r^2\varepsilon^2 n^r}{\binom{k}{r}\rho n^r\left(1+o(1)-\exp(-\sigma_r\varepsilon)-\frac{\sigma_r\varepsilon}{10}\right.} \leq \frac{\frac{3}{4}\sigma_r^2\varepsilon^2}{o(1)+\sigma_r\varepsilon-\frac{\sigma_r^2\varepsilon^2}{2}-\frac{\sigma_r\varepsilon}{10}} < 2\sigma_r\varepsilon, \tag{8}$$

with probability at least $\frac{1}{3}(1+o(1))$. Combining (4), (5) and (8) we get that there exists $\mathcal{K} \in \boldsymbol{\mathcal{K}}$ satisfying (a), (b) and (c).

Lemma 2. *Let $G = ((V_i)_{i=1}^k, E)$ be a k-partite r-graph satisfying assumptions of Lemma 1. Then there exists a system $\mathcal{S}$ of k-gons of G such that*

(*a*) $\left|\bigcup_{S\in\mathcal{S}} S\right| = |E|(1-o(1))$

(*b*) $|\mathcal{S}| \leq |E|\binom{k}{r}^{-1}(1+o(1))$.

Proof. Let $\delta \geq 0$ be a given real. We will show that there exists (provided $n = |V_1| = \cdots = |V_k|$ is sufficiently large) a system of k-gons of G which

$$\text{contains all but at most } \delta/2|E| \text{ edges} \tag{9}$$

and such that

$$|\mathcal{S}| \leq \frac{|E|\left(1+\frac{\delta}{2}\right)}{\binom{k}{r}}. \tag{10}$$

We shall construct our system $\mathcal{S}$ inductively. We shall also construct an auxiliary sequence $G_0, G_1, \ldots, G_t$ of r-graphs such that $E(G_0) \supset E(G_1) \supset \cdots \supset E(G_t)$ (t will be specified later). Set $G_0 = G$ and $\varepsilon = (\delta/4\sigma_r)$. According to Lemma 1 there exists a system $\mathcal{K}_0 = \mathcal{K}$ of k-gons of G_0 with the properties of Lemma 1. Suppose that we have constructed $G_j = ((V_i)_{i=1}^k, E_j)$, $E_j \subset E$ and a system $\mathcal{S}_j$ of k-gons, covering edges of $E - E_j$ so that

(a) $\sigma^L(G_j) \sim \sigma_l \exp\left[-\sigma_r \varepsilon j\left(\binom{k}{r} - \binom{l}{r}\right)\right] n^{k-l}$ for every L complete in G_j, $k > |L| = l \geq k$,

(b) $\rho_I(G_j) \sim \rho \exp(-\sigma_r \varepsilon j) n^r$ for every $I \in [\{1, 2, \ldots, k\}]^r$

(c) $|\mathcal{S}_j| \leq \left(1 + \frac{\delta}{2}\right) \frac{|E - E_j|}{\binom{k}{r}}$.

Then, according to Lemma 1, we can select a system $\mathcal{K}_j$ of k-gons from G_j such that if we put $G_{j+1} = G_j$—edges of all selected k-gons the following holds:

(a′) $\sigma^L(G_{j+1}) \sim \sigma_l n^{k-l} \exp\left[-\varepsilon\sigma_r(j+1)\left(\binom{k}{r} - \binom{l}{r}\right)\right]$ for any L complete in G_{j+1}, $k > |L| = l \geq r$,

(b′) $\rho_I(G_{j+1}) \sim \rho n^r \exp(-\varepsilon\sigma_r(j+1))$ for any $I \in [\{1, 2, \ldots, k\}]^r$.

Moreover, we have

$$|\{\{K^1, K^2\}, K^1, K^2 \in \mathcal{K}_j, K^1 \cap K^2 \neq \varnothing\}| \leq 2\sigma_r \varepsilon \left| \bigcup_{K \in \mathcal{K}_j} K \right|.$$

If we choose for each couple $\{K^1, K^2\}$, $K^1, K^2 \in \mathcal{K}_j$, $K^1 \cap K^2 \neq 0$ at least one of its elements and delete all such k-gons from $\mathcal{K}_j$ we get a system of at least

$$|\mathcal{K}_j| - 2\sigma_r \varepsilon \left| \bigcup_{K \in \mathcal{K}_j} \right|$$

pairwise disjoint k-gons covering at most $|\bigcup_{K \in \mathcal{K}_j} K|$ edges. Thus we get

$$|\mathcal{K}_j| \leq \frac{|\bigcup_{K \in \mathcal{K}_j} K|}{\binom{k}{r}} + 2\sigma_r \varepsilon \left| \bigcup_{K \in \mathcal{K}_j} K \right|.$$

Set $\mathcal{S}_{j+1} = \mathcal{S}_j \cup \mathcal{K}_j$; as $\mathcal{S}_{j+1}$ covers clearly all edges of $E - E_{j+1}$ we get that

(c′)
$$\begin{aligned} |\mathcal{S}_{j+1}| &= |\mathcal{S}_j| + |\mathcal{K}_j| \\ &\leq \left(1 + \frac{\delta}{2}\right) \frac{|E - E_j|}{\binom{k}{r}} + \frac{1}{\binom{k}{r}} \left(1 + 2\sigma_r \varepsilon \binom{k}{r}\right) \left| \bigcup_{K \in \mathcal{K}_j} K \right| \\ &< \left(1 + \frac{\delta}{2}\right) \frac{|E - E_j|}{\binom{k}{r}} + \frac{1}{\binom{k}{r}} \left(1 + \frac{\delta}{2}\right) |E_j - E_{j+1}| \\ &\leq \left(1 + \frac{\delta}{2}\right) \frac{|E - E_{j+1}|}{\binom{k}{r}}. \end{aligned}$$

Set $t=\lceil 1/\sigma_r\varepsilon \ln 2/\sigma\rceil$ and repeat this procedure t-times. The system $\mathcal{S}=\mathcal{S}_t$ covers all edges of $E-E_t$ and as $|E_t|\leqslant \exp(-\sigma_r\varepsilon t)|E|\leqslant(\delta/2)|E|$ we get that (9) holds. Moreover we have

$$|\mathcal{S}_t|\leqslant\left(1+\frac{\delta}{2}\right)\frac{|E-E_t|}{\binom{k}{r}}\leqslant\left(1+\frac{\delta}{2}\right)\frac{|E|}{\binom{k}{r}}$$

and thus (10) holds as well.

LEMMA 3. *Let $p>k>r$ be given positive integers. Let $A_1, A_2, \ldots, A_p$ be pairwise disjoint sets of the same (large) cardinality n. Then there exists a decomposition*

$$[\{A_i\}_{i=1}^p]^r=\bigcup\{E_j, J\in[\{1,2,\ldots,p\}]^k\} \tag{11}$$

such that for every $J, J'\in[\{1,2,\ldots,k\}]^p$, $J\neq J'$

(α) $E_J\cap E_{J'}\neq\varnothing$,

(β) $E_J\subset[\{A_i\}_{i\in J}]^r$, *and the r-graph $H(J)$ defined by $V(H(J))=\bigcup_{i\in J}A_i$, $E(H(J))=E_J$ satisfies*

(γ) $\rho_I(H(J))\sim\frac{1}{t}\,n^r$, *for every* $I\in[J]^r$,

(δ) $\sigma^L(H(J))\sim\sigma_l n^{k-l}$, *for any L complete*

$$\sigma_l=\left(\frac{1}{t}\right)^{\binom{k}{r}-\binom{l}{r}}\quad\textit{and}\quad k\geqslant|L|=l\geqslant r,$$

where

$$t=\binom{p-r}{k-r}.$$

PROOF. For every $e\in[\{A_i\}_{i=1}^p]^r$ let $X(e)$ be the set of all k-subsets J of $\{1,2,\ldots,p\}$ with the property that $\{i;\, A_i\cap e\neq 0\}\subset J$. Clearly $|X(e)|=t$. To every $e\in[\{A_i\}_{i=1}^p]^r$ choose independently on all other choices an element $\boldsymbol{\phi}(e)$ of $X(e)$. For every $J\in[\{1,2,\ldots,p\}]^k$ let $\boldsymbol{H}(J)$ be a random r-graph defined by

$$V(\boldsymbol{H}(J))=\bigcup_{i\in J}A_i$$

$$E(\boldsymbol{H}(J))=\boldsymbol{E}_J=\{\boldsymbol{\phi}(e)\subset\{A_i\}_{i\in J};\, e\in[\{A_i\}_{i\in J}]^r\}$$

Obviously

$$[\{A_i\}_{i=1}^r]^r=\bigcup\{\boldsymbol{E}_J;\, J\in[\{1,2,\ldots,p\}]^\kappa\} \tag{12}$$

yields a partition satisfying (α) and (β). We show that with probability $1-o(1)$ the partition (12) satisfies (γ) and (δ) as well. Let L, $|L|=l<k$ be a fixed subset of $\bigcup_{i\in J}A_i$ such that $|L\cap A_i|\leqslant 1$ for every $i\in J$. Let A_L^J denote the event that L is complete subset of $\boldsymbol{H}(J)$ and for $v\in A_{i_0}(i_0\in J-\{i;\, L\cap A_i\neq 0\})$ let $B_{L\cup\{v\}}^J$ denote the event that

$$[L\cup\{v\}]^r-[L]^r\subset\boldsymbol{H}(J).$$

Then

$$\operatorname{Prob}(B_{L\cup\{v\}}^J|A_L^J)=\operatorname{Prob}(B_{L\cup\{v\}}^J)=\left(\frac{1}{t}\right)^{\binom{l}{k-1}},$$

and these probabilities are independent for different $v\in A_{i_0}$ and fixed L and J. Thus, according to (3) we get that for fixed i_0, J, L and $\delta>0$

$$|\{v\in A_{i_0};[L\cup\{v\}]^r-[L]^r\in\boldsymbol{H}(J)\}\left|-\left(\frac{1}{t}\right)^{\binom{l}{r-1}}n\right|\geqslant\delta n \tag{13}$$

holds with the probability smaller than $\exp(-c_1 n)$, where $c_1 > 0$ is independent on n. As there are only polynomially many (in n) choices for i_0, J and L (n is large and $p, k, r, \delta > 0$ are fixed) we get that there exists $c_2 > 0 (c_2 < c_1)$ such that the probability of the event that there is i_0, J and L such that holds is bounded by $\exp(-c_2 n)$. Similarly one can show that there exists $c_3 > 0$ such that the probability that

$$\left| \rho_I(\boldsymbol{H}(J)) - \frac{1}{t} n^r \right| \geq \delta n^r$$

for some I and J is bounded by $\exp(-c_3 n)$. Hence the partition satisfies (γ) and (δ) with probability at least $1 - \exp(-c_3 n) - \exp(-c_2 n)$. Thus for n sufficiently large there exists a decomposition satisfying (α), (β) and

$$(\gamma') \quad \left| \rho_I(H(J)) - \frac{1}{t} n^r \right| < \delta n^r,$$

$$(\delta') \quad \left| |\{v \in A_{i_0}; [L \cup \{v\}]^r - [L]^r \in H(J)\}| - \left(\frac{1}{t}\right)^{\binom{l}{r-1}} \right| < \delta \left(\frac{1}{t}\right)^{\binom{l}{r-1}} n, \tag{14}$$

for any choice of I, J, L and i_0. As (14) holds for any L, (L is complete in $H(J)$)$|L| = l$, $r \leq l < k$, we get by induction that L is contained in at least

$$(1-\delta)^{k-l} \left(\frac{1}{t}\right)^{\binom{k}{r} - \binom{l}{r}} n^{k-l} (1-\delta)^{k-l} \left(\frac{1}{t}\right)^{\sum_{j=l}^{k-1} \binom{j}{r-1}} n^{k-l}$$

and at most

$$(1+\delta)^{k-l} \left(\frac{1}{t}\right)^{\binom{k}{r} - \binom{l}{r}} n^{k-l}$$

k-gons of $H(J)$. This together with (γ') and with the fact that $\delta > 0$ may be considered arbitrarily small yields (γ) and (δ).

Proof of the Theorem. Let $\delta > 0$ be positive real and k, r positive integers. Take

$$p = \left[\frac{2r(r-1)}{\delta} \binom{k}{r} \right].$$

Take large integer N (without loss of generality we shall suppose that N is divisible by p) and set $n = N/p$. Consider p pairwise disjoint sets $A_1, A_2, \ldots, A_p$ of the same cardinality n. Take the decomposition (the existence of which is ensured by Lemma 3)

$$[\{A_i\}_{i=1}^p]^r = \bigcup \{E_J; J \in [\{1, 2, \ldots, p\}]^k\}$$

Thus for each $J \in [1, 2, \ldots, p\}]^k$ we have that the r-graph $H(J) = ((A_j)_{j \in J}, E_J)$ satisfies the assumptions of Lemma 2. Hence we get, that for $N \geq N(\delta, k, r)$ there exists a system $\mathcal{S}(J)$ of k-gons such that

$$\text{(1)} \qquad \left| \bigcup_{S \in \mathcal{S}(J)} S \right| \geq |E_J| \left(1 - \frac{\delta}{8\binom{k}{r}} \right)$$

and

$$\text{(2)} \qquad |\mathcal{S}(J)| \leq |E_j| \left(1 + \frac{\delta}{8}\right) \Big/ \binom{k}{r}$$

To every from at most $\left(\delta/8\binom{k}{r}\right)|E_J|$ uncovered edges e choose a k-subset of $\bigcup_{j\in J} A_j$ which contains e and let ν_J be a set consisting of all k-gons of $\mathcal{S}(J)$ and all chosen k-sets. We have

$$|\nu_J| \leq \frac{|E_J|\left(1+\frac{\delta}{8}\right)}{\binom{k}{r}} + \frac{|E_J|\frac{\delta}{8}}{\binom{k}{r}} = \frac{|E_J|}{\binom{k}{r}}\left(1+\frac{\delta}{4}\right).$$

The set $\nu = \bigcup\{\nu_J, J\in[\{1,2,\dots,p\}]^k\}$ is a system of at most

$$\binom{p}{k}\left(|E_J|\Big/\binom{k}{r}\right)\left(1+\frac{\delta}{4}\right) \sim \binom{p}{k}\frac{n^r}{t}\left(1+\frac{\delta}{4}\right) < \left(\binom{pn}{r}\Big/\binom{k}{r}\right)\left(1+\frac{\delta}{2}\right)$$

k-sets (for the last inequality we used again that n is large) having the property that any $e\in[\{A_i\}_{i=1}^p]^r$ is contained in some element of ν. Now we examine elements of $[A]^r - [\{A_i\}_{i=1}^p]^r (A=\bigcup_{i=1}^p A_i)$. There are at most

$$\binom{pn}{r} - \binom{p}{r}n^r < \binom{pn}{r}\left(1-\left(1-\frac{r-1}{p}\right)^r\right) < \binom{pn}{r}\frac{r-1}{p}r$$

of them. For each such r-set e choose an k-set (subset of A) that contains e. This gives at most

$$\binom{pn}{r}\frac{r-1}{p}r < \binom{pn}{r}\delta/2\binom{k}{r}$$

new k-sets. After adding all such k-sets to ν we get an r-dense system $\mathcal{F}$ of k-sets of $\bigcup_{i=1}^p A_i$ such that

$$|\mathcal{F}| < \binom{N}{r}(1+\delta)\Big/\binom{k}{r}.$$

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V. Rödl
FJFI, ČVUT, Katedra Matematiky, Husova 5, 11000 Praha, Tchechoslovaque