MATHEMATICS

MARKOV CHAINS AND INTUITIONISM. III

NOTE ON CONTINUOUS FUNCTIONS WITH AN APPLICATION TO MARKOV CHAINS

BY

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0. The purpose of this paper is to consider continuous functions which are defined on some intervals and to derive limitproperties at the endpoints of such an interval if some conditions are given.

In section 3 we prove a limitproperty which has no classical counterpart.

In two papers [1] and [2] (Numbers in brackets refer to the references given at the end of this paper) some properties of Markov Chains have been discussed from the intuitionistic point of view and compared with the classical ones. In section 5 the theorem of section 2 will be applied to the transition probabilities of Markov Chains and it gives a new property about the existence of the limit at $t=0$ of the transition functions.

For the terminology used in this paper the reader is referred to [2] and to the references quoted there.

1. Theorem. Let $f(t)$ be a continuous function which is defined on $[\delta, \infty)$ for every real number $\delta > 0$.

If

$$\lim_{t \downarrow 0} f(t) \tag{1}$$

exists then we have:

(i) $\lim\limits_{t \downarrow x} f(t)$ exists for every $x \nless 0$;

(ii) by $g(x) \overset{\text{df}}{=} \lim\limits_{t \downarrow x} f(t)$ a function $g(x)$ is defined on $[\delta, \infty)$ for every real number $\delta \nless 0$ and $g(x)$ is continuous on that interval.

Remark.

On account of the continuity of $f(t)$ on $[\delta, \infty)$ for every $\delta > 0$ the existence of $\lim\limits_{t \downarrow x} f(t)$ and hence of $g(x)$ is assured at every point $x > 0$. Furthermore the value of $g(0)$ is given by (1).

From the classical point of view these remarks are sufficient to prove the existence of $g(x)$ on $[0, \infty)$, but from the intuitionistic point of view these remarks cannot be applied to real numbers $x \nless 0$ for which we have no proof of

$$[x=0] \vee [x>0].$$

Therefore we have to give a proof which includes this case.

Proof. Let x be a real number with $x \nless 0$, then we may suppose that this real number x is given by a sequence $\{\varrho_n\}$ of intervals ϱ_n of which the endpoints r_n' resp. r_n'' are rational numbers. Without lose of generality we may suppose that these endpoints satisfy the relations:

(2) $$(\forall n)\ [r_n' \nless 0) \wedge (r_{n+1}' \nless r_n') \wedge (r_n' < r_n'') \wedge (r_{n+1}'' \ngtr r_n'')]$$

and

$$\lim_{n\to\infty} (r_n'' - r_n') = 0.$$

Let the sequence $\{g_n(x)\}$ be defined by

$$g_n(x) = f(r_n'') \qquad (n = 1, 2, \ldots),$$

then we prove:

(j) $$\lim_{n\to\infty} g_n(x) \text{ exists},$$

(jj) $$\lim_{n\to\infty} g_n(x) = \lim_{t \downarrow x} f(t).$$

To this aim we prove:

$$(\forall \varepsilon)(\exists N)(n, m > N \Rightarrow |g_n(x) - g_m(x)| < 2\varepsilon).$$

We arbitrarily choose the real number $\varepsilon_1 > 0$. Then the relation (1) guarantees that a natural number k can be calculated such that:

(3) $$0 < t < 2^{-k} \Rightarrow |f(t) - a| < \varepsilon_1,$$

where $a = \lim_{t \downarrow 0} f(t)$.

Evidently an index N can be calculated such that:

(4) $$n \nless N \Rightarrow 0 \ngtr r_n'' - r_n' < 2^{-k-2}.$$

The numbers r_N'' and 2^{-k} are rational numbers, hence we have the disjunction

$$(r_N'' \nless 2^{-k}) \vee (r_N'' < 2^{-k}),$$

which implies that only the two cases:

$$A : r_N'' \nless 2^{-k};\ B : r_N'' < 2^{-k}$$

need to be considered.

Case A.

$r_N'' \nless 2^{-k}$ and (4) imply: $r_N' > 2^{-k-2}$ and from (2) we see: $x > 0$. Using the continuity of $f(t)$ it is now easily seen that:

(5) $$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f(r_n'') = \lim_{t \downarrow x} f(t) = g(x).$$

Case B.

The relations: $r_N'' < 2^{-k}$, (3) and

$$|f(r_n'') - f(r_m'')| \ngtr |f(r_n'') - a| + |f(r_m'') - a|$$

lead to:

$$(6) \qquad |g_n(x) - g_m(x)| < 2\varepsilon_1$$

for all $n, m > N$.

Now we have proved that for every real number $x \nless 0$, for which $\{\varrho_n\}$ is a defining sequence, and for every real number $\varepsilon > 0$ a natural number N can be calculated such that

$$|g_n(x) - g_m(x)| < 2\varepsilon$$

for all n, $m > N$ and the convergence of the sequence $\{g_n(x)\}$ is guaranteed by Cauchy's general convergence principle.

We now prove that $\lim g_n(x)$ is independent of the sequence $\{\varrho_n\}$. Let $\{\sigma_n\}$ with $\sigma_n = [s'_n, s''_n]$ be an other sequence which defines x and satisfies the relations (2) and let $\{g'_n(x)\}$ be the sequence which corresponds to $\{\sigma_n\}$.

From (2) it follows that an index N' can be calculated such that

$$n \nless N' \Rightarrow s''_n - s'_n < 2^{-k-2}.$$

Without lose of generality we may suppose: $N = N'$.

If $x > 0$ the uniquences of the limit easily follows from the continuity and there remains to consider the case:

$$(r''_N < 2^{-k}) \wedge (s''_N < 2^{-k}).$$

In this case the uniquenes is implied by

$$|g_n(x) - g'_n(x)| = |f(r''_n) - f(s''_n)| \ngtr |f(r''_n) - a| + |a - f(s''_n)| < 2\varepsilon_1.$$

Now we know that $g(x)$ is defined on $[0, \infty)$, hence by Brouwer's fan theorem is a continuous function (even uniformly continuous on every closed subinterval of $[0, \infty)$).

2. Theorem. Let $f(t)$ be a continuous function for every $t > 0$ such that $f(t)$ is uniformly continuous on $[\delta, \infty)$ for every $\delta > 0$.

Then we have:

The existence of $\lim\limits_{t \downarrow 0} f(t)$ is equivalent to uniform continuity of $f(t)$ on $(0, \infty)$.

Proof. Let us put $a = \lim\limits_{t \downarrow 0} f(t)$, then we have:

$$(1) \qquad (\forall\varepsilon)(\exists\delta)(0 < t < \delta \Rightarrow |f(t) - a| < \tfrac{1}{2}\varepsilon).$$

We arbitrarily choose a real number $\varepsilon_1 > 0$ and we calculate a corresponding real number δ_1 according to (1), then from (1) follows

$$t_1, t_2 \in (0, \delta_1) \Rightarrow |f(t_1) - f(t_2)| < \varepsilon_1.$$

We know that $f(t)$ is uniformly continuous on $[\tfrac{1}{2}\delta_1, \infty)$, hence we can calculate a real number $\delta_2 > 0$ such that:

$$[t_1, t_2 \in [\tfrac{1}{2}\delta_1, \infty)] \wedge [|t_1 - t_2| < \delta_2] \Rightarrow (|f(t_1) - f(t_2)| < \tfrac{1}{2}\varepsilon_1).$$

Let the real number δ_3 be defined by $\delta_3 = \min(\frac{1}{2}\delta_1, \delta_2)$, then we have:

$$[t_1, t_2 \in (0, \infty) \wedge |t_1 - t_2| < \delta_3] \Rightarrow (|f(t_1) - f(t_2)| < \varepsilon_1),$$

i.e. $f(t)$ is uniformly continuous on $(0, \infty)$.

Now we prove the inverse part of the theorem. In this case we know that $f(t)$ is uniformly continuous on $(0, \infty)$, hence

(2) $$(\forall\varepsilon)(\exists\delta)[(t_1 > 0) \wedge (t_2 > 0) \wedge (|t_1 - t_2| < \delta) \Rightarrow f(t_1) - f(t_2)| < \varepsilon].$$

We choose the real number $\varepsilon_1 > 0$ arbitrarily and according to (2) we calculate a corresponding value $\delta(\varepsilon_1)$ of δ.

Let $\{t_n\}$ be a sequence of real numbers such that

(3) $$(\forall n)(t_n > t_{n+1}) \text{ and } \lim_{n\to\infty} t_n = 0.$$

The relations (2) and (3) imply:

$$(\exists N)(n, m > N \Rightarrow |f(t_n) - f(t_m)| < \varepsilon_1).$$

The real number ε_1 was chosen arbitrarily, hence by Cauchy's general convergence principle we conclude that $\lim_{n\to\infty} f(t_n)$ exists. The uniqueness of the limit simply follows from the uniform continuity.

3.1. Let R represent the continuum.

We consider the following theorem which is trivially true from the classical point of view.

Let $f(\cdot)$ be a function which satisfies:

(i) $f(\cdot)$ is defined on R,

(ii) $f(t)$ is continuous for every $t \neq a \in R$,

(iii) $f(t)$ is continuous from the right at $t = a$,

then we have:

$$\lim_{t\downarrow x} f(t) \text{ exists for every } x \in R.$$

The proof can be seen immediately by using

$$(x < a) \vee (x = a) \vee (x > a) \text{ and } \lim_{t\downarrow x} f(t) = f(x).$$

3.2. From the intuitionistic point of view theorem 3.1. is an immediate consequence of Brouwer's fan theorem and (i) implies (ii) and (iii) and $f(t)$ is continuous everywhere.

3.3. Now we reformulate theorem 3.1. in a somewhat different way, which is an equivalent formulation from the classical point of view.

Let $f(\cdot)$ be a function such that

(j) $f(t)$ is defined for every $t \nless a$ and continuous for $t > a \in R$,

(jj) $f(t)$ is defined and continuous for every $t < a$,

(jjj) $f(t)$ is continuous from the right at $t = a$, then we have:

$$\lim_{t\downarrow x} f(t) \text{ exists for every } x \in R.$$

However, this theorem cannot be proved from the intuitionistic point of view.

3.4. Counter example

Let $f(t)$ be defined by

$$f(t)=2 \text{ for } t \nless a$$
$$f(t)=1 \text{ for } t < a.$$

Now we consider a real number for which we have no proof of

$$(\varrho<a) \vee (\varrho=a).$$

If $\varrho<a$, then $\lim\limits_{t \downarrow \varrho} f(t)=1$, but if $\varrho=a$ then $\lim\limits_{t \downarrow a} f(t)=2$.

This means that as long as we have no proof of $(\varrho=a) \vee (\varrho<a)$ we cannot calculate $\lim\limits_{t \downarrow \varrho} f(t)$.

4. We now prove a theorem which has no counterpart in the classical theory as becomes clear from section 3.3.

Theorem. Let $f(t)$ be a function which is defined for all real numbers t which satisfy

$$(t \nless a) \vee (t<a).$$

If $\lim\limits_{t \downarrow x} f(t)$ exists for every real number x, then we have:

$$\lim_{t \uparrow a} f(t)= \lim_{t \downarrow a} f(t).$$

Proof. We define the function $g(x)$ by:

$$g(x)= \lim_{t \downarrow x} f(t).$$

We know that $g(x)$ is defined for all real numbers, hence from Brouwer's fan theorem it follows that $g(x)$ is a continuous function.

In particular the function $g(x)$ is continuous at $x=a$, hence:

$$(\forall k)(\exists l)(|a-x|<2^{-l} \Rightarrow |g(x)-g(a)|<2^{-k}).$$

Now we choose a natural number k_1 and we calculate a natural number l_1 such that

$$x \in (a-2^{-l_1}, a) \Rightarrow |g(x)-g(a)|<2^{-k_1}. \tag{1}$$

However, the function $f(\cdot)$, which is defined on $(-\infty, a)$, is continuous at every point $x \in (a-2^{-l_1}, a)$, hence

$$\lim f(t)=g(x)=f(x) \text{ for every } x \in (a-2^{-l_1}, a). \tag{2}$$

The relations (1) and (2) imply:

$$x \in (a-2^{-l_1}, a) \Rightarrow |f(x)-g(a)|<2^{-k_1}.$$

The natural number k_1 was chosen arbitrarily, hence we have proved:

$$\lim_{x \uparrow a} f(x)=g(a)= \lim_{x \downarrow a} f(x),$$

5. An application to Markov Chains.

Let a stationary Markov Chain be given by the matrix $(p_{ij}(\cdot))$, where $p_{ij}(t)$ is defined for $i, j = 1, 2, \ldots$ and $t \in (0, \infty)$.

By using classical methods DOOB [3] has proved:

If $(p_{ij}(\cdot))$ is a transition matrix such that all functions $p_{ij}(\cdot)$ are Lebesgue measurable functions, then all $p_{ij}(\cdot)$ are uniformly continuous functions on $[\delta, \infty)$ for every $\delta > 0$.

Furthermore he proved:

All transition probability functions $p_{ij}(\cdot)$ are continuous on the open interval $(0, \infty)$ if and only if $\lim_{t \downarrow 0} p_{ij}(t)$ exists for all i and j.

As we saw in [2] the first property can be proved from the intuitionistic point of view without supposing the measurability of the transition functions and from a counterexample it became clear that the second property cannot be proved (nowadays) from the intuitionistic point of view. Instead of this property we now prove:

Theorem: If the transition probability functions $p_{ij}(\cdot)$ are defined for all $t > 0$ and all natural numbers i and j then the existence of

$$\lim_{t \downarrow 0} p_{ij}(t)$$

is equivalent to uniform continuity of $p_{ij}(\cdot)$ on $(0, \infty)$.

Proof. In [2] we have proved: each $p_{ij}(\cdot)$ is an uniformly continuous function on $[\delta, \infty)$ for every $\delta > 0$, hence we can apply the theorem which we proved in section 2 and the proof is finished.

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