

## Transversal Theory

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**1. Introduction.** Transversal theory is a branch of combinatorial mathematics which is only just beginning to emerge as a reasonably connected and coherent subject. Whether this is yet rich enough or mature enough to be called a 'theory' may be a matter for debate; indeed, it is by no means certain that this part of mathematics may not finally be classified under some broader, more comprehensive title. However, what is beyond dispute is the fact that during the last two decades a large number of papers have been published which include some reference to the so-called *marriage theorem* (Theorem 2.1), which is the starting point for transversal theory. These papers deal with surprisingly diverse problems and their only connecting link seems to be this common reference to the marriage theorem. The arguments employed have generally had an ad hoc flavour although some of these have been highly original. Transversal theory is a depository for developing those mathematical ideas of the marriage theorem type which frequently recur and which seem to belong to some more general framework.

Two books on the subject have been published recently by Crapo and Rota [11] and Mirsky [44] although these were written from rather differing viewpoints. The first part of this article will be expository and cover ground which is familiar to most combinatorial mathematicians. In the second part I shall describe some more recent work done on infinite transversals. The earlier bibliography, detailed proofs and a historical commentary can be found in Mirsky's book. Apart from the new result in set theory mentioned in § 6, I shall not dwell upon the applications of transversal theory to other branches of mathematics, but refer the reader interested in this aspect to the article by Harper and Rota [31]. Instead I shall try to give emphasis to those results which are either new or which have influenced the development of the subject.

**2. Early results.** The letter  $F$  will always denote the system  $\langle F_i \mid i \in I \rangle$  of subsets of a set  $S$  having index set  $I$ . The sets  $F_j$  ( $j \in I$ ) are the *members* of the system but these are not necessarily different subsets of  $S$ . We write  $|F| = |I|$  to denote the cardinality of  $F$ . If  $F = \langle F_i \mid i \in I \rangle$  and  $G = \langle G_j \mid j \in J \rangle$  are two systems, then we define  $F = G$  and  $F + G$  as follows:  $F = G$  means that there is a bijection  $f: I \rightarrow J$  such that  $F_i = G_{f(i)}$  ( $\forall i \in I$ );  $F + G$  denotes the system  $H = \langle H_\varepsilon \mid \varepsilon \in K \rangle$ , where  $K = (I \times \{0\}) \cup (J \times \{1\})$  and  $H_{(i,0)} = F_i$  ( $\forall i \in I$ ),  $H_{(j,1)} = G_j$  ( $\forall j \in J$ ).

A *transversal function* of  $F$  is an injective choice function for  $F$ , that is a function  $\varphi: I \rightarrow S$  such that  $\varphi(i) \neq \varphi(j)$  ( $i \neq j$ ) and  $\varphi(i) \in F_i$  ( $i \in I$ ). The element  $\varphi(i)$  is the *representative* of  $F_i$  in  $\varphi$  and  $\langle \varphi(i) \mid i \in I \rangle$  is a *system of distinct representatives* for  $F$ . A *transversal* of  $F$  is the range  $T = \{\varphi(i) \mid i \in I\}$  of a transversal function and a *partial transversal* is a transversal of some subsystem  $F \upharpoonright K = \langle F_i \mid i \in K \rangle$  ( $K \subset I$ ). We denote by  $\text{TR}(F)$  the set of all transversals of  $F$  and by  $\text{PTR}(F)$  the set of all partial transversals.

A system  $F$  has the *transversal property*,  $F \in \mathcal{T}$ , if and only if  $F$  has a transversal. Many problems in combinatorial mathematics reduce to the question of whether or not a certain system  $F$  has the transversal, or some similar type of property. Here I mention just two such related properties which will be considered in § 5 in the discussion of infinite systems. A system  $F$  has property  $\mathcal{B}$ ,  $F \in \mathcal{B}$ , if and only if there is a set  $B$  such that  $B \cap F_i \neq \emptyset \neq F_i \setminus B$  ( $\forall i \in I$ ). This property was first considered by Miller [40] (the letter  $\mathcal{B}$  standing for Bernstein).  $F$  has property  $\mathcal{B}_1$  (the selector property) if there is a set  $B$  such that  $|F_i \cap B| = 1$  ( $\forall i \in I$ ). For other generalizations of these see [19]. The most primitive statement about transversals is the axiom of choice (which we assume): *If  $F$  is a system of nonempty pairwise disjoint sets, then  $F \in \mathcal{T}$ .*

An obvious *necessary* condition for  $F$  to have a transversal is that

$$(2.1) \quad |F(K)| \geq |K| \quad (\forall K \subset I)$$

where  $F(K) = \bigcup_{i \in K} F_i$ , and the marriage theorem states that this condition is also sufficient in the case of *finite* systems.

**THEOREM 2.1.** *If  $|F| < \aleph_0$  then  $F \in \mathcal{T}$  if and only if (2.1) holds.*

This was proved by P. Hall [27] and condition (2.1) is usually referred to as Hall's condition. König had earlier proved an equivalent result [36], [37], [38] which he expressed in the language of bipartite graphs. There is a natural representation for a set system  $F$  as a bipartite graph. We can assume without loss of generality that  $I \cap S = \emptyset$  and then  $F$  defines a bipartite graph  $G_F = (V, E)$  with vertex set  $V = I \cup S$  and edge set  $E = \{\{i, x\} \mid i \in I, x \in F_i\}$ . A *matching* in a graph  $G = (V, E)$  is a set of pairwise disjoint edges  $W \subset E$ ; and, for  $X \subset V$ , an *X-matching* is a matching  $W$  such that every vertex of  $X$  is incident with some edge of  $W$ . It is easy to see that the set system  $F$  has a transversal if and only if the corresponding bipartite graph  $G_F$  has an *I-matching*. König showed that if  $n < \aleph_0$  and  $G$  is any bipartite graph, then  $G$  has a matching of size  $n$  if and only if  $|C| \geq n$  whenever  $C$  is a *covering set* (i.e., a set of vertices incident with every edge of  $G$ ). Since  $(I \setminus K) \cup F(K)$  is a covering

set of  $G_F (K \subset I)$ , it follows from (2.1) that, if  $|I| < \aleph_0$ , then  $G_F$  has a matching of size  $|I|$  and hence an  $I$ -matching.

This formulation of the transversal property in terms of matchings in bipartite graphs is frequently useful and gives proper emphasis to the dual roles played by the index set  $I$  and the ground set  $S$ . The terminology also suggests why Theorem 2.1 is sometimes called the marriage theorem. If  $I$  is a set of boys and  $F_i$  is the set of  $i$ 's girl friends ( $i \in I$ ), then a transversal of  $F$  (or a matching of  $G_F$ ) corresponds to a marriage arrangement in which each boy marries one of his girl friends. While this might be considered satisfactory for the boys ( $I$ ), it is most unlikely that it would be considered so by the girls in  $S$  left without husbands. Perhaps, therefore, we should instead seek criteria for the existence of a more socially satisfying *perfect matching*, that is a matching which is simultaneously an  $I$ -matching and an  $S$ -matching in  $G_F$ . But it is easily seen that a necessary and sufficient condition for this is that there should exist *some*  $I$ -matching ( $W$ ) and *some*  $S$ -matching ( $W'$ ) (consider the graph with edge set  $W \cup W'$ ). Therefore this reduces immediately to the one-sided problem of deciding which system  $F \in \mathcal{T}$ .

For those with more ambitious appetities, there is another natural generalization of Theorem 2.1 in which the  $i$ th boy demands a harem of size  $h_i$  [30].

**THEOREM 2.2.** *If  $|F| < \aleph_0$  and  $h_i$  is a nonnegative integer ( $i \in I$ ), then there are disjoint sets  $X_i \subset F_i$  ( $i \in I$ ) such that  $|X_i| = h_i$  if and only if  $|F(K)| \geq \sum_{i \in K} h_i$  ( $\forall K \subset I$ ).*

This follows immediately from Theorem 2.1 by considering an augmented system having  $h_i$  copies of  $F_i$  ( $i \in I$ ). This is the simplest of a number of modifications that can be effected on a set system in order to exploit a self-strengthening characteristic of Theorem 2.1 (see [44, Chapter 3]).

A more important early extension of Theorem 2.1 was obtained by Marshall Hall [28] who showed that the condition (2.1) is also sufficient in the case when  $|F|$  is arbitrary but each  $F_i$  ( $i \in I$ ) is finite. The  $2^{|I|} - 1$  conditions of (2.1) are mutually independent for a finite system of sets, but for an infinite system of *finite* sets (2.1) is equivalent to the smaller set of conditions

$$(2.1') \quad |F(K)| \geq |K| \quad (\forall K \subseteq I),$$

where  $K \subseteq I$  means that  $K$  is a finite subset of  $I$ . In view of this, Marshall Hall's theorem can be stated in the following way.

**THEOREM 2.3.** *Let  $F$  be a system of finite sets. Then  $F \in \mathcal{T}$  if and only if  $F_0 \in \mathcal{T}$  ( $\forall F_0 \subseteq F$ ).*

There are almost as many published proofs of this result as there are for Theorem 2.1. Algebraists use a variant of Zorn's lemma, topologists recognize it as a corollary of Tychonoff's theorem on the product of compact spaces, logicians employ Gödel's compactness theorem for the first order predicate calculus (see §5), while combinatorialists use Rado's selection lemma ([23], [28], [30], [32], [44]).

We do not know of any criteria analogous to Hall's condition (2.1) for the prop-

erties  $\mathcal{B}$  or  $\mathcal{B}_1$ . Indeed we are extremely ignorant about these properties for finite system. For example, if  $m(n)$  is the smallest number of sets of size  $n$  which do *not* have property  $\mathcal{B}$ , then  $m(1) = 1$ ,  $m(2) = 3$ ,  $m(3) = 7$  and  $m(4)$  is unknown. However, standard compactness arguments yield results for properties  $\mathcal{B}$  and  $\mathcal{B}_1$  similar to Theorem 2.3.

**THEOREM 2.4.** *If  $F$  is a system of finite sets and  $\mathcal{C} \in \{\mathcal{B}, \mathcal{B}_1\}$ , then  $F \in \mathcal{C}$  if and only if  $F_0 \in \mathcal{C} (\forall F_0 \subseteq F)$ .*

**3. Abstract independence.** Whitney was the first to study the abstract properties of linear independence and in his pioneering paper [68] he established the equivalence to different sets of axioms for this notion. The ones which most clearly reveal the underlying motivation of vectors in a vector space are the following. A *pre-independence structure* (Whitney used the term *matroid*) on a set  $S$  is a nonempty set  $\mathcal{M} \subset \mathcal{P}(S) = \{X \mid X \subset S\}$  satisfying the conditions:

- I1.  $A \subset B \in \mathcal{M} \Rightarrow A \in \mathcal{M}$  (hereditary).
- I2.  $A, B \in \mathcal{M}, |B| = |A| + 1 < \aleph_0 \Rightarrow (\exists b \in B \setminus A)(A \cup \{b\} \in \mathcal{M})$  (exchange).

A set  $X \subset S$  is *independent* or *dependent* according as  $X \in \mathcal{M}$  or  $X \in \mathcal{P}(S) \setminus \mathcal{M}$ . Since Whitney's paper, quite a lot of work has been done on the notion of abstract independence and other axiom schemes have been given; in particular, the theory was greatly extended by Tutte ([61], [62]) who exploited various analogies and applications to graph theory.

Whitney only considered the case of finite  $\mathcal{M}$ , but many basic results can be extended to infinite structures if one assumes some additional finiteness type of condition. The most common of these is

- I3.  $\mathcal{M}$  has finite character.

If  $\mathcal{M}$  satisfies I1—I3 we call it an *independence structure* on  $S$ ; it is determined by its finite members. One of the first deductions to be made from I1,2 is that if  $\mathcal{M}$  is a finite pre-independence structure, then the maximal independent sets (*bases*) all have the same (finite) cardinality. If  $\mathcal{M}$  is infinite there need not be any maximal independent sets, and even when there are they need not have the same cardinality [13]. However, if I3 is assumed then it is easy to see that any independent set is contained in a basis and moreover the bases all have the same cardinality [57].

It follows from the above that if  $\mathcal{M}$  is a pre-independence structure on  $S$ , then there is an associated rank function

$$(3.1) \quad \rho: \mathcal{P}(S) \rightarrow \{0, 1, 2, \dots, \infty\}$$

which is defined by

$$\rho(A) = \sup\{|X| \mid X \in \mathcal{M} \cap \mathcal{P}(A)\} \quad (A \subset S).$$

The basic property of  $\rho$ , which follows easily from the definition, is that it satisfies

$$(3.2) \quad \rho(A) \leq \rho(B) \quad (A \subset B \subset S),$$

$$(3.3) \quad \rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B) \quad (A, B \subset S).$$

There is an intimate connection between increasing submodular functions and

matroids ([55], [18]): *If  $\rho$  satisfies (3.1)—(3.3), then  $\mathcal{M}_\rho = \{X \subset S \mid \rho(X) \cong |X|\}$  satisfies  $I1, I2$  (although the rank function of  $\mathcal{M}_\rho$  is not necessarily  $\rho$ ).*

It is natural to ask under what conditions a set system  $F$  should have a transversal which is independent in some independence structure on  $S$ . Rado ([56], [57]) was the first to consider this problem and he obtained the following extension of Theorems 2.1 and 2.3.

**THEOREM 3.1.** *Let  $F$  be a system of finite subsets of  $S$  and let  $\mathcal{M}$  be an independence structure on  $S$  with rank function  $\rho$ . Then  $\mathcal{M} \cap \text{TR}(F) \neq \emptyset$  if and only if  $\rho(F(K)) \geq |K|$  ( $\forall K \in I$ ).*

This theorem admits the same kind of extensions as Theorem 2.1 and has numerous applications (e.g., [2], [3], [65]).

We deduce immediately from Theorem 3.1 the following analogue of Marshall Hall's theorem (Theorem 2.3): *If  $F$  is a system of finite subsets of  $S$  and  $\mathcal{M}$  is an independence structure on  $E$ , then the statements*

$$(3.4) \quad \mathcal{M} \cap \text{TR}(F) \neq \emptyset$$

and

$$(3.5) \quad \mathcal{M} \cap \text{TR}(F_0) \neq \emptyset \quad (\forall F_0 \in F)$$

are equivalent. Rado [57] proved that this equivalence is actually a characterization of independence structures.

**THEOREM 3.2.** *The nonempty set  $\mathcal{M} \subset \mathcal{P}(S)$  is an independence structure on  $E$  if and only if the statements (3.4) and (3.5) are equivalent for every system  $F$  of finite subsets of  $S$ .*

As we have already hinted, (pre-) independence structures abound in combinatorial mathematics apart from the more obvious algebraic ones, but for transversal theory the most important example is the following observation of Edmonds and Fulkerson [17].

**THEOREM 3.3.** *The set of partial transversals of  $F$ ,  $\text{PTR}(F)$ , is a pre-independence structure.*

This result is not difficult to prove, but it was important for the development of the subject since it initiated a new approach for subsequent research. In general,  $\text{PTR}(F)$  does not satisfy  $I3$ , but it does if  $F$  satisfies the local finiteness condition  $|F^{-1}(x)| < \aleph_0$  ( $\forall x \in S$ ), where  $F^{-1}(x) = \{i \in I \mid x \in F_i\}$  [46]. Theorem 3.3 (and the fact that the bases of a finite matroid have equicardinality) immediately gives the following result for finite set systems ([33], [39]).

**THEOREM 3.4.** *If  $F \in \mathcal{T}$  and  $P \in \text{PTR}(F)$ , then there is  $T \in \text{TR}(F)$  such that  $P \subset T$ .*

For infinite systems this simple argument fails and the proof [51] depends upon an extension of the Banach mapping theorem due to Ore. There is an important practical consequence of Theorem 3.4. To check (2.1) for a large finite system would be both expensive and uninformative, but Theorem 3.4 shows that there is an

efficient procedure for actually finding a maximal partial transversal of  $F$  which does not involve backtracking [29].

We call a (pre-) independence structure  $\mathcal{M}$  transversal if  $\mathcal{M} = \text{PTR}(F)$  for some  $F$ . Not all (pre-) independence structures are transversal, but the problem of deciding whether one is or not is not always easy (see [6]). However, transversal structures do arise in natural ways. For example, if  $G = (V, E)$  is a graph, then the matching matroid of  $G$ ,  $\mathcal{M}_G = \{X \subset V \mid \exists \text{ an } X\text{-matching in } G\}$ , is transversal [17]. While it is easily seen that  $\mathcal{M}_G$  is a pre-independence structure, it is by no means obvious that it is transversal.

The sum

$$\mathcal{M} = \sum_{i \in I} \mathcal{M}_i = \left\{ \bigcup_{i \in I} X_i \mid X_i \in \mathcal{M}_i \right\}$$

of a system  $\langle \mathcal{M}_i \mid i \in I \rangle$  of pre-independence structures on  $S$  is also a pre-independence structure (and if  $|I| < \aleph_0$  and each  $\mathcal{M}_i$  satisfies I3 then so does  $\mathcal{M}$ ). The rank function for  $\mathcal{M}$  is given by

$$(3.6) \quad \rho(A) = \min_{X \subset A} \left( \sum_{i \in I} \rho_i(X) + |A \setminus X| \right) \quad (A \subset S),$$

where  $\rho_i$  is the rank function of  $\mathcal{M}_i$ . This important formula was first stated, for finite  $I$ , by Nash-Williams [48] (it is also implicit in Edmonds [15]); the infinite case is proved in [3], [55]. While this result is not difficult to establish (e.g., see [66] for an elegant deduction of (3.6) from Theorem 3.1), it provides a useful general technique for solving a variety of problems (e.g., [45]).

As an illustration of the use of (3.6) we give an example due to Nash-Williams [48]. Consider the cycle matroid  $\mathcal{M}_G = \{X \subset E \mid X \text{ is acyclic}\}$  on the edge set of a graph  $G = (V, E)$ . If  $G$  is finite, then the rank of a set  $X \subset E$  is  $|V| - t(X)$ , where  $t(X)$  is the number of connected components of the graph  $(V, X)$ . The graph  $G$  contains  $k$  edge-disjoint spanning trees provided that  $E$  has rank  $k(|V| - 1)$  in the matroid sum  $\sum_{i=1}^k \mathcal{M}_i$ , where  $\mathcal{M}_i = \mathcal{M}_G$  ( $1 \leq i \leq k$ ). Thus, by (3.6), we see that a necessary and sufficient condition for this is that  $k(|V| - 1) \leq k(|V| - t(X)) + |E \setminus X|$  ( $\forall X \subset E$ ). Expressed differently, this condition states that

$$(3.7) \quad e(P) \geq k(|P| - 1),$$

where  $P = \{V_1, \dots, V_i\}$  is any partition of  $V$  into disjoint, nonempty sets and  $e(P)$  is the number of edges of  $G$  joining distinct  $V_j$ 's. This result had earlier been proved by Tutte [63] and Nash-Williams [47] by more direct but very involved methods and this use of the rank formula is a good example of the elegance and insight which is sometimes gained through generalization. The argument just used fails for infinite graphs, although Nash-Williams [50] has conjectured that (3.7) is sufficient for the general case. A more general problem would be to find necessary and sufficient conditions for the existence of pairwise disjoint bases  $B_i$  of  $\mathcal{M}_i$  ( $i \in I$ ), when the  $\mathcal{M}_i$  are matroids on an infinite set  $S$ .

In this context it should be mentioned that Edmonds (see [15]) has suggested a more general setting for transversal theory by defining a 'transversal' for a system

of independence structures  $\langle \mathcal{M}_i \mid i \in I \rangle$  to be a set  $T$  which is the disjoint union of bases  $B_i$  of  $\mathcal{M}_i$  ( $i \in I$ ). The original situation is regained when  $\mathcal{M}_i$  is taken to be the discrete matroid  $\{X \subset F_i \mid |X| \leq 1\}$  on  $F_i$  ( $i \in I$ ). Many of the basic results of transversal theory extend to this more general setting provided the  $\mathcal{M}_i$  are rank finite. For example, a generalization of Theorem 3.4 is that any partial transversal of a rank-finite system  $\langle \mathcal{M}_i \mid i \in I \rangle$  can be extended to a complete transversal provided one exists. For a fuller discussion of this see Brualdi [5].

So far we have only considered the existence of transversals of a single set system, but it is useful to consider the analogous problem when there are two or more systems. For example, Theorems 3.1 and 3.3 together immediately give the following extension of the marriage theorem (first proved in [25] in the context of flows in networks).

**THEOREM 3.5.** *The finite systems  $F = \langle F_1, \dots, F_n \rangle$ ,  $G = \langle G_1, \dots, G_n \rangle$  have a common transversal if and only if*

$$|F(K) \cap G(L)| \geq |K| + |L| - n \quad (K, L \subset \{1, 2, \dots, n\}).$$

A transfinite analogue for Theorem 3.5 of the Schroeder-Bernstein type is the following theorem proved by Pym [54] and Brualdi [4].

**THEOREM 3.6.** *The systems  $F = \langle F_i \mid i \in I \rangle$  and  $G = \langle G_i \mid i \in I \rangle$  have a common transversal if  $F$  has a common transversal with some subsystem of  $G$  and  $G$  has a common transversal with some subsystem of  $F$ .*

It would be useful to have a more quantitative type of condition for the existence of a common transversal of two infinite systems. More generally, when do two infinite matroids have a common basis? This is not known even for independence structures (for a partial solution see [5]).

Unfortunately, there is no result like Theorem 3.5 known which guarantees the existence of a common transversal for three or more systems. A more general problem is to find conditions for three pre-independence structures to have a common independent set of a given size. A solution to these problems would have several important consequences. For example, it would enable us to characterize those directed graphs having a Hamiltonian path [67].

**4. Systems with infinite members.** The problem of extending Theorem 2.1 to arbitrary systems remains as the central problem of transversal theory and is a prototype for similar questions in combinatorial set theory.

It is easily seen that Hall's condition (2.1) is not sufficient for  $F \in \mathcal{T}$  even for systems having a single infinite member, e.g., consider  $F = \langle \omega, \{0\}, \{1\}, \dots \rangle$ . Actually Rado and Jung [58] gave an extension of Theorem 2.1 to cover this case. Call a subsystem  $F \upharpoonright K$  of  $F$  *critical* if  $\text{TR}(F \upharpoonright K) = \{F(K)\}$ ; for finite  $K$  this is equivalent to  $F \upharpoonright K \in \mathcal{T}$  and  $|F(K)| = |K|$ . Suppose  $F$  is a system of finite sets and  $A$  is an infinite set. Then the result of [58] is that  $F + \langle A \rangle \in \mathcal{T}$  if and only if  $F \in \mathcal{T}$  and

$$(4.1) \quad A \not\subset \bigcup_{F \upharpoonright K \text{ critical}} F(K).$$

Extensions of this have been obtained by several authors ([17], [24], [69], [12], [60]) providing necessary and sufficient conditions for  $F \in \mathcal{F}$  in the case when  $G$  has arbitrarily many finite sets and a finite number of infinite sets.

Recently, Damerell and I [12] settled a conjecture of Nash-Williams [49] giving necessary and sufficient conditions for any denumerable system of sets to have a transversal. For  $X \subset S$ , let  $I(X) = \{i \in I \mid F_i \subset X\}$  and put

$$(4.2) \quad \begin{aligned} m_0(X) &= |X| - |I(X)| && \text{if } |X| < \infty, \\ &= \infty && \text{if } |X| = \infty. \end{aligned}$$

An obvious necessary condition (essentially (2.1)) for  $F \in \mathcal{F}$  is that  $m_0(X) \geq 0$  ( $\forall X \subset S$ ). In fact, for a finite set  $X \subset S$ ,  $m_0(X)$  measures the number of ‘spare’ elements in  $X$  which would be left over after choosing representatives for the sets  $F_i \subset X$ . For infinite  $X$ ,  $m_0(X)$  is simply a first approximation to this number of ‘spare’ elements in the sense that in this case there could possibly be infinitely many elements left over after choosing representatives for the sets  $F_i \subset X$ . Nash-Williams’ idea was to find successively better and better estimates for the number of ‘spare’ elements in the following way. If  $T = \langle T_n \mid n < \omega \rangle$  is an increasing sequence of subsets of  $X$  such that

$$(4.3) \quad T_0 \subset T_1 \subset \dots \subset X = \bigcup_{n < \omega} T_n,$$

then put  $D(T) = I(X) \setminus \bigcup_{n < \omega} I(T_n)$ . A function  $f : \mathcal{P}(S) \rightarrow \{0, \pm 1, \pm 2, \dots, \pm \infty\}$  will be called a *valuation* on  $S$ . If  $f$  is a valuation on  $S$ , denote by  $A(f, X)$  the set of all sequences  $T = \langle T_n \mid n < \omega \rangle$  satisfying (4.3) and such that  $f(T_n) = f(T_0) < \infty$  ( $n < \omega$ ). For  $T \in A(f, X)$  write  $\bar{f}(T) = f(T_0)$ . Now we define a transfinite sequence of valuations  $m_\alpha$  ( $\alpha \geq 0$ ) by induction on  $\alpha$  as follows. Suppose  $\alpha > 0$  and that  $m_\beta$  has been defined for  $\beta < \alpha$ . For  $X \subset S$  we put  $m_\alpha(X) = \inf_{\beta < \alpha} m_\beta(X)$  if  $\alpha$  is a limit ordinal, and for  $\alpha = \beta + 1$  put

$$\begin{aligned} m_\alpha(X) &= \inf_{T \in A(m_\beta, X)} (\bar{m}_\beta(T) - |D(T)|) && \text{if } A(m_\beta, X) \neq \emptyset, \\ &= \infty && \text{if } A(m_\beta, X) = \emptyset. \end{aligned}$$

Then we have the following result [12].

**THEOREM 4.1.** *If  $|F| = \aleph_0$ , then  $F \in \mathcal{F}$  if and only if*

$$(4.4) \quad m_\omega(X) \geq 0 \quad (\forall X \subset S).$$

Steffens [60] considered the following more qualitative type of condition which is somewhat similar to (4.1) and very easy to state:

$$(4.5) \quad F_i \not\subset F(K) \quad \text{whenever } i \in I \setminus K \text{ and } F \upharpoonright K \text{ is critical.}$$

Clearly (4.5) is necessary for  $F \in \mathcal{F}$  and Podewsky and Steffens have recently proved the following theorem [52].

**THEOREM 4.2.** *If  $|F| = \aleph_0$ , then  $F \in \mathcal{F}$  if and only if (4.5) holds.*

Theorem 4.1 and 4.2 both fail for nondenumerable systems. A good test case is the system  $F' = \langle \alpha \mid \omega \leq \alpha < \omega_1 \rangle$  which has no transversal by an elementary theo-



rem on regressive functions. However, both (4.4) and (4.5) are satisfied for this system.

On the other hand, both Theorems 4.1 and 4.2 can be extended to give necessary and sufficient conditions for the existence of transversals of denumerable systems in some independence structure  $\mathcal{M}$  on  $S$ . For Theorem 4.1 the only change needed is to replace  $|X|$  by  $\rho(X)$  in (4.2), where  $\rho$  is the rank function. The proof of [12] carries over with only minor modifications. In order to state the appropriate generalization of Theorem 4.2, call a subsystem  $F \upharpoonright K$   $\mathcal{M}$ -critical if  $\mathcal{M} \cap \text{TR}(F \upharpoonright K) \neq \emptyset$  and if  $B$  is a maximal independent subset of  $F(K)$  whenever  $B \in \mathcal{M} \cap \text{TR}(F \upharpoonright K)$ . Then it is easily shown [42] that, if  $|F| \leq \aleph_0$ , then  $\mathcal{M} \cap \text{TR}(F) \neq \emptyset$  if and only if

(4.5')  $F_i$  does not depend upon  $F(K)$  whenever  $i \in I \setminus K$  and  $F \upharpoonright K$  is  $\mathcal{M}$ -critical.

**5. Compactness theorems.** Let  $\kappa, \lambda, \mu$  denote infinite cardinals. The cofinality cardinal of  $\kappa$  is  $\text{cf } \kappa$  and the successor of  $\kappa$  is  $\kappa^+$ . We write  $F \in S(\kappa, \lambda)$  if  $|F| = \kappa$  and  $|F_i| = \lambda$  ( $\forall i \in I$ ). Expressions like  $S(\kappa, \leq \lambda)$ ,  $S(\kappa, < \lambda)$  have natural interpretations. We say  $F$  has property  $\mathcal{T}(\mu)$  if  $F' \in \mathcal{T}$  ( $\forall F' \subset F, |F'| \leq \mu$ ). Let  $T(\kappa, \lambda, \mu)$  be an abbreviation for the assertion:

$$F \in S(\kappa, \lambda) \ \& \ F \in \mathcal{T}(\mu) \Rightarrow F \in \mathcal{T}.$$

Then Marshall Hall's theorem (Theorem 2.3) asserts that  $T(\kappa, < \aleph_0, < \aleph_0)$  is true for every  $\kappa$ . It is natural to investigate if  $T(\kappa, \lambda, \mu)$  holds for other triples and W. Gustin (see [19], [20]) in the 1950's asked if

$$(5.1) \quad \neg T(\aleph_2, \aleph_0, \aleph_1)$$

is true. Erdős and Hajnal [21] noted that (5.1) holds in  $L$ . More generally, an easy consequence of a result of Jensen [34] is the following theorem [43].

**THEOREM 5.1.** *If  $\kappa$  is regular and not weakly compact and if  $\lambda < \kappa$ , then  $V = L \Rightarrow \neg T(\kappa, \lambda, < \kappa)$ ,*

The hypothesis  $V = L$  is not needed to prove (5.1). For example, the system  $F = \langle F_{\alpha\beta} \mid \omega \leq \alpha < \omega_1 \leq \beta < \omega_2 \rangle$ , where  $F_{\alpha\beta} = \alpha \times \{\alpha, \beta\} = \bigcup_{\nu < \alpha} \{(\nu, \alpha), (\nu, \beta)\}$ , satisfies  $F \in S(\aleph_2, \aleph_0) \cap T(\aleph_1)$  and  $F \notin \mathcal{T}$ . More generally, Shelah and I proved the following theorem [43].

**THEOREM 5.2.** *If  $\kappa$  is regular, then*

$$\neg T(\kappa, \lambda, < \kappa) \Rightarrow \neg T(\kappa^+, \lambda, < \kappa^+).$$

Since  $\neg T(\kappa^+, \kappa, \kappa)$  holds (consider  $\kappa^+$  identical sets of size  $\kappa$ ) we deduce from this that  $\neg T(\aleph_{\alpha+n}, \aleph_\alpha, < \aleph_{\alpha+n})$  ( $\alpha \geq 0, 1 \leq n < \omega$ ). However, this leaves several questions unanswered. For example, we cannot deduce from Theorem 5.2 whether  $\neg T(\kappa, \aleph_0, < \kappa)$  holds for  $\kappa \geq \aleph_\omega$ . Theorem 5.1 shows that we cannot prove the falsity of this for  $\kappa = \mu^+$ , but (rather surprisingly) it is false for singular  $\kappa$ . Very recently Shelah (unpublished) has proved the following result.

**THEOREM 5.3.** *If  $\text{cf } \kappa < \kappa$  and  $\lambda < \kappa$ , then  $T(\kappa, \lambda, < \kappa)$ .*

It is easily seen that this theorem of Shelah is best possible in the sense that  $\lambda$  cannot be replaced by  $< \kappa$ . More precisely, we have that  $\text{cf } \kappa < \kappa \Rightarrow \neg T(\kappa, < \kappa, < \kappa)$ .

To see this consider the system

$$F = \langle \{\alpha\} \mid \alpha \in \kappa \setminus C \rangle + \langle [\kappa_\rho, \kappa_{\rho+1}) \mid \rho < \mu \rangle + \langle C \rangle,$$

where  $\mu = \text{cf } \kappa < \kappa$ ,  $C = \{\kappa_\rho \mid \rho < \mu\}$  is a closed, cofinal subset of  $\kappa$  and  $[\kappa_\rho, \kappa_{\rho+1}) = \{\alpha \mid \kappa_\rho \leq \alpha < \kappa_{\rho+1}\}$ . It is easily seen that  $F \notin \mathcal{T}$  whereas  $F' \in \mathcal{T}$  for every proper subsystem  $F' \subsetneq F$ .

Theorem 5.3 shows that the regularity of  $\kappa$  is an essential hypothesis in Theorem 5.1. So also is the condition that  $\kappa$  not be weakly compact. We have the following very simple theorem.

**THEOREM 5.4.** *If  $\kappa$  is weakly compact then  $T(\kappa, < \kappa, < \kappa)$ .*

This can be proved in the same manner that Henkin [32] proved Marshall Hall's theorem. One of the several equivalent characterizations for  $\kappa$  to be weakly compact is that the infinitary propositional calculus which permits the conjunction of  $< \kappa$  formulae is  $\kappa$ -compact. Suppose  $F \in \mathcal{S}(\kappa, < \kappa) \cap \mathcal{T}(< \kappa)$ . We can assume that  $F = \langle F_i \mid i < \kappa \rangle$  and that  $F_i \subset \kappa$ . Consider the set of  $\kappa$  sentences

$$S = \left\{ \bigvee_{x \in F_i} p_{xi} \mid i < \kappa \right\} \cup \left\{ \neg(p_{xi} \wedge p_{xj}) \mid x < \kappa, i \neq j < \kappa \right\},$$

where  $p_{xi}$  is a propositional variable (with intended meaning " $x \in F_i$ "). The hypothesis ensures that any subcollection of  $< \kappa$  sentences of  $S$  has a model, and hence  $S$  has a model if  $\kappa$  is weakly compact, i.e.,  $F \in \mathcal{T}$ . In a similar way, as Jech remarked, one can prove a more exact analogue of Theorem 2.3 for large cardinals: *If  $\lambda$  is supercompact and  $\kappa \geq \lambda$ , then  $T(\kappa, < \lambda, < \lambda)$ .* It should be possible to prove

$$(5.2) \quad T(\kappa, < \kappa, < \kappa) \Rightarrow \kappa \text{ weakly compact,}$$

but at present this is still open.

Theorems 5.1 and 5.2 show that one cannot, in general, decide if  $F \in \mathcal{T}$  by examining all small subsystems of  $F$ . However, we do have the following compactness type of result [43].

**THEOREM 5.5.** *If  $F_0 \in \mathcal{S}(\kappa, < \aleph_0)$ ,  $F_1 \in \mathcal{S}(\lambda, \leq \lambda)$  and  $F = F_0 + F_1$ , then*

$$(5.3) \quad F \in \mathcal{T} \Leftrightarrow F \in \mathcal{T}(\lambda).$$

For example, this enables us to extend Theorems 4.1 and 4.3 to the case where  $F$  contains countably many denumerable sets and an arbitrary number of finite sets. Čudnovskii [9] has obtained the following more general theorem: *If  $\mu, \lambda \geq \nu \geq \omega$  and the infinitary language  $L_{\nu^+, \omega}$  is  $(\mu, \lambda)$ -compact, then (5.3) holds if  $F_0 \in \mathcal{S}(\kappa, < \aleph_0)$  and  $F_1 \in \mathcal{S}(\mu, \leq \nu)$ .*

I conclude this section by mentioning some related results about the properties  $\mathcal{B}$  and  $\mathcal{B}_1$  introduced in §2. First I state one of Miller's original results since there remains an interesting unsolved problem. Miller [40] proved that if  $F \in \mathcal{S}(\kappa, \geq \lambda)$  and  $n < \aleph_0$ , then

$$(\forall F' \subset F) (|F'| > \lambda \Rightarrow |\bigcap F'| < n) \Rightarrow F \in \mathcal{B}.$$

This result is easily seen to be best possible in the sense that  $n$  cannot be replaced by  $\aleph_0$ . For example, let  $A = \langle A_i \mid i < \omega \rangle$  be a system of  $\aleph_0$  disjoint denumerable sets, let  $\{T_\rho \mid \rho < 2^{\aleph_0}\}$  be all the transversals of  $A$  and let  $\{C_\rho \mid \rho < 2^{\aleph_0}\}$  be any set of  $2^{\aleph_0}$  almost disjoint (i.e.,  $|C_\rho \cap C_\sigma| < |C_\rho|$  for  $\rho \neq \sigma$ ) infinite subsets of  $\omega$ . Then the system  $F = A + \langle T_\rho \upharpoonright C_\rho \mid \rho < 2^{\aleph_0} \rangle \in \mathcal{S}(2^{\aleph_0}, \aleph_0)$  and

$$(5.4) \quad |F_i \cap F_j| < \aleph_0 \quad (i \neq j),$$

but  $F \notin \mathcal{B}$ . One of the problems stated in [19] which still remains unsolved is whether (under the assumption that  $2^{\aleph_0} > \aleph_1$ ) there is  $F \in \mathcal{S}(\aleph_1, \aleph_0)$  such that (5.4) holds and  $F \notin \mathcal{B}$ .

We say  $F$  has property  $\mathcal{B}(\mu)$  if  $F' \in \mathcal{B} (\forall F' \subset F, |F'| \leq \mu)$ . Let  $\mathcal{B}(\kappa, \lambda, \mu)$  denote the assertion:  $F \in \mathcal{S}(\kappa, \lambda) \ \& \ F \in \mathcal{B}(\mu) \Rightarrow F \in \mathcal{B}$ . Similarly, we define  $\mathcal{B}_1(\mu)$  and  $\mathcal{B}_1(\kappa, \lambda, \mu)$ . Essentially the same proof used to establish Theorem 5.4 above also gives that  $B(\kappa, < \kappa, < \kappa)$  and  $B_1(\kappa, < \kappa, < \kappa)$  are true if  $\kappa$  is weakly compact. Similar to Gustin's problem (5.1), Erdős and Hajnal [19] asked if the statements

$$(5.5) \quad \neg B(\aleph_2, \aleph_0, \aleph_1),$$

$$(5.6) \quad \neg B_1(\aleph_2, \aleph_0, \aleph_1)$$

are true. Węglorz [64] proved (5.6) assuming  $2^{\aleph_1} = \aleph_2$  (i.e., (5.6) is consistent) and recently Čudnovskii [8] proved this without any additional assumption. The same authors also proved (Węglorz assumed GCH, Čudnovskii without GCH) the following theorem.

**THEOREM 5.6.**  $B_1(\kappa, < \kappa, < \kappa) \Leftrightarrow \kappa$  is weakly compact.

The corresponding problems for property  $\mathcal{B}$ , like (5.2), remain open. In this connection, I should like to mention one additional new result due to Komjath and Hoffman [35] which gives a connection between the transversal property and property  $\mathcal{B}$ .

**THEOREM 5.7.** If  $F$  is a system of infinite sets, then  $F \in \mathcal{T} \Rightarrow F \in \mathcal{B}$ .

**6. Almost disjoint transversals.** In this final section I shall discuss some recent results in set theory which relate to questions of the form: "how many almost disjoint transversals does a set system have?" Such questions were first considered in [22] and [41], and recently K. Prikry and J. Baumgartner used results of this kind to give elementary proofs of a remarkable new result (Theorem 6.1) of J. Silver [59].

Let  $\kappa$  be a singular cardinal not cofinal with  $\omega$ , i.e.,  $\omega < \lambda = \text{cf } \kappa < \kappa$ , and let  $C = \{\kappa_\rho \mid \rho < \lambda\}$  be any closed cofinal set of cardinals in  $\kappa$ . In [22] Erdős, Hajnal and I proved the following result: If  $\mu^\lambda < \kappa$  ( $\mu < \kappa$ ) and  $S$  is a stationary subset of  $\lambda$ , and if  $T$  is a set of almost disjoint transversals of the system  $F = \langle \kappa_\rho \mid \rho \in S \rangle$ , then  $|T| \leq \kappa$ . The elementary proofs given by Prikry and Baumgartner of Silver's theorem can be described in terms of the following extension of this result.

**LEMMA 6.1.** If  $\mu^\lambda < \kappa$  ( $\mu < \kappa$ ) and  $S$  is stationary in  $\lambda$ , and if  $T$  is a set of almost disjoint transversals of the system  $F = \langle \kappa_\rho^+ \mid \rho \in S \rangle$  then  $|T| \leq \kappa^+$ .

After Cohn [10] proved the independence of the continuum hypothesis, it was

natural to investigate what possible values  $2^{\aleph_\alpha}$  could assume. Easton [14] proved that if  $h$  is any ordinal valued function satisfying (i)  $\alpha \leq \beta \Rightarrow h(\alpha) \leq h(\beta)$  and (ii)  $\text{cf}(\aleph_{h(\alpha)}) > \aleph_\alpha$ , then it is consistent (with ZFC) that  $2^{\aleph_\alpha} = \aleph_{h(\alpha)}$  ( $\aleph_\alpha$  regular). In view of this arbitrariness for the possible values of  $2^\mu$  for regular  $\mu$ , it was therefore very surprising when Silver [59] recently announced the following theorem.

**THEOREM 6.2.** *If  $\omega < \text{cf } \kappa < \kappa$  and  $A = \{\alpha < \kappa \mid \alpha \text{ cardinal and } 2^\alpha = \alpha^+\}$  is stationary in  $\kappa$ , then  $2^\kappa = \kappa^+$ .*

In particular, this shows that if GCH holds below  $\aleph_\omega$  (i.e.,  $2^{\aleph_\alpha} = \aleph_{\alpha+1}$  ( $\alpha < \omega_1$ )), then  $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$ .

Silver's original proof uses sophisticated model theory but Prikry and Baumgartner gave an elementary combinatorial proof based upon Lemma 6.1. To obtain Silver's theorem from the lemma we argue as follows. If  $A$  is stationary in  $\kappa$ , then  $\mu^\lambda < \kappa$  ( $\mu < \kappa$ ) and  $A \cap C$  is stationary, i.e.,  $S = \{\rho < \lambda \mid 2^{\kappa_\rho} = \kappa_\rho^+\}$  is stationary in  $\lambda$ . Since  $|\mathcal{P}(\kappa_\rho)| = \kappa_\rho^+$  for  $\rho \in S$ , we can write  $\mathcal{P}(\kappa_\rho) = \{x_\rho^\nu \mid \nu < \kappa_\rho^+\}$  ( $\rho \in S$ ). Then, for each  $X < \kappa$ , there is a transversal function  $\varphi_X$  of  $F = \langle \kappa_\rho^+ \mid \rho \in S \rangle$  defined by  $\varphi_X(\rho) = \nu \Leftrightarrow X \cap \kappa_\rho = x_\rho^\nu$ . Clearly  $\varphi_X, \varphi_Y$  have almost disjoint ranges if  $X \neq Y < \kappa$  and therefore, by the lemma,  $|\mathcal{P}(\kappa)| \leq \kappa^+$ .

**PROOF OF LEMMA 6.1.** We will assume that  $T$  is a set of  $\kappa^{++}$  almost disjoint transversal functions and deduce a contradiction.

Note that  $S_0 = \{\rho \in S \mid \rho \text{ a limit ordinal}\}$  is also stationary in  $\lambda$ . For  $\psi, \varphi \in T$  put  $S(\psi, \varphi) = \{\rho \in S_0 \mid \psi(\rho) < \varphi(\rho)\}$ , and let  $G(\varphi) = \{\psi \in T \mid S(\psi, \varphi) \text{ is stationary in } \lambda\}$ .  $G$  is a set mapping on  $T$  (i.e.,  $\varphi \notin G(\varphi)$ ) and

$$(\forall \psi, \varphi \in T) (\psi \neq \varphi \Rightarrow \psi \in G(\varphi) \text{ or } \varphi \in G(\psi))$$

since  $S(\psi, \varphi) \cup S(\varphi, \psi)$  is a final section of  $S_0$ . Therefore, by a well-known theorem on set mappings (e.g., [26]), it follows that  $|G(\varphi_0)| \geq \kappa^+$  for some  $\varphi_0 \in T$ .

Since  $\varphi_0(\rho) < \kappa_\rho^+$  ( $\forall \rho \in S_0$ ), there is an injective map  $h_\rho: \varphi_0(\rho) \rightarrow \kappa_\rho$ . Also, if  $\psi \in G(\varphi_0)$  and  $\rho \in S(\psi, \varphi_0)$ , then there is  $\sigma_\psi(\rho) < \rho$  such that  $h(\psi(\rho)) < \kappa_{\sigma_\psi(\rho)}$  (since  $\{\kappa_\rho \mid \rho < \lambda\}$  is closed and  $\rho$  is a limit ordinal). Now  $\sigma_\psi$  is regressive on the stationary set  $S(\psi, \varphi_0)$  and hence there are  $A_\psi \subset S$  and  $\rho_\psi < \lambda$  such that  $|A_\psi| = \lambda$  and  $\sigma_\psi(\rho) < \rho_\psi$  ( $\forall \rho \in A_\psi$ ). There are only  $2^\lambda \cdot \lambda < \kappa^+$  different pairs  $(A, \zeta)$  with  $A \subset \lambda, \zeta < \lambda$ , and hence there is  $G' \subset G(\varphi_0)$  such that  $|G'| = \kappa^+$  and  $(A_\psi, \rho_\psi) = (A, \zeta)$  ( $\forall \psi \in G'$ ). Since  $\kappa_\zeta^\lambda \leq \kappa$  it follows that there are  $\psi_1, \psi_2 \in G'$  such that  $\psi_1(\rho) = \psi_2(\rho)$  ( $\forall \rho \in A$ ) and this is a contradiction since  $|A| = \lambda$  and the members of  $T$  are pairwise disjoint.

It should be mentioned that Prikry has since obtained more general results than Lemma 6.1 and Theorem 6.2 by using refinements of the above argument. He also proved the following interesting companion result.

**THEOREM 6.3.** *Suppose that  $T$  is a set of almost disjoint transversals of the system  $F = \langle F_\rho \mid \rho < \kappa \rangle$ , where  $|F_\rho| < 2^{\aleph_\rho}$  ( $\rho < \kappa$ ) and  $\omega < \text{cf } \kappa \leq \kappa < 2^{\aleph_\omega}$ . If  $2^{\aleph_\omega}$  is real-valued measurable, then  $|T| < 2^{\aleph_\omega}$ .*

Obviously, the condition that  $\text{cf } \kappa > \omega$  is essential.

Finally, I conclude by mentioning a strong result of the Silver type which was obtained (independently) by Hajnal and Galvin by using an extension of these ideas on almost disjoint transversals.

**THEOREM 6.4.** *If  $\omega < \text{cf } \kappa < \kappa = \aleph_\alpha$  and  $2^\mu < \kappa$  ( $\forall \mu < \kappa$ ), then  $2^\kappa < \aleph_{(\omega^{1+\aleph_1})}$ .*

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