MULTIPLIER IDEALS, MILNOR FIBERS, AND OTHER SINGULARITY INVARIANTS

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Contents

Trivia: in how many different ways can the log canonical threshold of a polynomial be computed? At least six ways in general, plus four more ways with some luck. This richness of approaches reflects the central place that log canonical thresholds and, more generally, multiplier ideals have in singularity theory. Singularity theory is a subject deeply connected with many areas of mathematics and emerged with the works of Milnor and Arnold. The Milnor fiber is the main object of interest in classical singularity theory. More sophisticated approaches were developed since then by Deligne, Malgrange, and Kashiwara, with the introduction of the vanishing cycles functor and its version in terms of $\mathcal{D}$-modules. The theory culminated with the introduction of mixed Hodge structures on Milnor fibers by Steenbrink, Varchenko, Navarro-Aznar, and M. Saito. Modern approaches to singularity theory evolved with the introduction of the motivic Milnor fiber of Denef-Loeser, of arc spaces and jet schemes, and connections with number theory were highlighted via the Monodromy Conjecture and the test ideals.

In these lectures we will review some of these constructions and will explain how multiplier ideals are related to them. The purpose will be to place on the general map of local singularity invariants the role played by multiplier ideals.

We will start in section 1 with the topological point of view: Milnor fibers and local systems. Although we will not prove anything, this part forms the geometric intuition for later. In section 2 we will use log resolutions to define the multiplier ideals, jumping numbers, and inner jumping multiplicities. In section 3 we introduce another point of view on

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singularities coming from mixed Hodge structures. This forms the bridge between multiplier ideals and Milnor fibers. More precisely, we show using the Denef-Loeser motivic Milnor fiber that inner jumping multiplicities are multiplicities in the Hodge spectrum. In the next section, we explicit the Hodge filtration on rank one local systems. We apply this to understand the Hodge spectrum of homogeneous polynomials and to show how multiplier ideals are related to Hodge numbers of finite abelian coverings. In section 4 we introduce another invariant coming from log resolutions, the topological zeta function, and state the conjecture relating it with the Milnor fiber. This is the Monodromy Conjecture. In the last section we introduce generalized b-functions, their relation with Milnor fibers, topological zeta functions, and multiplier ideals. The example of hyperplane arrangements will be discussed in detail throughout the exposition.

For the complete answer to the trivia question above see the survey [1]. This survey is a guide to results and literature covering topics wider than these lectures. For a thorough reference list, please consult this survey. From these notes we left out the analytic approach to multiplier ideals and their connections with Frobenius map and test ideals. These are covered by the lectures of S. Boucksom and K. Smith.

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1. Milnor fibers and local systems

In this section we describe topological point of views on singularities. While topological invariants of singularities are difficult to compute unless tied with more algebraic approaches, they form the ground upon which geometric intuition builds in local singularity theory.

Let f be a polynomial in $\mathbb{C}[x_1, \ldots, x_n]$ vanishing at the origin in $\mathbb{C}^n$. More generally we can let f be a hypersurface singularity germ at the origin in $\mathbb{C}^n$. Let ϵ be a very small positive real number and let B_ϵ be the ball of radius ϵ around the origin in $\mathbb{C}^n$. Let

$$M_t := f^{-1}(t) \cap B_\epsilon .$$

Theorem 1.1. (Milnor, Hamm) *For small values of ϵ and even smaller values of $|t|$, the diffeomorphism class of M_t is constant.*

Definition 1.2. M_t is called a *Milnor fiber of f at the origin in $\mathbb{C}^n$*. Fix once and for all a Milnor fiber M_t and we will denote its diffeomorphism class by $M_{f,0}$. The cohomology vector spaces $H^i(M_{f,0}, \mathbb{C})$ of the Milnor fiber admit a $\mathbb{C}$-linear action T called *monodromy* generated by going once along a loop starting at t around 0.

Theorem 1.3. (Brieskorn) *The eigenvalues of the monodromy action T are roots of unity.*

Example 1.4. If the origin is a *nonsingular point* of f, that is if the derivatives $\partial f/\partial x_1$, ..., $\partial f/\partial x_n$ do not all vanish at 0, then $M_{f,0}$ is contractible to a point. So the reduced cohomology $\widetilde{H}^i(M_{f,0}, \mathbb{C}) = 0$.

If the origin is a singular point of f, how does one compute the Milnor cohomology, the monodromy, and its eigenvalues? Here are some examples, but we will talk later about how this is accomplished in general by algebraic approaches.

Example 1.5. **(Milnor)** If the origin is an *isolated singular point* of f, that is if the derivatives $\partial f/\partial x_1$, ..., $\partial f/\partial x_n$ vanish simultaneously at 0 but at no other point in a small open neighborhood of 0 in $f^{-1}(0)$, then the Betti numbers of the Milnor fiber are

$$\dim_{\mathbb{C}} H^j(M_{f,0}, \mathbb{C}) = \begin{cases} 0 & \text{for } j \neq 0, n-1, \\ 1 & \text{for } j = 0, \\ \dim_{\mathbb{C}} \mathbb{C}[[x_1, \dots, x_n]] / \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) & \text{for } j = n-1. \end{cases}$$

The last value for $j = n-1$ is denoted $\mu(f)$ and called the *Milnor number* of f.

Example 1.6. Let $f = y^2 - x^3$ be the cusp. Then the Milnor number $\mu(f) = 2$. The monodromy on $H^1(M_{f,0}, \mathbb{C}) = \mathbb{C}^2$ is diagonalizable with eigenvalues $e^{2\pi i \cdot 1/6}$ and $e^{2\pi i \cdot 5/6}$.

Example 1.7. **(Milnor)** If f is a homogeneous polynomial of degree d, the Milnor fiber of f at 0 is, up to diffeomorphism,

$$M_{f,0} = f^{-1}(1) \subset \mathbb{C}^n.$$

Let $h : f^{-1}(1) \to f^{-1}(1)$ be the map given by $a \mapsto e^{2\pi i/d} \cdot a$. Then the monodromy on the Milnor cohomology is $T = h^*$. The monodromy T is diagonalizable and the eigenvalues are d-th roots of unity. This example will lead us to introduce the following.

Definition 1.8. A *local system* $\mathcal{V}$ on a complex manifold X is a locally constant sheaf of finite dimensional complex vector spaces. The *rank* of $\mathcal{V}$ is the dimension of a fiber of $\mathcal{V}$. If X is a nonsingular algebraic variety, a local system on X will mean a local system on the underlying complex manifold.

Example 1.9. Rank one local systems on X together with their tensor product form a group. This group is $\mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}^*)$. A *unitary* local system of rank one corresponds to an element of $Hom(H_1(X, \mathbb{Z}), S^1)$. For example, $\mathbb{C}_X$ is unitary, and so is any rank one local system of finite order. Higher rank local systems are given by representations of the fundamental group of X. For higher rank, unitary local systems are given by unitary representations of the fundamental group.

Example 1.7 - continued. We show now how local systems arise naturally in questions related to Milnor fibers. Let U be the complement in $\mathbb{P}^{n-1}$ of the zero locus of f. Use the shorter notation M for $M_{f,0} = f^{-1}(1)$. Let

$$p : M \to U$$

be the natural projection. The group $\langle h \rangle = \mathbb{Z}/d\mathbb{Z}$ acts on M freely, the quotient can be identified with U, and p is the quotient map. The direct image of the constant sheaf $\mathbb{C}_M$ is a rank d local system $p_*\mathbb{C}_M$ on U. There is a decomposition

$$p_*\mathbb{C}_M = \oplus_{k=1}^d \mathcal{V}_k,$$

where $\mathcal{V}_k$ is the rank one local system on U given by the $e^{-2\pi ik/d}$-eigenspaces of fibers of the local system $p_*\mathbb{C}_M$. Then, since p is finite, by the Leray spectral sequence one has

$$H^i(M, \mathbb{C})_{e^{-2\pi ik/d}} = H^i(U, \mathcal{V}_k)$$

for $1 \leq k \leq d$. In other words, by 1.7, the eigenspaces of the Milnor monodromy are computed by rank one local systems in this case. It will be useful later to know that the monodromy of $\mathcal{V}_k$ around a general point of an irreducible component of the complement of U is given by multiplication by $e^{2\pi ikm/d}$, where m is the vanishing order of f along this component.

Example 1.10. Let f be a *hyperplane arrangement*, that is f a product of linear polynomials. We can assume that f is *central*, or in other words a product of linear forms. Using the notation of the previous example, Orlik-Solomon showed that the cohomology algebra $H^*(U, \mathbb{Z})$ is a *combinatorial invariant* of f, that is it only depends on how the hyperplanes intersect but not on their position. Rybnikov showed that the fundamental group $\pi_1(U)$ is not a combinatorial invariant of f. One can think of cohomology $H^*(U, \mathcal{V})$ of rank one local systems on U as lying somewhere between $H^*(U, \mathbb{Z})$ and $\pi_1(U)$. This is the reason why the following folklore conjecture is the current "holy grail" of the theory of hyperplane arrangements.

Conjecture 1.11. *The dimensions of the monodromy eigenspaces, and hence the Betti numbers of the Milnor fiber, of a hyperplane arrangement are combinatorial invariants.*

Even the simplest unknown case, the cone over a planar line arrangement with at most triple points, is surprisingly difficult.

Example 1.9 is a particular case of a more general situation in the following sense. Let X be a nonsingular complex projective variety, $D = \cup_{i\in S} D_i$ a divisor on X with irreducible components D_i, and let $U = X - D$. Let G be a finite abelian group, and $G^* = \mathrm{Hom}(G, \mathbb{C}^*)$ the dual group of G.

Definition 1.12. A map $Y \to X$ is a *G-cover* if it is a finite map together with a faithful action of G on Y such that the map exhibits X as the quotient of Y via G. Two covers are said to be equivalent if there is an isomorphism between them commuting with the cover maps.

The first part of the following is a classical result in topology. The second, due to Grauert-Remmert, is its algebraic counterpart.

Theorem 1.13. (Grauert-Remmert) *The morphisms of $H_1(U, \mathbb{Z})$ onto G are in one-to-one correspondence with the equivalence classes of unramified G-covers of U. These, in turn, are in one-to-one correspondence with equivalence classes of normal G-covers of X unramified above U. The group G is recovered as the group of automorphisms of the cover commuting with the cover map.*

Example 1.14. The canonical surjection

$$H_1(U, \mathbb{Z}) \to H_1(U, \mathbb{Z}/N\mathbb{Z})$$

defines an unramified cover U_N of U, and a corresponding normal cover X_N of X. These are called *congruence covers.*

Corollary 1.15. *The equivalence classes of unramified G-covers of U, or equivalently of normal G-covers of X unramified above U, are into one-to-one correspondence with subgroups G^* of the group $\mathrm{Hom}(H_1(U,\mathbb{Z}), S^1)$ of rank one unitary local systems on U.*

Proof. It is a standard exercise in duality that there is an one-to-one equivalence between surjections $H_1(U,\mathbb{Z}) \to G$ and subgroups $G^* \subset \mathrm{Hom}(H_1(U,\mathbb{Z}), S^1)$. $\square$

Example 1.9 -continued. An algebro-geometric description of rank one local systems on a nonsingular projective variety X is as the group $\mathrm{Pic}^\tau(X) \times H^0(X, \Omega^1_X)$ of so-called "Higgs line bundles". Here $\mathrm{Pic}^\tau(X) = \ker[c_1 : \mathrm{Pic}(X) \to H^2(X,\mathbb{R})]$ and it consists of finitely many disjoint copies of the Picard variety $\mathrm{Pic}^0(X)$. To a local system $\mathcal{V}$ given by $\rho \in \mathrm{Hom}(H_1(X,\mathbb{Z}), \mathbb{C}^*)$ one attaches (E,ϕ) in $\mathrm{Pic}^\tau(X) \times H^0(X, \Omega^1_X)$, where, as a holomorphic line bundle, $E = \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{V}$, and ϕ is the $(1,0)$-part of $\log|\rho|$ viewed as a cohomology class via $H^1(X,\mathbb{R}) \cong \mathrm{Hom}(H_1(X,\mathbb{Z}), \mathbb{R})$. Under this equivalence, the unitary local systems correspond to the elements of $\mathrm{Pic}^\tau(X)$. We will describe an algebro-geometric description of the unitary local systems of rank one on nonsingular quasi-projective varieties later, see Theorem 4.2.

One can construct many natural susbspaces of rank one local systems, such as the following.

Definition 1.16. The *characteristic subvarieties* of the space of rank one local systems on X are the sets of type

$$\{\mathcal{V} \mid \dim H^m(X,\mathcal{V}) \geq i\},$$

with m and i fixed.

In the nonsingular projective case, the characteristic varieties have quite a rigid structure involving subtori of $\mathrm{Pic}^0(X)$. We have just seen in the above example the relation of this group with local systems. A quasi-projective version of this structure result will be mentioned later.

Theorem 1.17. (Green-Lazarsfeld, Arapura, Simpson) *Let X be a nonsingular projective complex variety. The characteristic varieties of X are finite unions of torsion-translated complex subtori of the space of rank one local systems.*

2. Multiplier ideals

Next, we describe a point of view on singularities coming from birational geometry: multiplier ideals. We introduce the jumping numbers and, a more local notion, the inner jumping numbers. Later we will describe a relation between multiplier ideals and the topological point of views, where inner jumping numbers play a major role.

Let X be a nonsingular complex variety.

Definition 2.1. A *log resolution* of a collection of closed subschemes $Z_1, \ldots, Z_r$ in X is a map $\mu : Y \to X$ such that Y is nonsingular, μ is birational and proper, and: the exceptional locus, the inverse image of each Z_i, the support of the determinant of the Jacobian of μ, and the union of these are simple normal crossings divisors.

Log resolutions always exist by Hironaka. Denote by $K_{Y/X}$ the divisor given by the determinant of the Jacobian of μ. Note that $K_{Y/X} = K_Y - \mu^* K_X$, where K are canonical divisors. Let $c_1, \ldots, c_r$ be positive real numbers. Let $\lfloor H \rfloor$ denote the round-down of the coefficients of the irreducible components of the divisor H. The following is the key result that we use. A proof can be found in [5], along with a history of the subsequent definition.

Theorem 2.2. *Let $\mu : Y \to X$ be a log resolution of $Z_1, \ldots, Z_r$ in X. Denote by H_i the divisor $\mu^{-1} Z_i$, the scheme-theoretic inverse image of Z_i. Then $\mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor c_1 H_1 + \ldots + c_r H_r \rfloor)$ is independent of the choice of μ and $R^i \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor c_1 H_1 + \ldots + c_r H_r \rfloor) = 0$ for $i > 0$.*

Definition 2.3. The *multiplier ideal* of $(X, c_1 \cdot Z_1 + \ldots + c_r \cdot Z_r)$ is

$$\mathcal{J}(X, c_1 \cdot Z_1 + \ldots + c_r \cdot Z_r) := \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor c_1 H_1 + \ldots + c_r H_r \rfloor) \subset \mathcal{O}_X.$$

This is an ideal sheaf since it is included in $\mu_* \mathcal{O}_Y(K_Y) = \mathcal{O}_X$.

Multiplier ideals should be viewed as invariants of singularities in the following sense: smaller multiplier ideals means worse singularities.

Example 2.4. (Howald) The multiplier ideals of a scheme Z in $X = \mathbb{C}^n$ defined by a monomial ideal I are

$$\mathcal{J}(X, cZ) = \langle x^u \mid u + \mathbf{1} \ \in \ \text{Interior}(cP(I)) \rangle,$$

where $P(I)$ is the convex closure in $\mathbb{R}^n_{\geq 0}$ of $\{u \in \mathbb{Z}^n \mid x^u \in I\}$.

Let Z be a closed subscheme of X.

Definition 2.5. The *jumping numbers of Z in X* are those numbers $c > 0$ such that

$$\mathcal{J}(X, cZ) \neq \mathcal{J}(X, (c - \epsilon)Z)$$

for all $\epsilon > 0$. The *log canonical threshold* of (X, Z) is denoted $lct\,(X, Z)$ and is the smallest jumping number.

Proposition 2.6. (Ein-Lazarsfeld-Smith-Varolin) *The list of jumping numbers contains finitely many numbers in any compact interval, all rational numbers, and is periodic.*

Exercise 2.7. Let $K_{Y/X} = \sum_i k_i E_i$ and $\mu^{-1} Z = \sum_i a_i E_i$ be the irreducible decompositions. Then

$$lct\,(X, Z) = \min_i \left\{ \frac{k_i + 1}{a_i} \right\}.$$

Exercise 2.8. Let $f = x^2 - y^3$. Then the jumping numbers are 5/6, 1, 11/6, 2, ...

Example 2.9. (Mustaţă, Teitler) Let $D = \{f = 0\}$ be a hyperplane arrangement in $X = \mathbb{C}^n$, with irreducible components $D_i = \{f_i = 0\}$. An *edge* of D is any intersection of D_i. The poset of edges, ordered by inclusion, is denoted $\mathcal{L}$. For an edge V we define

$$r_V := \operatorname{codim} V, \quad a_V := \#\{D_i \supset V\},$$

where the last count takes into account multiplicities. A hyperplane arrangement A is *indecomposable* if there is no linear change of coordinates on $\mathbb{C}^n$ such that A can be written as the product of two non-constant polynomials in disjoint sets of variables. An edge V is called *dense* if the hyperplane arrangement $D_V = \{f_V = 0\}$ given by the image of $\cup_{D_i \supset V} D_i$ in $\mathbb{C}^n/V$ is indecomposable. For example, D_i is a dense edge for every i.

Then the multiplier ideals are

$$\mathcal{J}(X, cD) = \bigcap_{\text{dense } V \in \mathcal{L}} I_V^{\lfloor ca_V \rfloor + 1 - r_V}.$$

This can be seen by letting $\mu : Y \to X$ be the successive blow up, by decreasing codimension, of the dense edges. Denoting by E_V the exceptional divisor corresponding to a dense edge V, we have

$$K_{Y/X} = \sum_{\text{dense } V \in \mathcal{L}} (r_V - 1) E_V, \quad \text{ and } \mu^* D = \sum_{\text{dense } V \in \mathcal{L}} a_V E_V.$$

Mustaţă conjectured that the jumping numbers of D are combinatorial invariants of the arrangement, that is they depend only on the poset $\mathcal{L}$ together with the function r and the multiplicities of the components of D. We will see how this is proved later.

To measure the contribution of a singular point $x \in Z$ to a jumping number c, we introduce the following.

Definition 2.10. For a point x in Z, the *inner jumping multiplicity of* c *at* x is the vector space dimension

$$m_x(X, c \cdot Z) := \dim_{\mathbb{C}} \mathcal{J}(X, (c - \epsilon)Z) / \mathcal{J}(X, (c - \epsilon)Z + \delta\{x\}),$$

where $0 < \epsilon \ll \delta \ll 1$. We say that c is an *inner jumping number* of (X, Z) at x if $m_x(X, c \cdot Z) \neq 0$.

We will prove that this multiplicity is well-defined, is finite, and that inner jumping numbers are jumping numbers.

Let $\mu : Y \to X$ be a common log resolution of Z and $\{x\}$ in X. Let $S = \{i \mid a_i \neq 0\}$ be the index set of the irreducible decomposition of $E = \mu^{-1} Z = \sum_i a_i E_i$. For any subset $I \subset S$, let $E^I = \bigcup_{i \in I} E_i$. Let $E^{\emptyset} = \emptyset$. For a positive integer d, let $S_d = \{i \in S \mid d | a_i\}$. Let $S_{d,x} = \{i \in S_d \mid \mu(E_i) = \{x\} \}$. Write $c = r/d$ with r and d nonnegative integers such that $\gcd(r, d) = 1$. Having fixed c, define

$$F = E^{S_{d,x}}, \quad G = E^{S_d \setminus S_{d,x}}.$$

By comparing coefficients, we have:

Lemma 2.11. *Let H be the effective divisor on Y such that $\mu^{-1} x = H$. Then*

$$\begin{aligned} K_{Y/X} - \lfloor cE \rfloor + G &= K_{Y/X} - \lfloor (c - \epsilon)E \rfloor - F \\ &= K_{Y/X} - \lfloor (c - \epsilon)E + \delta H \rfloor, \end{aligned}$$

where $0 < \epsilon \ll \delta \ll 1$.

Lemma 2.12. (1) *$\mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor cE \rfloor + G)$ is independent of the log resolution and $R^i \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor cE \rfloor + G) = 0$ for $i > 0$.*

(2) *For $0 < \epsilon \ll 1$, $\mu_*(\mathcal{O}_F(K_{Y/X} - \lfloor (c - \epsilon)E \rfloor))$ is independent of the log resolution and $R^i \mu_*(\mathcal{O}_F(K_{Y/X} - \lfloor (c - \epsilon)E \rfloor)) = 0$ for $i > 0$.*

(3) $\mu_*(\mathcal{O}_G(K_{Y/X} - \lfloor cE \rfloor + G))$ *is independent of the log resolution and* $R^i\mu_*(\mathcal{O}_G(K_{Y/X} - \lfloor cE \rfloor + G)) = 0$ *for* $i > 0$.

Proof. By Lemma 2.11,

$$\mu_*\mathcal{O}_Y(K_{Y/X} - \lfloor cE \rfloor + G) = \mathcal{J}(X, (c-\epsilon)Z + \delta\{x\})$$

for $0 < \epsilon \ll \delta \ll 1$. This shows (1) by Theorem 2.2. For all small $\epsilon > 0$, consider the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(A - F) \longrightarrow \mathcal{O}_Y(A) \longrightarrow \mathcal{O}_F(A) \longrightarrow 0,$$

where $A = K_{Y/X} - \lfloor (c-\epsilon)E \rfloor$. By (1) and Lemma 2.11, the first sheaf pushes forward to a multiplier ideal on X and has no higher direct images. The second sheaf pushes forward to $\mathcal{J}(X, (c-\epsilon)Z)$ and has no higher direct images by Theorem 2.2. This implies (2). (3) is similar. $\square$

Let

$$\mathcal{K}(X, cZ) = \mathcal{J}(X, (c-\epsilon)Z)/\mathcal{J}(X, cZ),$$

and

$$\mathcal{K}_x(X, cZ) = \mathcal{J}(X, (c-\epsilon)Z)/\mathcal{J}(X, (c-\epsilon)Z + \delta\{x\}),$$

for $0 < \epsilon \ll \delta \ll 1$. Remark that $\mathcal{K}(X, cZ) = \mu_*(\mathcal{O}_{F+G}(K_{Y/X} - \lfloor (c-\epsilon)E \rfloor))$. Similarly, the proof of Lemma 2.12-(2) gives:

Proposition 2.13. *With the above notation, we have:*

(1) $\mathcal{K}_x(X, cZ) = \mu_*(\mathcal{O}_F(K_{Y/X} - \lfloor (c-\epsilon)E \rfloor))$, *where* $0 < \epsilon \ll 1$.
(2) $m_x(X, cZ) = \chi(Y, \mathcal{O}_F(K_{Y/X} - \lfloor (c-\epsilon)E \rfloor)$, *where* χ *is the sheaf Euler characteristic.*

Proposition 2.14. *If* c *is an inner jumping number of* (X, Z) *at* x *then* c *is a jumping number of* (X, Z).

Proof. Consider the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_G(K_{Y/X} - \lfloor cE \rfloor + G) &\longrightarrow \mathcal{O}_{F+G}(K_{Y/X} - \lfloor (c-\epsilon)E \rfloor) \\ &\longrightarrow \mathcal{O}_F(K_{Y/X} - \lfloor (c-\epsilon)E \rfloor) \longrightarrow 0. \end{aligned}$$

None of the three sheaves has higher direct images for μ and the last two sheaves push-forward to $\mathcal{K}(X, cZ)$ and, respectively, $\mathcal{K}_x(X, cZ)$. If $\mathcal{K}_x(X, cZ) \neq 0$ then $\mathcal{K}(X, cZ) \neq 0$. $\square$

Remark 2.15. If x is an isolated singularity of Z, and if X is replaced by a small open neighborhood of x if necessary, then all non-integral jumping numbers are inner jumping numbers. This is because we can assume $G = \emptyset$ and so $\mathcal{K}(X, cZ) = \mathcal{K}_x(X, cZ)$.

Exercise 2.16. Let D be the plane cuspidal cubic. Then $m_0(\mathbb{C}^2, 5/6 \cdot D) = 1$. However, $m_0(\mathbb{C}^2, D) = 0$ although 1 is a jumping number.

3. Hodge filtration I: Milnor fibers

Next, we will see how mixed Hodge structures give yet another point of view on singularities. Using them we describe a relation between multiplier ideals and Milnor fibers and local systems. We should mention that there is now a more general theorem, relating multiplier ideals with $\mathcal{D}$-modules. Although we will not talk about it in these lectures, we will see some consequences of this later.

We will keep things very simple and talk only about the Hodge filtration. This means we will be talking about vector spaces V with a decreasing filtration $F^\bullet V$ of subspaces. We let $Gr_F^p V = F^p V/F^{p+1}V$ be the graded p-piece of the filtration.

Example 3.1. If X is a smooth projective complex variety, the Hodge filtration and its graded pieces on $H^m(X,\mathbb{C})$ are

$$F^p H^m(X,\mathbb{C}) = \bigoplus_{i\geq p} H^{m-i}(X,\Omega_X^i) \quad \text{and} \quad Gr_F^p H^m(X,\mathbb{C}) = H^{m-p}(X,\Omega_X^p).$$

Example 3.2. If U is a smooth quasi-projective complex variety of dimension n, then

$$Gr_F^p H^m(U,\mathbb{C}) = H^{m-p}(Y,\Omega_Y^p(\log E)),$$

where: Y is any smooth compactification of U with complement $E = Y - U$ a simple normal crossings divisor; $\Omega_Y^1(\log E)$ is the $\mathcal{O}_X$-module locally generated by $dx_1/x_1, \ldots, dx_r/x_r$, $dx_{r+1},\ldots,dx_n$ where $x_1,\ldots,x_r$ are local equations for the irreducible components of E; and $\Omega_Y^p = \Lambda^p\Omega_Y^1$.

Example 3.3. (Deligne, Timmerscheidt) More generally, let U be a smooth quasi-projective complex variety of dimension n and $\mathcal{V}$ a unitary local system on U. Then

$$Gr_F^p H^{p+q}(U,\mathcal{V}) = H^q(Y,\Omega_Y^p(\log E)\otimes\overline{\mathcal{V}}),$$

where $\overline{\mathcal{V}}$ is a vector bundle called the "canonical Deligne extension" of $\mathcal{V}$.

We will come back and make this example very explicit, but let's see first where all this leading to.

Theorem 3.4. (Steenbrink, Varchenko, Navarro Aznar, M. Saito) *Let f be a hypersurface germ. There is a canonical mixed Hodge structure on the cohomology of the Milnor fiber of f, compatible with semi-simple part of the monodromy.*

We will not give the construction of the Hodge filtration of this theorem. But we will still be able to understand concretely what it says. Let us assume for example that f is a homogeneous polynomial. We have seen in 1.7 that the Milnor fiber is the smooth quasi-projective variety given by $f-1$. Moreover, we have seen that the Milnor fiber cohomology is given by unitary local systems of rank one on U, the complement in $\mathbb{P}^{n-1}$ of the zero locus of f. The canonicity of the Hodge filtration of the Milnor fiber means that this is the same filtration as the one given by 3.3. We will make 3.3 explicit in the next section.

There is yet another way to understand concretely what the Hodge filtration on the Milnor fiber is, provided that one is happy to work only with numerical invariants. Let us describe this now.

Definition 3.5. (Steenbrink) Let f be a hypersurface germ at the origin in $\mathbb{C}^n$. The *Hodge spectrum of* f *at* 0 is

$$Sp(f,0) = \sum_{\alpha \in \mathbb{Q}} n_{\alpha,0}(f) \cdot t^{\alpha},$$

where the *spectrum multiplicities*

$$n_{\alpha,0}(f) := \sum_{i\in\mathbb{Z}} (-1)^{i-n+1} \dim_{\mathbb{C}} \mathrm{Gr}_F^{\lfloor n-\alpha \rfloor} \widetilde{H}^i(M_{f,0},\mathbb{C})_{e^{-2\pi i \alpha}}$$

record the generalized Euler characteristic on the $\lfloor n-\alpha \rfloor$-graded piece of the Hodge filtration on the $\exp(-2\pi i\alpha)$-monodromy eigenspace on the reduced cohomology of the Milnor fiber.

It turns out that one can write the Hodge spectrum more economically as:

Proposition 3.6.

$$Sp(f,0) = \sum_{\alpha\in\mathbb{Q}\cap(0,n)} \sum_{\lfloor n-\alpha \rfloor \leq i < n} (-1)^{i-n+1} \dim_{\mathbb{C}} \mathrm{Gr}_F^{\lfloor n-\alpha \rfloor} H^i(M_{f,0},\mathbb{C})_{e^{-2\pi i\alpha}} \cdot t^{\alpha}.$$

Let us give a formula for the Hodge spectrum in terms of log resolutions. Given a positive integer d, fix for once and for all a generator of $\mathbb{Z}/d\mathbb{Z}$. For a complex variety Z with an action of $\mathbb{Z}/d\mathbb{Z}$, and for a rational number $\alpha \in [0,1)$ there is a well-defined class

$$[Z,\alpha] = \sum_{i\in\mathbb{Z}} (-1)^i [(H_c^i(Z,\mathbb{C})_{e^{2\pi i\alpha}}, F)]$$

in the Grothendieck ring of the abelian category of filtered vector spaces. This class is a generalized Euler characteristic and it behaves as expected: satisfies additivity and a Mayer-Vietoris formula. We will see how this works in practice. To get numerical invariants we will further apply to such classes the functor

$$gr^p(H,F) := \dim Gr_F^p H.$$

Let $\mathbb{L}$ be the class such that $gr^p(H\cdot\mathbb{L}^m) = gr^{p-m}(H)$.

Let X be a nonsingular complex variety of dimension n and f a regular function defining a divisor D. Let $\mu: Y \to X$ be a log resolution of (X,D). Let $\mu^*D = \sum_i a_i E_i$, where E_i are the irreducible components. Fix $\alpha \in [0,1)$ a rational number, and write $\alpha = r/d$ where r and d are nonnegative integers with $\gcd(r,d) = 1$. If $\alpha = 0$, let $d = 1$. Let $J = \{i \mid a_i \neq 0\}$ and $J_d = \{i \in J \mid d|a_i\}$. For $I \subset J$, let $E_I = \cap_{i\in I} E_i$ and $E^I = \cup_{i\in I} E_i$. Let $E_{\emptyset} = Y$ and $E^{\emptyset} = \emptyset$. For $I \subset J$, let $E_I^o = E_I - E^{J\setminus I}$.

Let $p: \tilde{Y} \to Y$ be the degree-d cyclic cover obtained by taking the d-th root of the local equations of the divisor $R = \sum_{i\in J\setminus J_d} a_i E_i$. That is, locally $\tilde{Y}$ is the normalization of the subvariety of $Y \times \mathbb{C}$ given by $y^d = s$, where s defines R. A more conceptual way to construct this covering is given in Exercise 4.14. For any locally closed subset $W \subset Y$ let $\widetilde{W} = p^{-1}(W)$.

Theorem 3.7. (Denef-Loeser) *Define*

$$S_{\alpha,x} := \sum_{I\subset J_d} [(E_I^o \cap \mu^{-1}(x))^{\sim}, \alpha](1-\mathbb{L})^{|I|-1}.$$

For $c \in \mathbb{Q} \cap (0,1]$ the multiplicity of c in the Hodge spectrum $Sp(f,x)$ is

$$n_{c,x}(f) = (-1)^{n-1} gr^{n-1}(S_{1-c,x}).$$

What Denef and Loeser did is more general, cf. [3]. The class $S_{\alpha,x}$ is defined via motivic integration in an appropriate Grothendieck ring of varieties. These classes, for varying α, recover the whole Hodge spectrum of f and the monodromy zeta function of f, see 5.3. They form the so-called "motivic Milnor fiber". The motivic Minor fiber can be obtained from a "motivic zeta function". This motivic zeta function recovers the topological zeta function which will be introduced later see 5.1.

The following connects multiplier ideals with the Milnor fiber. More precisely, it connects the inner jumping multiplicities from 2.10 with the Hodge spectrum multiplicities.

Theorem 3.8. (B.) *Let X be a nonsingular complex variety. Let D be a hypersurface in X and $x \in D$ be a point. Let f be any local equation of D at x. Then for any $c \in (0,1]$,*

$$m_x(X, cD) = n_{c,x}(f).$$

Corollary 3.9. *For all $\alpha \in (0,1]$,*

1. *α appears in the Hodge spectrum of f if and only if α is an inner jumping number of (X,D) at x;*
2. *(M. Saito) the multiplicity $n_\alpha(f)$ is ≥ 0;*
3. *(Varchenko) if x is an isolated singularity of D and $\alpha \neq 1$, then, replacing X by an open neighborhood of x if necessary, α appears in the Hodge spectrum if and only if α is a jumping number.*

Example 3.10. Let $f = x^2 - y^3$. Then $Sp(f,0) = t^{5/6} + t^{7/6}$.

Proof of Theorem 3.8. We will use Theorem 3.7. We can assume that f is a regular function on X and D is the divisor of f. Let $\mu : Y \to X$ be a common resolution of the point x and of the divisor D.

Let α be a rational number in $[0,1)$. Let $M = \{I \subset J_d \mid I \cap J_{d,x} \neq \emptyset\}$. We have

$$S_{\alpha,x} = \sum_{I \in M} [\widetilde{E_I^o}, \alpha](1-\mathbb{L})^{|I|-1} = \star,$$

because $\mu^{-1}(x)$ is a divisor whose components have nonzero multiplicity in $\mu^* D$. By additivity,

$$\star = \sum_{I \in M} ([\widetilde{E_I}, \alpha] - [(E_I \cap E^{J \setminus I})\tilde{}, \alpha])(1-\mathbb{L})^{|I|-1}.$$

Using Mayer-Vietoris on $(E_I \cap E^{J\setminus I})\tilde{}$,

$$\star = \sum_{I \in M} \sum_{L \subset J \setminus I} (-1)^{|L|} [\widetilde{E}_{I \cup L}, \alpha](1-\mathbb{L})^{|I|-1}.$$

If $L \not\subset J_d$ then $E_{I\cup L}$ is included in $E_I \cap E^{J \setminus J_d}$. This implies that the $\mathbb{Z}/d$-action on $\widetilde{E}_{I\cup L}$ factors through a $\mathbb{Z}/d'$-action with $d' < d$. This further implies that $[\widetilde{E}_{I\cup L}, \alpha] = 0$. Therefore

$$\star = \sum_{I \in M} \sum_{L \subset J_d \setminus I} (-1)^{|L|} [\widetilde{E}_{I \cup L}, \alpha](1-\mathbb{L})^{|I|-1}.$$

Then

$$gr^{n-1}(S_{\alpha,x}) = \sum_{I\in M}\sum_{L\subset J_d\setminus I}\sum_{i=0}^{|I|-1}(-1)^{|L|+i}\binom{|I|-1}{i}gr^{n-1-i}[\widetilde{E}_{I\cup L},\alpha] = \star\star\,.$$

The varieties $\widetilde{E}_{I\cup L}$ are locally for the Zariski topology quotients of nonsingular varieties by finite abelian groups. Since $I\cap J_{d,x}\neq\emptyset$, they are finite over closed subvarieties of $\mu^{-1}(x)$ and therefore projective. It is known then that $gr^{n-1-i}[\widetilde{E}_{I\cup L},\alpha]=0$ if $n-1-i>\dim\widetilde{E}_{I\cup L}$. Since the divisors E_i are in normal crossing, the dimension of $\widetilde{E}_{I\cup L}$ is $n-|I|-|L|$. Hence in the summation above it suffices to take $i=|I|-1$ and $L=\emptyset$, i.e.

$$\star\star = \sum_{I\in M}(-1)^{|I|-1}gr^{n-|I|}[\widetilde{E}_I,\alpha].$$

The Hodge theory of $\widetilde{E}_I$ behaves as if $\widetilde{E}_I$ is smooth, a fact which is holds, more generally, for varieties with rational singularities. So

$$\star\star = \sum_{I\in M}(-1)^{|I|-1}\sum_i(-1)^i\dim H^i(\widetilde{E}_I,\mathcal{O}_{\widetilde{E}_I})_{1-\alpha},$$

where $H^*(\widetilde{E}_I,\mathcal{O}_{\widetilde{E}_I})_{1-\alpha}$ is the part of the cohomology of $\mathcal{O}_{\widetilde{E}_I}$ on which the our fixed generator of $\mathbb{Z}/d\mathbb{Z}$ acts by multiplication with $e^{2\pi i(1-\alpha)}$. Since the covering $p:\widetilde{E}_I\to E_I$ is a finite morphism, we can replace $\mathcal{O}_{\widetilde{E}_I}$ by $p_*\mathcal{O}_{\widetilde{E}_I}$ in $\star\star$. By Exercise 4.14, the eigensheaf decomposition of $p_*\mathcal{O}_{\widetilde{E}_I}$ gives

$$\star\star = \sum_{I\in M}(-1)^{|I|-1}\chi(\mathcal{O}_{E_I}\otimes\mathcal{O}_Y(\lfloor(1-\alpha)\mu^*D\rfloor)),$$

where χ stands for the sheaf Euler characteristic. The varieties E_I are projective and nonsingular by assumption. Let ω_{E_I} be the canonical-dualizing sheaf. By duality,

$$\begin{aligned}\star\star &= (-1)^{n-1}\sum_{I\in M}\chi\left(\omega_{E_I}\otimes\mathcal{O}_Y(-\lfloor(1-\alpha)\mu^*D\rfloor)\right)\\ &= (-1)^{n-1}\sum_{\emptyset\neq L\subset J_{d,x}}\left[\chi\left(\omega_{E_L}\otimes\mathcal{O}_Y(-\lfloor(1-\alpha)\mu^*D\rfloor)\right)+\right.\\ &\qquad\left.+\sum_{\emptyset\neq K\subset J_d\setminus J_{d,x}}\chi(\omega_{E_{L\cup K}}\otimes\mathcal{O}_Y(-\lfloor(1-\alpha)\mu^*D\rfloor)\right]\,.\end{aligned}$$

Let $G=E^{J_d\setminus J_{d,x}}$. Using a (coherent) Mayer-Vietoris for $G\cap E_L$ for the last sum,

$$\begin{aligned}\star\star = (-1)^{n-1}\sum_{\emptyset\neq L\subset J_{d,x}}&\left[\chi\left(\omega_{E_L}\otimes\mathcal{O}_Y(-\lfloor(1-\alpha)\mu^*D\rfloor)\right)+\right.\\ &\left.+\chi\left(\omega_{G\cap E_L}\otimes\mathcal{O}_Y(-\lfloor(1-\alpha)\mu^*D\rfloor)\right)\right],\end{aligned}$$

where $\omega_{G\cap E_L}$ is the dualizing sheaf of $G\cap E_L$. By adjunction for $G\cap E_L$ in E_L,

$$\star\star = (-1)^{n-1}\sum_{\emptyset\neq L\subset J_{d,x}}\chi(\omega_{E_L}\otimes\mathcal{O}_Y(-\lfloor(1-\alpha)\mu^*D\rfloor+G)).$$

Let $F = E^{J_{d,x}}$. Using a (coherent) Mayer-Vietoris for F,

$$\star\star = (-1)^{n-1}\chi(\omega_F \otimes \mathcal{O}_Y(-\lfloor (1-\alpha)\mu^* D\rfloor + G)).$$

By adjunction for $F \subset Y$, $\omega_F \simeq \mathcal{O}_F \otimes \mathcal{O}_Y(K_Y + F)$ and therefore

$$\star\star = (-1)^{n-1}\chi(\mathcal{O}_F \otimes \mathcal{O}_Y(K_{Y/X} - \lfloor (1-\alpha-\epsilon)\mu^* D\rfloor)),$$

for small $\epsilon > 0$. Letting $\alpha = 1 - c$, the claim follows by Proposition 2.13. □

Next, we go back to making 3.3 explicit. As we noted already, this will give an explicit description of the Hodge filtration on the Milnor fiber of homogeneous polynomials. Not surprisingly, multiplier ideals will appear again.

4. Hodge filtration II: local systems

We now describe explicitly the Hodge filtration on the cohomology of unitary local systems. We will work only with rank one unitary local systems and we will use a geometric description of these as a black-box. One of the outcomes is a relation between multiplier ideals and the Hodge filtration on local systems. We prove then that Hodge spectrum and the jumping numbers of a hyperplane arrangement are combinatorial invariants. We also show how multiplier ideals relate to Hodge numbers of finite abelian coverings.

Let X be a smooth complex projective variety of dimension n. Let D be a reduced effective divisor on X with irreducible decomposition $D = \cup_{i\in S} D_i$, for a finite set of indices S. Let $U = X - D$ be the complement of D in X.

Definition 4.1. The group of *realizations of boundaries* of X on D is

$$\mathrm{Pic}^\tau(X,D) := \left\{ (L,\alpha) \in \mathrm{Pic}(X) \times [0,1)^S : c_1(L) = \sum_{i\in S} \alpha_i \cdot [D_i] \ \in H^2(X,\mathbb{R}) \right\},$$

where the group operation is

$$(L,\alpha)\cdot(L',\alpha') = (L \otimes L' \otimes \mathcal{O}_X(-\lfloor \alpha+\alpha'\rfloor \cdot D)), \{\alpha+\alpha'\}).$$

Here $\alpha \cdot D$ means the divisor $\sum_{i\in S}\alpha_i D_i$, and $\lfloor . \rfloor$ (resp. $\{.\}$) is taking the round-down (resp. fractional part) component-wise.

Note that the inverse of (L,α) is (M,β) where $M = L^\vee \otimes \mathcal{O}_X(\sum_{\alpha_i\neq 0} D_i)$, and β_i is 0 if $\alpha_i = 0$ and is $1-\alpha_i$ otherwise. The black-box that we will use from [2] is:

Theorem 4.2. (**Mochizuki, B.**) *Let X be a smooth projective variety, D a divisor on X, and let $U = X - D$. There is a natural canonical group isomorphism*

$$\mathrm{Pic}^\tau(X,D) \xrightarrow{\sim} \mathrm{Hom}(H_1(U,\mathbb{Z}), S^1)$$

between realizations of boundaries of X on D and unitary local systems of rank one on U.

Using a second compactification of U, we get an isomorphism between the realizations of boundaries of the two compactifications. Let us state next the precise form of this isomorphism. Fix a log resolution

$$\mu : Y \to X$$

of (X, D) which is an isomorphism above U. Let $E = Y - U$ with irreducible decomposition $E = \cup_{j \in S'} E_j$.

Proposition 4.3. *The map* $\mathrm{Pic}^\tau(X, D) \to \mathrm{Pic}^\tau(Y, E)$ *given by*

$$(L, \alpha) \mapsto (\mu^* L - \lfloor \mu^*(\alpha D) \rfloor, \beta)$$

is an isomorphism, where β_j *is the fractional part of the coefficient of* E_j *in* $\mu^*(\alpha D)$.

The following can be taken as the definition of the isomorphism between $\mathrm{Pic}^\tau(Y, E)$ and $\mathrm{Hom}(H_1(U, \mathbb{Z}), S^1)$ if one knows what the canonical Deligne extension $\overline{\mathcal{V}}$ of a local system $\mathcal{V}$ is. Or, the other way around, which is what we want, it can be taken as the definition of the canonical Deligne extension.

Lemma 4.4. *Let* $\mathcal{V}$ *be a rank one unitary local system on* U. *Then* $\mathcal{V}$ *corresponds to* (M, β) *in* $\mathrm{Pic}^\tau(Y, E)$ *where* $M = \overline{\mathcal{V}} \otimes \mathcal{O}_Y(\sum_{\beta_j \neq 0} E_j)$ *and* $\beta_j \in [0, 1)$ *is such that the monodromy of* $\mathcal{V}$ *around a general point of* E_j *is multiplication by* $\exp(2\pi i \beta_j)$.

Exercise 4.5. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree d. Let $f = \prod_{i \in S} f_i^{m_i}$ be the irreducible decomposition of f, and d_i be the degree of f_i. Denote by D (resp. D_i) the hypersurface defined by f (resp. f_i) in $X = \mathbb{P}^{n-1}$. Let $U = X - D$. Let $\mathcal{V}_k$ be the local systems on U from Example 1.7 computing the Milnor fiber of f. Let $(L^{(k)}, \alpha^{(k)})$ be the elements in $\mathrm{Pic}^\tau(X, D)$ corresponding to $\mathcal{V}_k$. Then, using the last remark of 1.7,

$$\alpha_i^{(k)} = \left\{ \frac{k m_i}{d} \right\}, \qquad L^{(k)} = \mathcal{O}_X \left(\sum_{i \in S} \alpha_i^{(k)} d_i \right).$$

Let us come back to making Example 3.3 very explicit. In fact we do something more in the next Theorem. Namely, we start with a smooth compactification of U such that the boundary divisor does not necessarily have only simple normal crossings. In this way we obtain a relation between the multiplier ideals of the divisor and the Hodge filtration of local systems.

Theorem 4.6. *Let* $\mathcal{V}$ *be a rank one unitary local system on* U *corresponding to* (L, α) *in* $\mathrm{Pic}^\tau(X, D)$. *Then:*

(a) $Gr_F^p H^{p+q}(U, \mathcal{V}^\vee) = H^{n-q}(Y, \Omega_Y^p(\log E)^\vee \otimes \omega_Y \otimes \mu^* L \otimes \mathcal{O}_Y(-\lfloor \mu^*(\alpha \cdot D) \rfloor))^\vee$;

(b) $Gr_F^p H^{p+q}(U, \mathcal{V}) = H^q(Y, \Omega_Y^{n-p}(\log E)^\vee \otimes \omega_Y \otimes \mu^* L \otimes \mathcal{O}_Y(-\lfloor \mu^*((\alpha - \epsilon) \cdot D) \rfloor))$, *for all small* $\epsilon > 0$, *if* $\alpha_i \neq 0$ *for all* $i \in S$;

(c) $Gr_F^0 H^q(U, \mathcal{V}^\vee) = H^{n-q}(X, \omega_X \otimes L \otimes \mathcal{J}(X, \alpha D))^\vee$;

(d) $Gr_F^n H^{n+q}(U, \mathcal{V}) = H^q(X, \omega_X \otimes L \otimes \mathcal{J}(X, (\alpha - \epsilon) D))$ *for all small* $\epsilon > 0$, *if* $\alpha_i \neq 0$ *for all* $i \in S$. *This space is* 0 *if* $q \neq 0$ *and* $L - (\alpha - \epsilon) D$ *is nef and big, by Nadel vanishing.*

Proof. (a) By 3.3,

$$Gr_F^p H^{p+q}(U, \mathcal{V}^\vee) = H^q(Y, \Omega_Y^p(\log E) \otimes \overline{\mathcal{V}^\vee}).$$

Consider the element corresponding to $\mathcal{V}$ from 4.3, and the corresponding tuple β. By calculating the inverse in the group $\mathrm{Pic}^\tau(Y, E)$ of this element, the dual local system $\mathcal{V}^\vee$

corresponds to $(\mu^*L^\vee \otimes \mathcal{O}_Y(\lfloor \mu^*(\alpha D)\rfloor + \sum_{\beta_j\neq 0} E_j), \gamma)$ in $\mathrm{Pic}^\tau(Y,E)$, where γ_j is 0 if $\beta_j = 0$ and is $1-\beta_j$ if $\beta_j \neq 0$. Hence, by Lemma 4.4,

$$\begin{aligned}\overline{\mathcal{V}^\vee} &= \mu^*L^\vee \otimes \mathcal{O}_Y\left(\lfloor \mu^*(\alpha D)\rfloor + \sum_{\beta_j\neq 0} E_j - \sum_{\gamma_j\neq 0} E_j\right)\\ &= \mu^*L^\vee \otimes \mathcal{O}_Y(\lfloor \mu^*(\alpha D)\rfloor).\end{aligned}$$

The conclusion follows by Serre duality.

(b) By Lemma 4.4,

$$\overline{\mathcal{V}} = \mu^*L \otimes \mathcal{O}_Y\left(-\lfloor \mu^*(\alpha D)\rfloor - \sum_{\beta_j\neq 0} E_j\right),$$

which we plug into 3.3. Now we use the isomorphism

$$\Omega_Y^p(\log E) \cong \Omega_Y^{n-p}(\log E)^\vee \otimes \omega_Y \otimes \mathcal{O}_Y\left(\sum_j E_j\right).$$

If $\alpha_i \neq 0$ for all i, that is if $\mathcal{V}$ is not the restriction to U of a local system over a larger open subset of X, then the coefficients of E_j in $\mu^*(\alpha\cdot D)$ are nonzero. Hence

$$-\lfloor \mu^*(\alpha D)\rfloor + \sum_{\beta_j=0} E_j = -\lfloor \mu^*((\alpha-\epsilon)D))\rfloor,$$

for all $0<\epsilon\ll 1$. The conclusion follows.

(c) We let $p=0$ in (a) and use the definition of multiplier ideals. The identification of H^{n-q} of the $\mathcal{O}_Y$-module from (a) with H^{n-q} of the $\mathcal{O}_X$-module $\omega_X\otimes L\otimes \mathcal{J}(X,\alpha D)$ is due to the triviality of the Leray spectral sequence that follows from the projection formula and the local vanishing from Theorem 2.2.

(d) We let $p=n$ in (b), then proceed as in (c). The vanishing for $q\neq 0$ is Nadel's vanishing theorem. $\square$

Exercise 4.7. Now we draw some conclusions about the Hodge spectrum of a homogeneous polynomial. We keep the notation of Exercise 4.5. Together with 3.6, it says that the only rational numbers that can have nonzero multiplicity in $Sp(f,0)$ are of the type

$$\alpha = \frac{k}{d} + p \ \in\ (0,n), \qquad \text{with } k,p\in\mathbb{Z},\ 1\le k\le d,\ 0\le p<n. \tag{1}$$

Let $\mu:(Y,E)\to(X,D)$ be a log resolution which is an isomorphism above U. With α,k,p, as in (1), define

$$\beta_i^{(k)} := \left\{-\frac{km_i}{d}\right\}, \qquad M^{(k)} := \mathcal{O}_X\left(\sum_{i\in S}\beta_i^{(k)} d_i\right),$$

$$\mathcal{E}_\alpha := \Omega_Y^{n-p-1}(\log E)^\vee \otimes \omega_Y \otimes \mu^*M^{(k)} \otimes \mathcal{O}_Y\left(-\lfloor \mu^*\left(\beta^{(k)}\cdot D_{red}\right)\rfloor\right),$$

where in the last sheaf the tensor products are over $\mathcal{O}_Y$. Let α be as in (1). Then the multiplicity of α in $Sp(f,0)$ is

$$n_{\alpha,0}(f) = (-1)^{n-p-1}\chi(Y, \mathcal{E}_\alpha).$$

Exercise 4.8. With the same notation as in the previous exercises, if $\alpha = k/d \in (0,1)$, then

$$n_{\alpha,0}(f) = \dim H^0(\mathbb{P}^{n-1}, \mathcal{O}(k-n) \otimes \mathcal{J}(\mathbb{P}^{n-1}, (k/d - \epsilon)\, D))\,.$$

This can also be proved using Theorem 3.8 instead of Theorem 4.6.

Theorem 4.9. (B. - M. Saito) *The jumping numbers and the Hodge spectrum of a hyperplane arrangement are combinatorial invariants.*

Proof. We can assume the arrangement is central. By the previous exercise, the Hodge spectrum can be written in terms of the Euler characteristic of vector bundles on a log resolution Y. By Hirzebruch-Riemann-Roch theorem, the Euler characteristic of a vector bundle is computed from the cohomology ring of Y. Let us use the canonical log resolution of a hyperplane arrangement from 2.9. Then, a theorem of De Concini and Procesi guarantees that the cohomology ring of Y is a combinatorial invariant.

For jumping numbers, it is enough to restrict to those in $(0,1)$. We can assume that the arrangement is essential, that is the smallest edge is the origin. Let f be the polynomial defining the arrangement D. We claim that $\alpha \in (0,1)$ is a jumping number if and only if there is an edge V such that α is a spectral number for f_V in $\mathbb{C}^n/V$, where f_V is defined in 2.9. Indeed, if α is a spectral number for f_V, then α is an inner jumping number for f_V at the origin in $\mathbb{C}^n/V$, by Theorem 3.8. So α is a jumping number of f_V by Proposition 2.14. Now, take a point $p \in V - \cup_{D_i \not\supset V} D_i$. After choosing a splitting of $V \subset \mathbb{C}^n$, we have locally around p, $D = D_V \times V \subset \mathbb{C}^n = \mathbb{C}^n/V \times V$ and $f = f_V \cdot u$, where u is a (locally) invertible function. Hence α is a jumping number of f. Conversely, suppose α is a jumping number for f. By 2.9, the support of the quotient

$$\mathcal{K} = \mathcal{J}(\mathbb{C}^n, (\alpha - \epsilon)D) \;/\; \mathcal{J}(\mathbb{C}^n, \alpha D)$$

is a union of edges. If $\mathcal{K}$ has zero-dimensional support, then $Supp(\mathcal{K}) = \{0\}$, and α is an inner jumping number. Thus it is spectral number of $f = f_0$. Otherwise, let V be a top-dimensional irreducible component of $Supp(\mathcal{K})$. Restricting f to a transversal at a general point of V, we obtain that α is an (inner) jumping number of f_V, and thus a spectral number of f_V. □

Example 4.10. The precise formula for the Hodge spectrum multiplicities of a hyperplane arrangement in terms of $\mathcal{E}_\alpha$ in 4.7, plus the explicit combinatorial description of the ring $H^*(Y, \mathbb{Z})$ due to De Concini-Procesi, leads to the following. Let f be a reduced central essential hyperplane arrangement in $\mathbb{C}^3$. That is, it is the cone over a line arrangement D in $\mathbb{P}^2$, not all lines passing through one point. Then $n_\alpha(f) = 0$ if $\alpha d \notin \mathbb{Z}$, and we have for

$\alpha = \frac{i}{d} \in (0,1]$ with $i \in [1,d] \cap \mathbb{Z}$

$$n_\alpha(f) = \binom{i-1}{2} - \sum_{m\geq 3} \nu_m \binom{\lceil im/d \rceil - 1}{2},$$

$$n_{\alpha+1}(f) = (i-1)(d-i-1) - \sum_{m\geq 3} \nu_m \left(\lceil im/d \rceil - 1\right)\left(m - \lceil im/d \rceil\right),$$

$$n_{\alpha+2}(f) = \binom{d-i-1}{2} - \sum_{m\geq 3} \nu_m \binom{m - \lceil im/d \rceil}{2} - \delta_{i,d},$$

where ν_m is the number of points of multiplicity m in D, and $\delta_{i,d} = 1$ if $i = d$ and 0 otherwise.

Example 4.11. (continuation of previous example) A number $\alpha \in (0,1)$ is a jumping number of f if and only if there is an integer $m \geq 3$ such that $m\alpha \in \mathbb{Z} \cap [2,m)$ and $\nu_m \neq 0$ or there is $i \in \mathbb{Z} \cap [3,d)$ such that $\alpha = \frac{i}{d}$ and $n_\alpha(f) \neq 0$.

Another fact that can be proved using the explicit description of the Hodge filtration on local systems from Theorem 4.6 is a formula, in terms of multiplier ideals, for the Hodge numbers $h^{q,0} = h^{0,q}$ of finite abelian coverings ramified over a given divisor. To get there, we start first with a geometric characterization of finite abelian covers, which is immediate from Theorem 4.2 and Corollary 1.15.

Corollary 4.12. *Let G be a finite abelian group. The equivalence classes of normal G-covers of X unramified above U are into one-to-one correspondence with the subgroups $G^* \subset \mathrm{Pic}^\tau(X,D)$.*

We can now prove the formula for Hodge numbers of finite abelian coverings. This was done by Libgober in the case $X = \mathbb{P}^N$ and proposed by him as a conjecture in the general case we are dealing with.

Theorem 4.13. (Libgober, B.) *Let $\pi : \widetilde{X} \to X$ be a normal G-cover of X unramified above U corresponding to an inclusion $G^* = \{(L_\chi, \alpha_\chi) \mid \chi \in G^*\} \subset \mathrm{Pic}^\tau(X,D)$. Let $H^0(Y, \Omega_Y^q)$ denote the space of global q-forms on a nonsingular model Y of $\widetilde{X}$. Then*

$$H^0(Y, \Omega_Y^q) \cong \bigoplus_{\chi \in G^*} H^{n-q}(X, \omega_X \otimes L_\chi \otimes \mathcal{J}(X, \alpha_\chi \cdot D)).$$

Proof. We have $H^q(Y, \mathcal{O}_Y) = Gr_F^0 H^q(\pi^{-1}U, \mathbb{C})$ by 3.2. This in turn equals

$$Gr_F^0 H^q(U, \pi_*\mathbb{C}) = \bigoplus_{\chi \in G^*} Gr_F^0 H^q(U, \mathcal{V}_\chi),$$

where $\mathcal{V}_\chi$ is represented by (L_χ, α_χ) in $\mathrm{Pic}^\tau(X,D)$. Now we use part (c) of Theorem 4.6. $\square$

Example 4.14. Since we are talking about finite abelian covers, let us indicate how one can prove a fact used in the proof of Theorem 3.8. With the notation of Theorem 3.8, let $I \subset J_d$ and let $W = E_I$. The $\mathbb{Z}/d\mathbb{Z}$-action gives an eigensheaf decomposition

$$p_*\mathcal{O}_{\widetilde{W}} = \bigoplus_{0 \leq j < d} \mathcal{O}_W \otimes \mathcal{O}_Y(\lfloor \frac{j}{d} \mu^* D \rfloor),$$

and the $\mathbb{Z}/d\mathbb{Z}$-action on each term is given by multiplication by $e^{2\pi ij/d}$. To prove this, we use that

$$\pi_*\mathcal{O}_{\widetilde{X}} = \bigoplus_{\chi\in G^*} L_\chi^{-1},$$

in Theorem 4.13. Now, $\widetilde{Y}$ is the normal $\mathbb{Z}/d\mathbb{Z}$-cover of Y unramified above $Y - E^J$ corresponding to the cyclic subgroup of $\mathrm{Pic}^\tau(Y, E)$ generated by

$$\left(\mathcal{O}_Y\left(-\lfloor\frac{1}{d}\mu^*D\rfloor\right), \left\{\frac{a_i}{d}\right\}_{i\in J}\right).$$

This is almost correct, and it would be enough if Y would be projective. If Y is not projective, we can compactify it by adding a simple normal crossings divisor.

Now that we have formulas for the birational Hodge numbers of finite abelian covers, one can ask what can be said about the behavior of these numbers when we consider the congruence covers. It turns out that they form quasi-polynomial functions.

Definition 4.15. A function f on the set $\{1, 2, 3, \ldots\}$ is a *quasi-polynomial* if there exists a natural number M and polynomials $f_i(x) \in \mathbb{Q}[x]$ for $1 \leq i \leq M$ such that $f(N) = f_i(N)$ if $N \equiv i \pmod M$.

Recall from 1.14 that U_N and X_N are the coverings of U and, respectively, X given by the surjections $H_1(U, \mathbb{Z}) \to H_1(U, \mathbb{Z}/N\mathbb{Z})$. The following was proved for the surface case by E. Hironaka, using intersection theory on singular surfaces, and generalized in [2].

Theorem 4.16. (E. Hironaka, B.) *Let $h^q(N)$ denote the Hodge numbers $h^{q,0} = h^{0,q}$ of any nonsingular model of the congruence cover X_N. Then, for every q, the function $h^q(N)$ is quasi-polynomial.*

The proof involves a multiplier ideal analog of the characteristic varieties, see 1.16. Define

$$V_i^q(X, D) := \{(L, \alpha) \in \mathrm{Pic}^\tau(X, D) : h^q(X, \omega_X \otimes L \otimes \mathcal{J}(\alpha\cdot D)) \geq i\}.$$

Let

$$B(X, D) := \left\{\alpha \in [0, 1)^S : c_1(L) = \alpha\cdot[D] \in H^2(X, \mathbb{R}) \text{ for some } L \in \mathrm{Pic}(X)\right\},$$

so that there is an exact sequence

$$0 \to \mathrm{Pic}^\tau(X) \to \mathrm{Pic}^\tau(X, D) \to B(X, D) \to 0.$$

To prove Theorem 4.16 we use the following quasi-projective version of the structure of characteristic varieties of Theorem 1.17.

Theorem 4.17. (B.) *There exists a decomposition of $B(X, D) \subset \mathbb{R}^S$ into a finite number of rational convex polytopes P such that for every q and i the subset $V_i^q(X, D)$ of $\mathrm{Pic}^\tau(X, D)$ is a finite union of sets of the form $P\times\mathcal{U}$, where $\mathcal{U}$ is a torsion translate of a complex subtorus of $\mathrm{Pic}^\tau(X)$. Any intersection of sets $P\times\mathcal{U}$ is also of this form. Pointwise, the subset of $V_i^q(X, D)$ corresponding to $P\times\mathcal{U}$ consists of the realizations $(L + M, \alpha)$ with $\alpha \in P$ and $M \in \mathcal{U}$, for some line bundle L depending on P which can be chosen such that (L, α) is torsion for some $\alpha \in P$.*

The proof is based on reduction to the projective case. Related results are given by Arapura and Libgober. For more details and for a more general statements, see [2].

Sketch of proof of Theorem 4.16. The subgroup $H_1(U, \mathbb{Z}/N\mathbb{Z})^*$ of $\mathrm{Pic}^\tau(X, D)$ is the N-torsion part. Denote by V_i^q the sets $V_i^q(X, D)$. By Theorem 4.13,

$$\begin{aligned} h^q(N) &= \sum_{(L,\alpha)\in \mathrm{Pic}^\tau(X,D)[N]} h^{n-q}(X, \omega_X \otimes L \otimes \mathcal{J}(X, \alpha \cdot D)) \\ &= \sum_{i\geq 1} i \cdot \# \left[(V_i^{n-q} - V_{i+1}^{n-q})[N] \right], \end{aligned}$$

where $S[N]$ denotes the set of N-torsion elements of $\mathrm{Pic}^\tau(X, D)$ lying in the subset S. Since $V_{i+1}^q \subset V_i^q$,

$$h^q(N) = \sum_{i\geq 1} \# V_i^{n-q}[N].$$

Now, using the structure of V_i^{n-q} from Theorem 4.17, one can boil it down to showing that, for a convex rational polytope Q in $\mathbb{R}^m$, the function

$$f(N) = \# \left[\mathbb{Z}^n \cap NQ \right]$$

is a quasi-polynomial in N. This is a theorem of E. Ehrhart. $\square$

5. Zeta functions

Next, we introduce with the help of log resolutions one more invariant of singularities, the topological zeta function. The connection with Milnor fibers is the statement of the well-known Monodromy Conjecture. We prove it for hyperplane arrangements. In a later section we will return to this invariant and state a stronger conjecture. Then we introduce the K-log canonical threshold, defined for complete fields K of characteristic zero. The p-adic zeta function, which is a p-adic analog of the topological zeta function, is related to this type of log canonical thresholds.

The setup is the familiar one: X is nonsingular complex algebraic variety, Z is a closed subscheme, $\mu : Y \to X$ denotes a log resolution of (X, Z), the scheme-theoretic inverse image of Z is $\sum_{i\in S} a_i E_i$, and $K_{Y/X} = \sum_{i\in S} k_i E_i$. Define for a subset I of the index set S the set $E_I^o := \cap_{i\in I} E_i - \cup_{j\in S\setminus I} E_j$.

Definition 5.1. (Denef-Loeser) The *topological zeta function* of (X, Z) is

$$Z^{top}(X, Z)(s) := \sum_{I\subset S} \chi(E_I^o) \cdot \prod_{i\in I} \frac{1}{a_i s + k_i + 1}.$$

When Z is a hypersurface given by a regular function f, we will use the notation $Z_f^{top}(s)$. The point is that this invariant is well-defined. Denef-Loeser have showed this first by using motivic integration. This a rather mysterious invariant of singularities, and its relation with the Milnor fiber is the subject of the following conjecture. We state it for a hypersurface, but a similar conjecture can be made for closed subschemes. The origin of this conjecture is in number theory and is due to Igusa. This is a geometer-friendly version due to Denef and Loeser, and inspired by a more general, motivic version.

Monodromy Conjecture (topological version). *If $s = c$ is a pole of $Z_f^{top}(s)$ then $e^{2\pi i c}$ is an eigenvalue of the monodromy on $H^i(M_{f,x}, \mathbb{C})$ for some i and some $x \in f^{-1}(0)$.*

A more general conjecture will be stated soon. It is an open question whether $-lct\,(X, Z)$ is the biggest pole of $Z^{top}(X, Z)(s)$. This is shown by Veys for $n = 2$.

Example 5.2. (Docampo) Let M be the space of all matrices with complex coefficients of size $r \times s$, with $r \leq s$. The *k-th generic determinantal variety* is the subvariety Z_k consisting of matrices of rank at most k. Then

$$Z^{top}(M, Z_k)(s) = \prod_{c \in \Omega} \frac{1}{1 - sc^{-1}} \ ,$$

where

$$\Omega = \left\{ -\frac{r^2}{k+1}, -\frac{(r-1)^2}{k}, -\frac{(r-2)^2}{k-1}, \ldots, -(r-k)^2 \right\}.$$

A typical, but limited, tool employed in the study of the poles of the topological zeta function is the following.

Definition 5.3. Let f be a hypersurface germ at the origin in $\mathbb{C}^n$. The *monodromy zeta function of f at* 0 is

$$Z(s) := \prod_{j \in \mathbb{Z}} \det(1 - sT, H^j(M_{f,0}, \mathbb{C}))^{(-1)^j}.$$

The *m-th Lefschetz number of f at* 0 is

$$\Lambda(m) := \sum_{j \in \mathbb{Z}} (-1)^j \operatorname{Trace}(T^m, H^j(M_{f,0}, \mathbb{C})).$$

The Lefschetz numbers recover the monodromy zeta function: if $\Lambda(m) = \sum_{i|m} s_i$ for $m \geq 1$, then $Z(s) = \prod_{i \geq 1} (1 - t^i)^{s_i/i}$. And the Lefschetz numbers can be read from a log resolution, see [4]:

Theorem 5.4. (A'Campo)

$$\Lambda(m) = \sum_{a_i | m} a_i \cdot \chi(E_i^o \cap \mu^{-1}(0)),$$

In the remaining part of this section we will prove the following.

Theorem 5.5. (B.-Mustaţă-Teitler) *The Monodromy Conjecture holds for hyperplane arrangements.*

Lemma 5.6. *If $g \in \mathbb{C}[x_1, \ldots, x_m]$ with $\deg g = d$ gives an indecomposable central hyperplane arrangement in $\mathbb{C}^m$, then $\{\exp(2\pi i k/d) \mid k = 1, \ldots, d\}$ is the union of the sets of eigenvalues of the monodromy action on $H^i(M_{g,0}, \mathbb{C})$ with $i = 0, \ldots, m-1$.*

Proof. By 1.7, each eigenvalue of the action of the monodromy on the cohomology of the Milnor fiber is of the form $\exp(2\pi i k/d)$. Conversely, let μ be the canonical log resolution obtained by blowing up the dense edges of g, as in 2.9. Then, since $\mu^{-1}(0) = E_0$, we have by 5.4 that $\Lambda(n)$ equals $d \cdot \chi(U)$ if d divides n, and 0 otherwise, where U is the complement of the zero locus of g in $\mathbb{P}^{m-1}$. Hence the monodromy zeta function of g at 0 is

$$Z(s) = (1 - t^d)^{\chi(U)}.$$

It is known that indecomposability is equivalent to $\chi(U) \neq 0$. Hence for every k, $\exp(2\pi ik/d)$ is an eigenvalue of the monodromy on $H^j(M_{g,0}, \mathbb{C})$, for some j. □

Proof of Theorem 5.5. We explain the case when the arrangement $D = \{f = 0\}$ is central. Using the canonical log resolution as in 2.9, the poles of $Z_f^{top}(s)$ are included in the set

$$\left\{ -\frac{r_V}{a_V} \mid V \in \mathcal{L} \text{ dense} \right\}.$$

If $V \in \mathcal{L}$ is dense, then D_V is indecomposable and is the zero locus of a product f_V of linear forms on $\mathbb{C}^n/V$, with $\deg(f_V) = a_V$. By Lemma 5.6, $\exp(-2\pi i r_V/a_V)$ is an eigenvalue of the monodromy on the Milnor cohomology of f_V at the origin in $\mathbb{C}^n/V$. Now, take a point $p \in V - \cup_{D_i \not\supset V} D_i$. After choosing a splitting of $V \subset \mathbb{C}^n$, we have locally around p, $D = D_V \times V \subset \mathbb{C}^n = \mathbb{C}^n/V \times V$ and $f = f_V \cdot u$, where u is a (locally) invertible function. Hence the Milnor fiber of f_V at the origin is a deformation retract of $M_{f,p}$. Therefore $\exp(-2\pi i r_V/a_V)$ is an eigenvalue of the monodromy on the cohomology of the Milnor fiber of f at p. □

For the rest of this section, we will describe a number theoretic point of view on singularities, the p-adic analog of the topological zeta function. Let us start with introducing a variant of the log canonical threshold.

Let K be a complete field of characteristic zero. For example, K can be $\mathbb{C}, \mathbb{R}, \mathbb{Q}_p$, or a finite extension of $\mathbb{Q}_p$. For a scheme X of finite type over K, we denote by X_K the associated K-analytic space consisting of the set $X(K)$ of K-points of X with the induced topology, together with the sheaf of K-analytic functions of $X(K)$. Let X be a smooth scheme of finite type over K (i.e. X_K is a K-analytic manifold, e.g. $X = K^n$), and Z a closed subscheme. By Hironaka, there exists a K-analytic log resolution $\mu : Y_K \to X_K$. Let $\mu^* Z_K = \sum_i a_i E_{i,K}$ and $K_{Y_K/X_K} = \sum_i k_i E_{i,K}$.

Definition 5.7. The *K-log canonical threshold* of (X, Z) is

$$lct_K(X, Z) = \min_i \left\{ \frac{k_i + 1}{a_i} \mid E_{i,K} \neq \emptyset \right\}.$$

Note that when $K \subset \mathbb{C}$, we have $lct_{\mathbb{C}}(X, Z) = lct(X_{\mathbb{C}}, Z_{\mathbb{C}})$, but in general $lct_{\mathbb{C}}(X, Z) \leq lct_K(X, Z)$.

Example 5.8. Let $f = (x^2 + y^2 + z^2)^2 + (xy^5 + y^6)$. Let X be the affine 3-space and D the divisor given by D. We want to compute $lct_{\mathbb{R}}(X, D)$ and compare it with $lct_{\mathbb{C}}(X, D)$. Blowup the origin. For example, in one affine chart, with $x = t_1 z$ and $y = t_2 z$, the total transform of f is $z^4[(t_1^2 + t_2^2 + 1)^2 + z^2(t_1 + t_2^6)]$. Let E_1 be the exceptional divisor and E_0 be the proper transform of D. Then the total transform of $D_{\mathbb{R}}$ is $E_{0,\mathbb{R}} + 4E_{1,\mathbb{R}}$ and this is a simple normal crossings divisor. The pullback of $dx \wedge dy \wedge dz$ is $z^2 dt_1 \wedge dt_2 \wedge dz$. Hence $lct_{\mathbb{R}}(X, D) = 3/4$. However, $E_{0,\mathbb{C}} + 4E_{1,\mathbb{C}}$ is not a simple normal crossings divisor, and one has to blowup the quadric $x^2 + y^2 + z^2$ inside $E_{1,\mathbb{C}} = \mathbb{P}^2$. Using the vanishing orders of the exceptional divisor of this blowup, we get $lct_{\mathbb{C}}(X, D) = 2/3$ which is $< lct_{\mathbb{R}}(X, D)$.

Let us focus on the affine case $X = \operatorname{Spec} K[x_1, \ldots, x_n]$, where K is a finite extension of $\mathbb{Q}_p$, and Z is given by $f = (f_1, \ldots, f_r)$ with $f_i \in K[x_1, \ldots, x_n]$. Let R be the valuation ring of K, and P be the maximal ideal of R. Let q be the cardinality of the residue field R/P. The absolute value on $X_K = K^n$ is defined by $|z|_K = \max_{1 \le i \le n} q^{-ord(z_i)}$.

Definition 5.9. (Igusa) The *p-adic zeta function of (X, Z) over K* is

$$Z^K(X, Z)(s) := \int_{R^n} |f(x)|_K^s dx,$$

with $s \in \mathbb{C}$, $\operatorname{Re}(s) > 0$, where dx is the Haar measure on K^n normalized such that R^n has measure 1.

Theorem 5.10. (Igusa, Meuser, Veys - Zuniga Galindo) *The p-adic zeta function $Z^K(X, Z)(s)$ admits a meromorphic continuation to the complex plane as a rational function of q^{-s}. The poles have the form*

$$-\frac{k_i + 1}{a_i} - \frac{2\pi\sqrt{-1}}{\log q}\frac{j}{a_i}, \quad j \in \mathbb{Z},$$

where a_i and k_i are the vanishing orders in a K-analytic log resolution of (X_K, Z_K) as in 5.7.

The proof is to use the K-analytic log resolution to reduce the computation of $Z^K(X, Z)(s)$ to the p-adic zeta function of a monomial.

The connection between K-log canonical thresholds and p-adic zeta functions is the following.

Theorem 5.11. (Igusa, Veys - Zuniga Galindo) *The biggest real part of a pole of $Z^K(X, Z)(s)$ is $-lct_K(X, Z)$.*

The proof is to show that $q^{lct_K(X,Z)}$ is the radius of convergence of $Z^K(X, Z)(s)$ as a function in q^{-s}. Again, this is reduced to a monomial computation via a K-analytic log resolution.

This result has the consequence that $lct_K(X, Z)$ can be computed by counting solutions modulo prime powers.

Corollary 5.12. *Let N_j be the number of solutions of $f_1(x) \equiv \ldots \equiv f_r(x) \equiv 0$ mod P^j in R/P^j. Then*

$$\limsup_{j \to \infty} (N_j q^{-nj})^{\frac{1}{j}} = q^{-lct_K(X,Z)}.$$

The proof is to notice that $\sum_{j \ge 0} N_j (q^{-n} t)^j = (1 - tZ^K(X, Z)(s))/(1 - t)$, where $t = q^{-s}$.

The original p-adic version of the Monodromy Conjecture, posed by Igusa, says that when Z is a hypersurface given a polynomial $f \in K[x_1, \ldots, x_n]$, if $s = c$ is a pole of $Z^K(X, Z)(s)$, then $exp(2\pi i Re(c))$ is an eigenvalue of the monodromy for f viewed as a polynomial with coefficients in $\mathbb{C}$.

6. B-functions

An algebraic way of obtaining eigenvalues of monodromy on Milnor fibers is via b-functions. We introduce generalized b-functions and discuss their relation with Milnor fibers, topological zeta functions, and multiplier ideals.

Let X be a nonsingular complex variety of dimension n and Z a closed subscheme. We denote by $\mathcal{D}_X$ the sheaf of algebraic differential operators $\mathcal{D}_X$ defined locally as

$$\mathcal{D}_X \underset{loc}{=} \mathbb{C}\left[x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right].$$

Definition/Theorem 6.1. (Bernstein, Sato) If $X = \mathbb{C}^n$ and Z is a hypersurface given by a polynomial f, there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that

$$b(s)f^s = Pf^{s+1}$$

for some operator $P \in \mathcal{D}_X[s]$. The monic polynomial of minimal degree among the $b(s)$ is the *b-function* of f and it is denoted $b_f(s)$.

Example 6.2. (Cayley) This is the oldest example of a nontrivial b-function. Let $f = det(x_{ij})$ be the determinant of an $n \times n$ matrix of indeterminates. Then $b_f(s) = (s+1)\ldots(s+n)$ and the differential operator from the definition of the b-function is $P = det(\partial/\partial x_{ij})$.

Example 6.3. Let $f = x^2 + y^3$. Then $b_f(s) = (s + \frac{5}{6})(s+1)(s+\frac{7}{6})$ and the differential operator from the definition of the b-function is $P = \partial_y^3/27 + y\partial_x^2\partial_y/6 + \partial_x^3 x/8$.

Theorem 6.4. (Malgrange, Kashiwara) *The roots of $b_f(s)$ are negative rational numbers. The set consisting of $e^{2\pi i c}$, where c are the roots of $b_f(s)$, is the set of eigenvalues of the Milnor monodromy at points along Z.*

Let us introduce a more general b-function. Let $f = (f_1, \ldots, f_r)$ be r polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. We define bijective $\mathcal{D}_X$-linear actions $t_1, \ldots, t_r$ on $\mathcal{O}_X[\prod_i f_i^{-1}, s_1, \ldots, s_r]\prod_i f_i^{s_i}$ as follows. For $1 \le i, j \le r$, let t_i act on s_j by leaving it alone if $i \neq j$, and replacing s_i with $s_i + 1$. For example: $t_j \prod_{i=1}^r f_i^{s_i} = f_j \prod_{i=1}^r f_i^{s_i}$. Define $s_{ij} := s_i t_i^{-1} t_j$.

Definition/Theorem 6.5. (B. - Mustaţă - Saito) If $X = \mathbb{C}^n$ and Z is a closed subscheme given by an ideal with generators $f = (f_1, \ldots, f_r)$, for any $g \in \mathcal{O}_X$ there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that

$$b(s_1 + \ldots + s_r) g \prod_{i=1}^r f_i^{s_i} = \sum_{k=1}^r P_k (f_k g \prod_{i=1}^r f_i^{s_i}),$$

for some operators P_k in $\mathcal{D}_X[s_{ij}]_{1 \le i,j \le r}$. The monic polynomial of minimal degree among the $b(s)$ is called the *generalized b-function* and is denoted $b_{f,g}(s)$, or in case $g = 1$, we denote it $b_f(s)$.

It is a fact that the generalized b-functions $b_{f,g}(s)$ do not depend on the choice of generators of the ideal of Z.

Definition/Theorem 6.6. (BMS) Let X be a nonsingular complex variety and Z a closed subscheme. Define the *b-function of* (X, Z)

$$b_{(X,Z)}(s)$$

to be the lowest common multiple of the polynomials $b_f(s)$ obtained from local equations of Z in different charts of X. The polynomial

$$b_Z(s) := b_{(X,Z)}(s - \operatorname{codim}(X, Z))$$

depends only on the scheme Z, and so it deserves the name *b-function of the scheme* Z.

In view of 6.4, the following is a generalization of the Monodromy Conjecture. We refer to the second part of [1] for the current status of these conjectures.

Strong Monodromy Conjecture (topological version). *Let X be a nonsingular complex variety and Z a closed subscheme. If $s = c$ is a pole of the topological zeta function $Z^{top}(X, Z)(s)$, then c is a root of $b_{(X,Z)}(s)$.*

Although b-functions are difficult to compute in general, the set of roots contains jumping numbers. We describe this next, along with the relation between b-functions and multiplier ideals.

Theorem 6.7. (BMS) *Let X be a nonsingular complex variety and Z a closed subscheme. Then the multiplier ideals of Z are locally given by*

$$\mathcal{J}(X, cZ) \underset{loc}{=} \{g \in \mathcal{O}_X \mid c < \alpha \text{ if } b_{f,g}(-\alpha) = 0\}$$

for $c > 0$, where $f = (f_1, \ldots, f_r)$ are local equations for Z in X.

The current proof of this theorem, as well as that of Theorem 6.5, goes through the theory of V-filtrations of Malgrange and Kashiwara, and through log resolutions, and we will not explain it. It would be interesting if a direct analytic proof, not involving log resolutions, can be found.

The following is a generalization from the hypersurface case of results of Kollár and Ein-Lazarsfeld-Smith-Varolin, which were obtained by analytic methods.

Corollary 6.8. *Let X be a nonsingular complex variety and Z a closed subscheme.*

(a) The biggest root of the b-function $b_{(X,Z)}(s)$ is $-lct(X, Z)$.

(b) If $lct(X, Z) \leq c < lct(X, Z) + 1$ and c is a jumping number of Z in X, then

$$b_{(X,Z)}(-c) = 0.$$

Sketch of Proof. We can assume $X = \mathbb{C}^n$ and Z is given by $f = (f_1, \ldots, f_r)$. Part (a) is immediate from Theorem 6.7. For part (b), note that it is enough to prove: if $b_{f,g}(-c) = 0$ and $c \in [lct(f), lct(f) + 1)$, then $b_f(-c) = 0$. We will point out how the V-filtration helps to prove this.

First, we claim that $b_{f,g}(s)$ is the minimal polynomial of the action of $s = s_1 + \ldots + s_r$ on

$$\mathcal{D}(g \prod_i f_i^{s_i}) / \sum_{k=1}^{r} t_k \mathcal{D}(g \prod_i f_i^{s_i}),$$

where $\mathcal{D} := \mathcal{D}_X[s_{ij}]_{1\leq i,j\leq r}$. This is a quotient of subspaces of $\mathcal{O}_X[\prod_i f_i^{-1}, s_1, \ldots, s_r]\prod_i f_i^{s_i}$. Denote $\mathcal{D}(g\prod_i f_i^{s_i})$ by N_g, and $\sum_{k=1}^r t_k\mathcal{D}(g\prod_i f_i^{s_i})$ by N_g'. We have that

$$\{\sum_k P_k f_k g \prod_i f_i^{s_i} \mid P_k \in \mathcal{D}\} = \{\sum_k P_k t_k g \prod_i f_i^{s_i} \mid P_k \in \mathcal{D}\} = N_g',$$

where the last equality follows from the commutator rules:

$$\begin{aligned} s_{ij}t_k &= t_k s_{ij} \quad \text{ if } i \neq k\,, \\ s_{kj}t_k &= t_k s_{kj} - t_j\,. \end{aligned}$$

Since $g\prod_i f_i^{s_i} \in N_g$, it follows that $b_{f,g}(s)$ divides the minimal polynomial of s on N_g/N_g'. Conversely, the commutator rule

$$s\cdot s_{ij} = s\cdot s_i t_i^{-1}t_j = s_{ij}\cdot(s_1 + \ldots + (s_i+1) + \ldots + (s_j-1) + \ldots s_r) = s_{ij}\cdot s$$

implies that $b_{f,g}(s)N_g \subset N_g'$. Hence the minimal polynomial of s on N_g/N_g' divides $b_{f,g}(s)$, and so they must be equal.

Denote by N and N' the spaces N_1 and N_1', respectively. We have an exact sequence of quotients of subspaces of N with an action of s that admits minimal polynomials:

$$0 \to \frac{N_g\cap N'}{N_g'} \to \frac{N_g}{N_g'} \to \frac{N}{N'} \tag{2}$$

Let $c_0 = lct\,(X,Z)$ and $c \in [c_0, c_0+1)$ such that $b_{f,g}(-c) = 0$. We need to show that $s+c$ does not divide the minimal polynomial of s on the space on the left. This is provided by the V-filtration as below. In imprecise terms, we could summarize this as saying that there exists a lift to N of the Jordan decomposition of s on N/N'.

Define a decreasing filtration $V^\alpha(N/N')$ of N/N', indexed by the roots α of $b_f(-s)$, by

$$V^\alpha(N/N') = \{n \in N/N' \mid \prod_{\beta\geq\alpha}(s+\beta)n = 0\},$$

where the product is over roots $\beta \geq \alpha$ of $b_f(-s)$ taken with their multiplicity. In particular $N/N' = V^{c_0}(N/N')$ by part (a). Similarly, using just like here the minimal polynomials of s, we have filtrations V^α on the other two quotients in (2).

The crucial point is that there exists a lift of this filtration to N. More precisely, there is a decreasing filtration $V^\alpha N$, parametrized by a discrete set of positive rational numbers α, such that:

- $s+\alpha$ is nilpotent on $Gr_V^\alpha N = V^\alpha N/V^{>\alpha}N$,
- $t_k V^\alpha N \subset V^{\alpha+1}N$, and
- $V^\alpha(N/N') = V^\alpha N/N' \cap V^\alpha N$.

We have then that $N = V^{c_0}N$ and $N' \subset V^{c_0+1}N$. For a subspace $M \subset N$, let $V^\alpha M = M \cap V^\alpha N$. Since $N_g' \subset N'$, we have $N_g' \subset V^{c_0+1}(N_g)$. Then

$$Gr_V^c\left(\frac{N_g\cap N'}{N_g'}\right) = Gr_V^c\,(N_g\cap N') = 0.$$

This means that $s+c$ does not divide the minimal polynomial of s on $N_g\cap N'/N_g'$, which is what we wanted to show. $\square$

Remark 6.9. The original p-adic version of the Strong Monodromy Conjecture can be stated using the real parts of poles of p-adic zeta functions. According to 5.11, a particular open case of this p-adic version is that

$$b_{(X,Z)}(-lct_K(X,Z)) = 0,$$

where K is any p-adic field, with a fixed embedding in $\mathbb{C}$, such that (X, Z) is defined over K. When $K = \mathbb{C}$, this is true as we have seen. When $K = \mathbb{R}$, this is also true, by M. Saito.

Remark 6.10. It is known that b-functions and $\mathcal{D}$-modules are suitable for algorithms based on Gröbner bases. Because of this, Theorem 6.7 is at the heart of all implemented algorithms for computation without case restriction of multiplier ideals, jumping numbers, and non-hypersurface log canonical thresholds, cf. [1].

Let us come back to the Strong Monodromy Conjecture and look at the case of hyperplane arrangements. For the various definitions see 1.10, 2.9.

Conjecture 6.11. (B. - Mustaţă - Teitler) *If $g \in \mathbf{C}[x_1, \dots, x_m]$ is a polynomial of degree d, such that $(g^{-1}(0))_{red}$ is an indecomposable, essential, central hyperplane arrangement, then $-\frac{m}{d}$ is a root of the Bernstein-Sato polynomial $b_g(s)$.*

Theorem 6.12. (BMT) *Let $f \in \mathbf{C}[x_1, \dots, x_n]$ define a hyperplane arrangement in $\mathbb{C}^n$. If Conjecture 6.11 holds for all f_V, where V are the dense edges, then the Strong Monodromy Conjecture holds for f.*

Proof. We explain the central case. We assume that for all $m, d > 0$, and for all polynomials g of degree d in m variables defining an indecomposable, essential, central hyperplane arrangement, we have $b_g(-m/d) = 0$. As in the proof of Theorem 5.5, the candidate poles of $Z_f^{top}(s)$ are $-r_V/a_V$, where V are the dense edges. Then $r_V/a_V = \dim(\mathbb{C}^n/V)/\deg f_V$, with f_V indecomposable (and automatically essential and central) in $\mathbb{C}^n/V$. By assumption, we have $b_{f_V}(-r_V/a_V) = 0$. On the other hand, $b_{f_V}(s)$ equals the local b-function of f in a small neighborhood of a point p as in the proof of Theorem 5.5. But $b_f(s)$ is the least common multiple of the local b-functions. Hence every candidate pole as above is a root of $b_f(s)$. $\square$

We will not prove the strongest results known for hyperplane arrangements about the Strong Monodromy Conjecture or Conjecture 6.11, see [1]. However, a lot examples are provided by the following combinatorial condition.

Definition 6.13. A hyperplane arrangement f is of *moderate type* if the following condition is satisfied:

$$\frac{r_V}{a_V} \leq \frac{r_{V'}}{a_{V'}} \quad \text{for any two dense edges } V \subset V'. \tag{3}$$

Theorem 6.14. (Saito) *If f is a hyperplane arrangement of moderate type, then for any dense edge V, $-r_V/a_V$ is a root of $b_f(s)$. So Conjecture 6.11 and the Strong Monodromy Conjecture hold in this case.*

Proof. We will show that r_V/a_V is a jumping number of f for any dense edge V. For any given dense edge V, we may assume that there is strict inequality in (3) by replacing V with an edge containing it without changing r_V/a_V. The conclusion follows directly from the formula for multiplier ideals in terms of dense edges of 2.9. $\square$

Remark 6.15. Even in the case of hyperplane arrangements defined by a reduced polynomial g, in general it is not the case that one can prove Conjecture 6.11 by showing that m/d is a jumping number for the multiplier ideals of g. For example, if $g = xy(x-y)(x+y)(x+z)$, then $3/5$ is not a jumping number as it follows from 4.11. However, this arrangement is decomposable and the Strong Monodromy Conjecture actually holds. In order to get an indecomposable example, we consider an arrangement with $d = 10$, $\nu_7 = 1$, $\nu_3 = 3$, and with $\nu_{m'} = 0$ for $7 \neq m' > 3$, recalling the notation of 4.11. In this case we see by 4.11 that $3/10$ is not a jumping number. For example, we may take

$$g = xy(x+y)(x-y)(x+2y)(x+3y)(x+4y)(2y+z)(x+2y+z)z.$$

On the other hand, $-3/10$ is indeed a root of $b_g(s)$, since Conjecture 6.11 holds for reduced arrangements in three variables, cf. [1].

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