The Largest Cartesian Closed Category of Domains, Considered Constructively

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Abstract

A conjecture of Smyth [10] is discussed which says that if D and $[D \to D]$ are effectively algebraic directed-complete partial orders with least element (cpo's), then D is an effectively strongly algebraic cpo, where it was left open what exactly is meant by an effectively algebraic and an effectively strongly algebraic cpo.

First, notions of an effectively strongly algebraic cpo and an effective SFP object are introduced. The effective SFP objects are just the constructive (computable) objects in the effectively given category [9] of indexed ω-algebraic cpo's.

Theorem *Every effective SFP object is an effectively strongly algebraic cpo, and vice versa. Moreover, this equivalence holds effectively.*

This shows that the given notion of an effective SFP object is stable. In effectivity considerations of ω-algebraic cpo's it is usual to require that the partial order be decidable on the compact elements. Here, we use a stronger assumption.

Theorem *If D is an indexed ω-algebraic cpo that has a computable completeness test and $[D \to D]$ is an ω-algebraic cpo, then D is an effective SFP object.*

An ω-algebraic cpo has a computable completeness test, if there is a procedure which decides for any two finite sets X and Y of compact cpo elements whether X is a complete set of upper bounds of Y. It is an open question whether this requirement can be weakened in the above result.

Corollary *The category of effective SFP objects and continuous maps is the largest Cartesian closed full subcategory of the category of ω-algebraic cpo's that have a computable completeness test.*

Next, it is studied whether this result also holds in a constructive framework (or, to be more precise, in the framework of recursive mathematics), where one considers categories with constructive domains as objects, that is, domains consisting only of the constructive (computable) elements of an indexed ω-algebraic cpo, and computable maps as morphisms. The notions of a weakly indexed full subcategory and of being constructively Cartesian closed are introduced. The effectivity requirements in these definitions are very weak.

Theorem *The category of constructive SFP domains is the largest constructively Cartesian closed weakly indexed full subcategory of the category of constructive domains that have a computable completeness test.*

Constructive (effective) versions of domain-theoretic results are very important both for the foundations of computer science as well as for applications, since programming languages specify *computable* maps and *computable* (*effectively given*) data structures. Moreover, effective versions of classical results are good approximations to what can be proved constructively and such results have turned out to be at the heart of computer science, at least under the viewpoint of developing correct programs. In this respect the results of the paper are relevant to the workshop.

Keywords: Effectively given domains, SFP domains, largest Cartesian closed category of domains

1 Introduction

In his seminal paper [10] Smyth showed that the category **SFP** introduced by Plotkin [8] is the largest Cartesian closed category of domains, thus confirming a conjecture of Plotkin. In this paper we treat Plotkin's conjecture for the case of effectively given domains.

For various reasons one mostly uses the term *domain* to mean ω-algebraic directed-complete partial order with least element in studies of programming language semantics. Unfortunately, the class of domains is not closed under an important construction needed e.g. for the interpretation of higher-type procedures: the space $[D \to E]$ of continuous maps between two domains D and E must not be a domain again.

To circumvent this problem, people often restrict to the bounded-complete domains, the class of which is closed under the function space construction. However, also this class is not closed under all constructions needed in semantics: the Plotkin or convex powerdomain of a bounded-complete domain is not, in general, bounded-complete. Powerdomains are used for the interpretation of nondeterministic programs. Plotkin therefore introduced the larger class of SFP domains and showed that it is closed under the construction of his powerdomain as well as the function space. Moreover, he conjectured that if D and $[D \to D]$ are domains, then D is SFP. The conjecture, proved by Smyth, indicates that the category **SFP** of SFP domains and continuous maps is the largest category of domains closed under the constructions of interest.

The question which category has to be considered instead of **SFP**, if the term *domain* is allowed to mean some more general kind of directed-complete partial order, has extensively been studied by Jung [3, 4, 5, 1]. If instead of the space of continuous maps one confines to the space of stable maps, the corresponding problem has been dealt with by Amadio [2] and Zhang [12].

In his paper Smyth conjectured that with respect to a natural notion of effectively algebraic and of effectively strongly algebraic cpo the following statement be true: If D and $[D \to D]$ are effectively algebraic cpo's, then D is an effectively strongly algebraic cpo. The study of effectiveness is important in a theory of the foundations of programming. "One reason", said Smyth [9], "has to do with the systematic study of the power of specification techniques. We cannot require of a general purpose programming language that it be able to specify (define) all number-theoretic functions, but only (at most) those which are partial recursive. A corresponding distinction must be made for all the 'data types' which one may wish to handle. And the problem is not simply that of picking out the computable functions over a given data type; we have the problem of specifying the data types themselves, and thus of determining the 'computable', or effectively given, data types (i.e. the types which should in principle be specifiable)."

We first introduce the notions of an effectively strongly algebraic domain and of an effective SFP domain and show that their (effective) equivalence, which shows that we have obtained a stable effectivity notion for SFP domains.

Plotkin introduced SFP domains as colimits of ω-chains of finite domains with embeddings as connecting morphisms. Then he proved that they are exactly the strongly algebraic domains, that is, those domains for which for any finite set X of compact elements, the least set containing X and closed under the operation of taking all minimal upper bounds of subsets of X is finite. Here, we encode the finite domains and the embeddings between them in a canonical way and consider effective ω-chains. These are such that for a given natural number n one can compute both the index of the nth domains and the index of the nth embedding in the chain. Effective SFP domains are then defined to be colimits of such effective chains. An effectively strongly algebraic domain is a strongly algebraic domain which has an indexing of its compact elements such that for any finite set X of compact elements a canonical index of the set of its minimal upper bounds can be computed from a canonical index of X.

Effective SFP domains have also been studied by Kanda in his dissertation [7]. But whereas in the present paper the effective SFP domains are the constructive objects of the category of indexed domains (that is, domains which come with a fixed numbering of their compact elements), in the sense that each object can be constructed in an effective way from its finite parts, this is not the case in Kanda's treatment, as he does not code the finite domains by canonical or explicit indices, from which the domains can easily be recovered. Instead he codes finite domains in the same way as effectively given domains in general. This coding contains only partial information about the domain. (See also the remark of Smyth in [11, Section 5].)

In effectivity considerations of domains it is usual to require that the domain order be decidable on the compact elements. Here, we use a stronger requirement. A domain is said to have a computable completeness test if there is a procedure which decides for any two finite sets X and Y of compact elements whether X is a complete set of upper bounds of Y. We show that if D and $[D \to D]$ are domains such that D has a computable completeness test, then D is an effective SFP domain. It is not known, whether the condition of having a computable completeness test can be weakened in this result. As in Smyth [10] it follows that the category of effective SFP domains and continuous maps is the largest Cartesian closed full subcategory of the category of domains having a computable completeness test.

Next, it is studied whether this result also holds in a constructive framework, or, to be more precise, in the framework of recursive mathematics. Here, one considers categories with constructive domains as objects, that is, domains consisting only of the constructive (computable) elements of an indexed ω-algebraic cpo, and computable maps as morphisms. It is shown that the category of constructive SFP domains is the largest constructively Cartesian closed weakly indexed full subcategory of the category of constructive domains having a computable completeness test.

The effectivity requirements that have to be satisfied by the category are rather weak compared with the conditions considered by Kanda [6] and Smyth [9] in their approaches to effectiveness in categories. We only require that for any two objects the corresponding morphism set is indexed in such a way that the universality statement in the definition of a categorical product holds effectively.

The rest of this paper is organized as follows. In Section 2 basic definitions and results from domain theory are given. Section 3 is its effective counterpart. Here, the definition of an effectively given SFP domain is given and some properties are derived. Smyth's conjecture is treated in Section 4.

2 Domains

Let $(D, \sqsubseteq)$ be a partial order with smallest element $\perp$. For a subset S of D, $\downarrow S = \{ x \in D \mid (\exists y \in S) x \sqsubseteq y \}$ is the *lower set* generated by S. The subset S is called *compatible* if it has an upper bound. S is *directed*, if it is nonempty and every pair of elements in S has an upper bound in S. D is a *directed-complete* partial order (cpo) if every directed subset S of D has a least upper bound $\bigsqcup S$ in D, and D is *bounded-complete* if every compatible subset has a least upper bound in D.

An element x of a cpo D is *compact* if for any directed subset S of D the relation $x \sqsubseteq \bigsqcup S$ always implies the existence of an element $u \in S$ with $x \sqsubseteq u$. We write D^0 for the set of compact elements of D. If D^0 is countable and for every $y \in D$ the set $\downarrow\{y\} \cap D^0$ is directed and $y = \bigsqcup \downarrow\{y\} \cap D^0$, the cpo D is *ω-algebraic* or, as we prefer to say, a *domain*. A standard reference for domain theory is [1].

The *product* $D \times E$ of two cpo's D and E is the Cartesian product of the underlying sets ordered coordinatewise. Obviously, $D \times E$ is a domain again with $(D \times E)^0 = D^0 \times E^0$, if D and E are domains.

Definition 2.1
A map $F\colon D \to E$ between cpo's D and E is *continuous* if it is monotone and for any directed subset S of D,

$$F(\bigsqcup S) = \bigsqcup F(s).$$

Let $[D \to E]$ denote the set of all continuous maps from D to E. Endowed with the *pointwise order*, that is $F \sqsubseteq G$ if $F(x) \sqsubseteq G(x)$, for all $x \in D$, it is a cpo again, but in general it need not be a domain. This means that the category **DOM** of domains and continuous maps is not Cartesian closed. Therefore one considers subclasses of domains which have this property, when using domains in programming language semantics, *e.g.* SFP domains. To introduce this kind of domains we need the following definitions.

Definition 2.2
An *embedding/projection* (F, G) from a cpo D to a cpo E is a pair of maps $F \in [D \to E]$ and $G \in [E \to D]$ such that $G \circ F = \mathrm{Id}_D$, the identity map on D, and $F \circ G \sqsubseteq \mathrm{Id}_E$. The map F is called *embedding* and G *projection.*

Note that the map G is uniquely determined by F, and vice versa [11]. Therefore, we also write F^R instead of G. Embeddings are one-to-one and preserve compactness [8].

Lemma 2.3
Let D and E be domains and $F\colon D \to E$. Then F is an embedding if and only if there is a monotone and one-to-one map $F_0\colon D^0 \to E^0$ such that for all $y \in E$ and all $u, u' \in D^0$, if $F_0(u), F_0(u') \sqsubseteq y$ then there exists some $\bar{u} \in D^0$ so that $u, u' \sqsubseteq \bar{u}$ and $F_0(\bar{u}) \sqsubseteq y$.

Suppose (F, F^R) is an embedding/projection from C to D and (G, G^R) is an embedding/projection from D to E. Then the composition of (F, F^R) and (G, G^R) is defined by

$$(G, G^R) \circ (F, F^R) = (G \circ F, F^R \circ G^R).$$

Let $\mathbf{DOM^e}$ denote the category of domains and embeddings.

By an ω-chain in $\mathbf{DOM^e}$ we understand a diagram of the form $\mathcal{D} = D_0 \overset{F_0}{\to} D_1 \overset{F_1}{\to} \dots$ (that is, a functor from ω to $\mathbf{DOM^e}$). As is well known, the category $\mathbf{DOM^e}$ is ω-cocomplete: every ω-chain in $\mathbf{DOM^e}$ has a colimit. Up to isomorphism this given by the set

$$D_\infty = \{\, x \in \Pi_{m\in\omega} D_m \mid (\forall m \in \omega) x_m = F_m^R(x_{m+1}) \,\}$$

endowed with the componentwise partial order, that is

$$x \sqsubseteq y \Leftrightarrow (\forall m \in \omega) x_m \sqsubseteq_{D_m} y_m.$$

Note that

$$D_\infty^0 = \{\, u \in D_\infty \mid (\exists m \in \omega) u_m \in D_m^0 \wedge (\forall n \geq m) u_{n+1} = F_n(u_n) \,\}.$$

Definition 2.4
An *SFP domain* is a colimit of an ω-chain in $\mathbf{DOM^e}$, where all domains in the chain are finite.

In [8] Plotkin gave an alternative, purely order-theoretic characterization of SFP domains, which is quite useful in many cases.

Definition 2.5
Let D be a partial order, X be a subset of D and $\mathrm{UB}(X)$ be the set of all upper bounds of X.

1. An element x of D is a *minimal upper bound* of X if it is an upper bound of X and it is nor strictly greater than any other upper bound of X.
2. A subset Y of $\mathrm{UB}(X)$ is *complete* for X if whenever $x \in \mathrm{UB}(X)$, then $x \sqsupseteq y$ for some $y \in Y$.

Let $\mathcal{U}(X)$ be the set of minimal upper bounds of X. Then $\mathcal{U}(X)$ is contained in every subset Y of $\mathrm{UB}(X)$ that is complete for X. Define $\mathcal{U}^*(X)$ to be the least set containing X and closed under $\mathcal{U}$. Then a domain D is called *strongly algebraic* if for each finite subset X of D^0, $\mathcal{U}(X)$ is complete and $\mathcal{U}^*(X)$ is finite.

Theorem 2.6
A domain is an SFP domain if and only if it is strongly algebraic.

As is well known, any partially ordered D set may be viewed as a category. The objects of the category are the elements of D and the set of morphisms between two objects x and y is a one-point set precisely when $x \sqsubseteq y$ and the empty set otherwise. Under this view-point ω-chains correspond to infinite increasing sequences and the colimits of such chains to least upper bounds of the sequences. Having this analogy in mind it is natural to ask what are the compact or finitary objects of a category.

Definition 2.7
An object A of a category $\mathbf{K}$ is *finitary* in $\mathbf{K}$ provided that, for any ω-chain $\mathcal{K} = (V_n, F_n)_{n\in\omega}$ in $\mathbf{K}$ with colimit $\big(V, (G_n)_{n\in\omega}\big)$, the following holds: for any morphism $H \in \mathbf{K}[A, V]$, and for sufficiently large n, there is a unique morphism $K \in \mathbf{K}[A, V_n]$ such that $H = G_n \circ K$.

Obviously, the finitary objects in $\mathbf{DOM^e}$ are just the finite domains.

3 Effectively given domains

In what follows, let $\langle\ ,\ \rangle : \omega^2 \to \omega$ be a recursive pairing function with corresponding projections π_1 and π_2 such that $\pi_i(\langle a_1, a_2\rangle) = a_i$, and let Δ be a standard coding of all finite subsets of natural numbers. We extend the pairing function in the usual way to an n-tuple encoding. Moreover, let $P^{(n)}$ ($R^{(n)}$) denote the set of all n-ary partial (total) recursive functions, and let W_i be the domain of the ith partial recursive function φ_i with respect to some Gödel numbering φ. We let $\varphi_i(a)\downarrow$ mean that the computation of $\varphi_i(a)$ stops and $\varphi_i(a)\downarrow \in C$ that it stops with value in C.

Let S be a nonempty set. A *(partial) numbering* ν of S is a partial map $\nu \colon \omega \rightharpoonup S$ (onto) with domain $\mathrm{dom}(\nu)$. The value of ν at $n \in \mathrm{dom}(\nu)$ is denoted, interchangeably, by ν_n and $\nu(n)$. The pair (S, ν) is called *numbered set*. Note that instead of numbering and numbered set, respectively, we also say *indexing* and *indexed set*.

Definition 3.1
Let ν and κ be numberings of the set S.

1. $\nu \leq \kappa$, read ν is *reducible* to κ, if there is some function $g \in P^{(1)}$ with $\mathrm{dom}(\nu) \subseteq \mathrm{dom}(g)$, $g(\mathrm{dom}(\nu)) \subseteq \mathrm{dom}(\kappa)$, and $\nu_m = \kappa_{g(m)}$, for all $m \in \mathrm{dom}(\nu)$.
2. $\nu \equiv \kappa$, read ν is *equivalent* to κ, if $\nu \leq \kappa$ and $\kappa \leq \nu$.

A map $F \colon S \to S'$ from a numbered set (S, ν) to a numbered set (S', ν') is *effective* if there is some function $f \in P^{(1)}$ such that $f(i)\downarrow \in \mathrm{dom}(\nu')$ and $F(\nu_i) = \nu'_{f(i)}$, for all $i \in \mathrm{dom}(\nu)$.

The following definition is essentially due to Smyth [9].

Definition 3.2
An *effectively given* category is a category $\mathbf{K}$ together with a total indexing ϱ of its finitary objects and a total indexing ϑ of the morphisms between finitary objects such that the following conditions hold:

1. The set $\{\, \langle m, n\rangle \mid \varrho_m = \varrho_n \,\}$ is recursive.
2. The set $\{\, m \in \omega \mid \vartheta_m$ is an identity morphism$\,\}$ is recursive.
3. There are functions $d, c \in R^{(1)}$ such that $\varrho_{d(m)}$ and $\varrho_{c(m)}$, respectively, are the domain and codomain of ϑ_m.
4. There is a function $\mathrm{comp} \in P^{(2)}$ such that for all $m, n \in \omega$ for which the codomain of ϑ_m is the domain of ϑ_n, $\mathrm{comp}(m, n)\downarrow$ and $\vartheta_n \circ \vartheta_m = \vartheta_{\mathrm{comp}(m,n)}$.

An ω-chain $(A_m, F_m)_{m\in\omega}$ of finitary objects in $\mathbf{K}$ is *effective* if there is a function $t \in R^{(1)}$ such that $A_m = \varrho_{\pi_1(t(m))}$ and $F_m = \vartheta_{\pi_2(t(m))}$, for all $m \in \omega$. A *constructive* object A of $\mathbf{K}$ is then a colimit of an effective ω-chain of finitary objects in $\mathbf{K}$.

We have already seen that the finite domains are just the finitary objects of the category $\mathbf{DOM^e}$. The same holds if we confine to the subcategory $\mathbf{IDOM^{ce}}$ of indexed domains and computable embeddings. Here, an *indexed domain* (D, δ) is a domain D with a fixed total numbering δ of its compact elements.

Definition 3.3
Let (D, δ) and (E, ε) be indexed domains. A map $F \in [D \to E]$ is *computable* if the set $\{\, \langle i, j\rangle \mid \varepsilon_j \sqsubseteq F(\delta_i) \,\}$ is recursively enumerable (r.e.).

The numbering of the compact elements is used to impose certain effectivity requirements on these elements. A condition that we shall always use is the decidability of the domain order.

Definition 3.4
A domain D with a total numbering δ of its compact elements is *effectively given* if the set $\{\, \langle i, j\rangle \mid \delta_i \sqsubseteq \delta_j \,\}$ is recursive.

Note that an embedding from an effectively given domain into another effectively given domain is computable exactly if its restriction to the compact elements is effective. Moreover, a Gödel number of the function witnessing effectivity can be computed from an index of the r.e. set witnessing the computability of the embedding, and vice versa.

If (D,δ) and (E,ε) are indexed domains, define the numbering $\delta\times\varepsilon$ by $(\delta\times\varepsilon)_{\langle i,j\rangle}=(\delta_i,\varepsilon_j)$. Then $(D\times E,\delta\times\varepsilon)$ is an indexed domain again. If (D,δ) and (E,ε) are effectively given, the same is true for $(D\times E,\delta\times\varepsilon)$.

Domain elements of particular interest are those which can be approximated effectively.

Definition 3.5
Let (D,δ) be an indexed domain. An element x of D is called *constructive* if the set $\{\, i\in\omega \mid \delta_i\sqsubseteq x \,\}$ is r.e.

Observe that computable maps map constructive elements to constructive elements. We denote the set of constructive elements of D by D_c. With respect to the restriction of the domain order it is a partial order, which we call *constructive domain*.

Let us now introduce canonical indexings of the finite domains and the embeddings between finite domains. In order not to have to deal with isomorphic copies we consider only finite domains that have natural numbers as elements. For $m\in\omega$ set

$$E_m=\{\,\langle m,a\rangle \mid a\in\pi_1(\Delta_m)\cup\pi_2(\Delta_m)\,\}$$

and order it by

$$\langle m,a\rangle\sqsubseteq_m\langle m,b\rangle\Leftrightarrow\langle a,b\rangle\in\Delta_m.$$

In case that E_m is a partial order with smallest element, all elements are compact. We enumerate them in the following way:

$$\eta^m_a=\begin{cases}\langle m,a\rangle & \text{if } \langle m,a\rangle\in E_m,\\ n_\perp & \text{otherwise,}\end{cases}\qquad (a\in\omega).$$

Here $n_\perp$ is the smallest element of E_m. Then (E_m,η^m) is an effectively given domain. Now, let $\zeta_m=(E_m,\eta^m)$, if E_m is a partial order with smallest element, and let $\zeta_m=(\{0\},\{\langle 0,0\rangle\},\lambda a.0)$, otherwise. Moreover, for natural numbers $\langle m,i,n\rangle$ such that there is some embedding $F\in[E_m\to E_n]$ with $\Delta_i=\{\,\langle a,b\rangle \mid \eta^n_b=F(\eta^m_a)\,\}$, define $\theta_{\langle m,i,n\rangle}=F$. In any other case set $\theta_{\langle m,i,n\rangle}=\mathrm{Id}_{\{0\}}$. Then ζ and θ, respectively, are numberings of the finite domains and the embeddings between these such that the category $\mathbf{IDOM^{ce}}$ is effectively given.

Definition 3.6
An effective SFP domain is a colimit of an effective ω-chain in $\mathbf{IDOM^{ce}}$.

Let $\mathcal{D}=\big((D_m,\delta^m),F_m\big)_{m\in\omega}$ be an effective ω-chain in $\mathbf{IDOM^{ce}}$. Set $F_{mn}=F_{n-1}\circ\cdots\circ F_m$, for $m<n$, and $F_{mm}=\mathrm{Id}_{D_m}$. Moreover, let $\mathrm{in}_m\colon D_m\to D_\infty$, defined by

$$\mathrm{in}_m(x)(n)=\begin{cases}F_{mn}(x) & \text{if } m\le n,\\ F^R_{nm}(x) & \text{otherwise,}\end{cases}$$

for $x\in D_m$, be the canonical embedding of D_m into D_∞. For $\langle m,a\rangle\in\omega$ set $\delta^\infty_{\langle m,a\rangle}=\mathrm{in}_m(\delta^m_a)$. Then δ^∞ is an indexing of D^0_∞ such that D_∞ is effectively given.

If D is a colimit of $\mathcal{D}$ there is a computable isomorphism $H\in[D\to D_\infty]$. Isomorphisms are embeddings and as we have seen in Lemma 2.3, these are determined by their values on the compact elements. Moreover, they are computable just if their restriction to the computable elements is effective. Let φ_i and φ_j witness that the restrictions of H and H^{-1}, respectively, to D^0 and D^0_∞ are effective. Moreover, let φ_c witness that the ω-chain $\mathcal{D}$ is effective. Then $\langle i,j,c\rangle$ is an *index* of D. This defines a partial indexing σ of the effective SFP domains.

As we shall see next, Plotkin's order-theoretic characterization of the SFP domains also holds in the effective setting.

Definition 3.7
A domain D with total numbering δ of its compact elements is *effectively strongly algebraic* if it is strongly algebraic and the operation $\mathcal{U}$ is effective, that is, there is some function $g\in R^{(1)}$ such that $\mathcal{U}(\delta(\Delta_i))=\delta(\Delta_{g(i)})$, for all $i\in\omega$.

Since $\delta_m \sqsubseteq \delta_n$ if and only if $\delta_n \in \mathcal{U}(\{\delta_m, \delta_n\})$, it follows that every effectively strongly algebraic domain (D, δ) is effectively given. If i is a Gödel number of the function g witnessing the effectivity of $\mathcal{U}$, then i is called an *index* of D. Let τ denote the indexing of the effectively strongly algebraic domains thus obtained.

Theorem 3.8
Every effective SFP domain is an effectively strongly algebraic domain, and vice versa. Moreover, this equivalence holds effectively, that is, $\sigma \equiv \tau$.

In the introduction it has already been mentioned that in his dissertation [7] Kanda studied SFP domains in an effective setting. But he does not work with canonical indexings of the finite domains and embeddings. As a consequence of this, the numbering of his effective SFP domains is weaker than the one used here. He has no equivalence between the numberings of the effective SFP domains and the effectively strongly algebraic domains, respectively, as above.

As is well known, the category **SFP** of SFP domains and continuous maps is Cartesian closed: the one-point domain $\{\bot\}$ is the terminal object, the domain product is the categorical product and the space of continuous maps between two SFP domains is the categorical exponent. Note that for two SFP domain D and E, $[D \to E]$ is an SFP domain again.

Definition 3.9
Let D and E be SFP domains. A finite subset T of $D^0 \times E^0$ is called *joinable* if

$$(\forall T' \subseteq T)[(\forall u \in \mathcal{U}^D(\mathrm{pr}_1(T')))(\exists v \in \mathcal{U}^E(\mathrm{pr}_2(T')))(u, v) \in T].$$

Here pr_i is the projection onto the ith component.

For elements $u \in D^0$ and $v \in E^0$ define the *step function* $(u \searrow v)\colon D \to E$ by

$$(u \searrow v)(x) = \begin{cases} v & \text{if } u \sqsubseteq x, \\ \bot & \text{otherwise,} \end{cases} \qquad (x \in D).$$

Then the compact elements of $[D \to E]$ are exactly the maps of the form $\bigsqcup\{\, (u_i \searrow v_i) \mid i \in I \,\}$, where $u_i \in D^0$ and $v_i \in E^0$, for $i \in I$, so that $\{\, (u_i, v_i) \mid i \in I \,\}$ is joinable.

If (D, δ) and (E, ε) are effective SFP domains, then it follows with Theorem 3.8 that the set $\{\, i \in \omega \mid \{\, (\delta_m, \varepsilon_n) \mid \langle m, n\rangle \in \Delta_i \,\} \text{ is joinable} \,\}$ is recursive. Thus we can define a numbering γ of $[D \to E]^0$ by setting

$$\gamma_i = \begin{cases} \bigsqcup\{\, (\delta_m \searrow \varepsilon_n) \mid \langle m, n\rangle \in \Delta_i \,\} & \text{if } \{\, (\delta_m, \varepsilon_n) \mid \langle m, n\rangle \in \Delta_i \,\} \text{ is joinable,} \\ (\bot_D \searrow \bot_E) & \text{otherwise,} \end{cases}$$

for $i \in \omega$. Then it is easily verified that $([D \to E], \gamma)$ is effectively given. It is even an effective SFP domain. In addition, we have the important property that an element F of $[D \to E]$ is constructive exactly if it is a computable map. Note that F is uniquely determined by its values on the computable elements.

Define a *constructive SFP domain* to be the constructive domain obtained from an effective SFP domain, then we achieve the following result.

Theorem 3.10
The categories **ESFP** of effective SFP domains and continuous maps and **CSFP** of constructive SFP domains and computable maps are both Cartesian closed.

4 The conjecture

In his paper [10] Smyth conjectured that the proof of his Theorem 1 may be used to show that with respect to appropriate effectivity notions the following statement be true:

> If D and $[D \to D]$ are effectively algebraic domains, then D is an effectively strongly algebraic domain.

In effectivity considerations of domains these usually have to be effectively given. The effectivity requirement in the definition of effectively given domains is quite weak. It is not clear to us how the set of minimal least of upper bounds of a finite set of compact elements can be computed in the case of such domains. We therefore strengthen this condition.

Definition 4.1
A domain D with a total numbering δ of its compact elements has a *computable completeness test* if the set

$$\{ \langle i,j \rangle \mid \delta(\Delta_j) \subseteq \mathrm{UB}(\delta(\Delta_i)) \wedge \delta(\Delta_j) \text{ is complete for } \delta(\Delta_i) \}$$

is recursive.

Since $\delta_i \sqsubseteq \delta_j$ if and only if $\{\delta_j\}$ is complete for $\{\delta_i, \delta_j\}$, every domain that has a computable completeness test is effectively given.

Let **EDOMCC** be the category of domains with a computable completeness test and continuous maps and **CDOMCC** the category of constructive domains obtained from domains with a computable completeness test and computable maps. Obviously, the object sets of both categories are closed under the construction of product domains, which is also the categorical product in these domains. Hence, both domains are Cartesian.

Proposition 4.2
1. **ESFP** is a proper full subcategory of **EDOMCC**.
2. **CSFP** is a proper full subcategory of **CDOMCC**.

Proof. Let (D, δ) be an effective SFP domain and note that

$$\begin{aligned}&\delta(\Delta_j) \subseteq \mathrm{UB}(\delta(\Delta_i)) \wedge \delta(\Delta_j) \text{ is complete for } \delta(\Delta_i)\\ &\qquad\qquad \Leftrightarrow \mathcal{U}(\delta(\Delta_i)) \subseteq \delta(\Delta_j) \wedge \delta(\Delta_j) \subseteq \mathrm{UB}(\delta(\Delta_i)).\end{aligned}$$

Since D is effectively strongly algebraic and hence also effectively given, the right hand side of this equivalence is recursive in i and j. Thus, D has a computable completeness test. ∎

Now, we can state and prove our version of Smyth's conjecture.

Theorem 4.3
If D and $[D \to D]$ are domains such that D has a computable completeness test, then D is effectively strongly algebraic.

The second important result in Smyth's paper says that **SFP** is the largest Cartesian closed full subcategory of **DOM**. For the proof he needed the next result.

Lemma 4.4
Let **K** be a full subcategory of the category **CPO** of cpo's and continuous maps. Then the following three statements hold:

1. If **K** has a terminal object T, then T is the one-point cpo.
2. If **K** has a terminal object and the product $A \times_{\mathbf{K}} B$ of objects A and B exists, then $A \times_{\mathbf{K}} B$ is isomorphic in **CPO** to the usual product $A \times B$.
3. If **K** has a terminal object and all products of pairs, and the exponent E^D of objects D and E exists, then E^D is isomorphic in **CPO** to the usual function space $[D \to E]$.

Now note that if D is a domain with a computable completeness test and D is isomorphic to a cpo E, then also E is a domain with a computable completeness test. With Theorem 4.3 we therefore obtain the following analogue of Smyth's result.

Theorem 4.5
ESFP is the largest Cartesian closed full subcategory of **EDOMCC**.

In the rest of this section we deal with the question whether a similar statement is true with respect to **CSFP** and **CDOMCC**. As we shall see such a statement holds, but only in an effective categorical setting. Whereas in the definition of effective SFP domains we used rather strong, though very natural, effectivity requirements, we shall now employ only very weak conditions.

Definition 4.6
Let $\mathbf{K}$ be a category and for any two objects A and B, $\alpha^{A,B}$ be a partial indexing of the morphism set $\mathbf{K}[A,B]$. Then $(\mathbf{K}, (\alpha^{A,B})_{A,B\in\mathrm{Ob}_{\mathbf{K}}})$ is called *weakly indexed*.

Definition 4.7
Let $(\mathbf{K}, (\alpha^{A,B})_{A,B\in\mathrm{Ob}_{\mathbf{K}}}), (\mathbf{K}', (\beta^{A,B})_{A,B\in\mathrm{Ob}_{\mathbf{K}'}})$ be weakly indexed categories and A, B be objects of $\mathbf{K}$.

1. The categorical product $(A \times B, \mathrm{pr}_A, \mathrm{pr}_B)$ of A and B is *constructive* if for any object C of $\mathbf{K}$ there is a function $\mathrm{prod}_C \in P^{(2)}$ such that for all $a \in \mathrm{dom}(\alpha^{C,A})$ and all $b \in \mathrm{dom}(\alpha^{C,B})$, $\alpha^{C,A\times B}_{\mathrm{prod}_C(a,b)}$ is the unique morphism in $\mathbf{K}[C, A\times B]$ with
$$\alpha^{C,A}_a = \mathrm{pr}_A \circ \alpha^{C,A\times B}_{\mathrm{prod}_C(a,b)} \quad \text{and} \quad \alpha^{C,B}_b = \mathrm{pr}_B \circ \alpha^{C,A\times B}_{\mathrm{prod}_C(a,b)}.$$
2. $(\mathbf{K}, (\alpha^{A,B})_{A,B\in\mathrm{Ob}_{\mathbf{K}}})$ is a *full subcategory* of $(\mathbf{K}', (\beta^{A,B})_{A,B\in\mathrm{Ob}_{\mathbf{K}'}})$ if $\mathbf{K}$ is a full subcategory of $\mathbf{K}'$ and for all objects A, B of $\mathbf{K}$, $\alpha^{A,B} \equiv \beta^{A,B}$.

Definition 4.8
A weakly indexed category $\mathbf{K}$ is *constructively Cartesian closed* if $\mathbf{K}$ contains a terminal object and for every pair of objects there is a constructive categorical product and a categorical exponent.

We now have to verify that **CDOMCC** and **CSFP** are weakly effective categories. Let (D,δ) and (E,ε), respectively, be both effective SFP domains or both domains that have a computable completeness. If $F\colon D \to E$ is computable and $W_i = \{\, \langle m,n\rangle \mid \varepsilon_n \sqsubseteq F(\delta_m)\,\}$, then we call i an *index* of the restriction of F to D_c. This defines partial indexings of $\mathbf{CSFP}[D,E]$ and $\mathbf{CDOMCC}[D,E]$, respectively.

Proposition 4.9
1. The category **CDOMCC** is weakly indexed.
2. The category **CSFP** is a constructively Cartesian closed weakly indexed full subcategory of **CDOMCC**.

In the framework of weakly effective categories Lemma 4.4 can be strengthened.

Lemma 4.10
Let $\mathbf{K}$ be a weakly indexed full subcategory of **CDOMCC**. Then the following three statements hold:

1. If $\mathbf{K}$ has a terminal object T, then T is the one-point cpo.
2. Let $\mathbf{K}$ have a terminal object. If the product $A \times_{\mathbf{K}} B$ of objects A and B exists, then $A \times_{\mathbf{K}} B$ is isomorphic in **CDOMCC** to the usual product $A \times B$.
3. Let $\mathbf{K}$ have a terminal object and all products of pairs. If the exponent E^D of objects D and E exists, and $\overline{E^D}$, $\bar{D}$ and $\bar{E}$, respectively, are objects in **EDOMCC** with $E^D = (\overline{E^D})_c$, $D = \bar{D}_c$ and $E = \bar{E}_c$, then $\overline{E^D}$ is isomorphic in **CPO** to the usual function space $[\bar{D} \to \bar{E}]$.

As above we now obtain our constructive analogue of Smyth's second result.

Theorem 4.11
CSFP is the largest constructively Cartesian closed weakly indexed full subcategory of **CDOMCC**.

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