# Some Arithmetic Aggregation Operators with Intuitionistic Trapezoidal Fuzzy Numbers and Their Application to Group Decision Making

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*Abstract*—**Methods for aggregating intuitionistic trapezoidal fuzzy information are investigated. Some operational laws of intuitionistic trapezoidal fuzzy numbers are introduced. Based on these operational laws, some aggregation operators, including intuitionistic trapezoidal fuzzy ordered weighted averaging (ITFOWA) operator and intuitionistic trapezoidal fuzzy hybrid aggregation (ITFHA) operator, are proposed. Properties of these intuitionistic trapezoidal fuzzy information aggregation operators are also analyzed. An approach to multiple attribute group decision making (MAGDM) with intuitionistic trapezoidal fuzzy information is developed based on the ITFWAA and the ITFHA operators. Finally, an illustrative example is given to verify the developed approach and to demonstrate its practicality and effectiveness.**

*Index Terms*—**Multiple attribute group decision making; intuitionistic trapezoidal fuzzy numbers; intuitionistic trapezoidal fuzzy ordered weighted averaging (ITFOWA) operator; intuitionistic trapezoidal fuzzy hybrid aggregation (ITFHA) operator**

## I. Introduction

Atanassov [1,2] introduced the concept of intuitionistic fuzzy set(IFS) characterized by a membership function and a non-membership function, which is a generalization of the concept of fuzzy set [3] whose basic component is only a membership function. The intuitionistic fuzzy set has received more and more attention since its appearance[2-16]. Gau and Buehrer [4] introduced the concept of vague set. But Bustince and Burillo [5] showed that vague sets are intuitionistic fuzzy sets. Xu[6-8] developed some aggregation operators with intuitionistic fuzzy information. Li [9] investigated MADM with intuitionistic fuzzy information and constructed several linear programming models to generate optimal weights for attribute. Lin [10] presented a new method for handling multiple attribute fuzzy decision making problems, where the characteristics of the alternatives are represented by intuitionistic fuzzy sets. The proposed method allows the degrees of satisfiability and non-satisfiability of each alternative with respect to a set of attribute to be represented by intuitionistic fuzzy sets, respectively. Furthermore, the proposed method allows the decision-maker to assign the degree of membership and the degree of non-membership of the attribute to the fuzzy concept “importance.” Liu and Wang [11] developed an evaluation function for the decision making problem to measure the degrees to which alternatives satisfy and do not satisfy the decision maker’s requirement. Then, they proposed the intuitionistic fuzzy point operators, and defined a series of new score functions for the MADM problems based on intuitionistic fuzzy point operators and evaluation function. Later, Atanassov and Gargov [17-18] further introduced the interval-valued intuitionistic fuzzy set (IVIFS), which is a generalization of the IFS. The fundamental characteristic of the IVIFS is that the values of its membership function and non-membership function are intervals rather than exact numbers. Xu[19-20] developed some aggregation operators with interval-valued intuitionistic fuzzy information. Xu [21] investigated the interval-valued intuitionistic fuzzy MADM with the information about attribute weights is incompletely known or completely unknown, a method based on the ideal solution was proposed. Wang[22] investigated the interval-valued intuitionistic fuzzy MADM with incompletely known weight information. A nonlinear programming model is developed. Then using particle swarm optimization algorithms to solve the nonlinear programming models, the optimal weights are gained. And ranking is performed through the comparison of the distances between the alternatives and idea/anti-idea alternative. Shu, Cheng and Chang[23] gave the definition and operational laws of intuitionistic triangular fuzzy number and proposed an algorithm of the intuitionistic fuzzy fault-tree analysis. Wang[24] gave the definition of intuitionistic trapezoidal fuzzy number and interval intuitionistic trapezoidal fuzzy number. Wang and Zhang [25] gave the definition of expected values of intuitionistic trapezoidal fuzzy number and proposed the programming method of multi-criteria decision-making based on intuitionistic trapezoidal fuzzy number with incomplete certain information. Wang and Zhang[26] developed the Hamming distance of intuitionistic trapezoidal fuzzy numbers and intuitionistic trapezoidal fuzzy weighted arithmetic averaging (ITFWAA) operator, then proposed multi-criteria decision-making method with incomplete certain information based on intuitionistic trapezoidal fuzzy number.

The aim of this paper is to propose some new arithmetic aggregation operators including including


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intuitionistic trapezoidal fuzzy ordered weighted averaging (ITFOWA) operator and intuitionistic trapezoidal fuzzy hybrid aggregation (ITFHA) operator, are proposed. Properties of these intuitionistic trapezoidal fuzzy information aggregation operators are also analyzed. An approach to multiple attribute group decision making (MAGDM) with intuitionistic trapezoidal fuzzy information is developed based on the ITFWAA and the ITFHA operators. Finally, some illustrative examples are given to verify the developed approach.

## II. PRELIMINARIES

In the following, we shall introduce some basic concepts related to intuitionistic trapezoidal fuzzy numbers.

**Definition 1([24-26]).** Let $\tilde{a}$ is an intuitionistic trapezoidal fuzzy number, its membership function is:

$$\mu_{\tilde{a}}(x)=\begin{cases}\dfrac{x-a}{b-a}\mu_{\tilde{a}}, & a\le x<b;\\ \mu_{\tilde{a}}, & b\le x\le c;\\ \dfrac{d-x}{d-c}\mu_{\tilde{a}}, & c<x\le d;\\ 0, & others.\end{cases} \tag{1}$$

its non-membership function is:

$$\nu_{\tilde{a}}(x)=\begin{cases}\dfrac{b-x+\nu_{\tilde{a}}(x-a_1)}{b-a_1}, & a_1\le x<b;\\ \nu_{\tilde{a}}, & b\le x\le c;\\ \dfrac{x-c+\nu_{\tilde{a}}(d_1-x)}{d_1-c}, & c<x\le d_1;\\ 0, & \text{others}.\end{cases} \tag{2}$$

where $0\le \mu_{\tilde{a}}\le 1; 0\le \nu_{\tilde{a}}\le 1$ and $\mu_{\tilde{a}}+\nu_{\tilde{a}}\le 1; a,b,c,d\in R$. Then $\tilde{a}=\left\langle\left([a,b,c,d];\mu_{\tilde{a}}\right),\left([a_1,b,c,d_1];\nu_{\tilde{a}}\right)\right\rangle$ is called an intuitionistic trapezoidal fuzzy number.

For convenience, let $\tilde{a}=\left([a,b,c,d];\mu_{\tilde{a}},\nu_{\tilde{a}}\right)$.

**Definition 2([24-26]).** Let $\tilde{a}_1=\left([a_1,b_1,c_1,d_1];\mu_{\tilde{a}_1},\nu_{\tilde{a}_1}\right)$ and $\tilde{a}_2=\left([a_2,b_2,c_2,d_2];\mu_{\tilde{a}_2},\nu_{\tilde{a}_2}\right)$ be two intuitionistic trapezoidal fuzzy number, and $\lambda\ge 0$, then

(1) $\tilde{a}_1+\tilde{a}_2=\left([a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2];\right.$

$\left.\mu_{\tilde{a}_1}+\mu_{\tilde{a}_2}-\mu_{\tilde{a}_1}\cdot\mu_{\tilde{a}_2},\nu_{\tilde{a}_1}\cdot\nu_{\tilde{a}_2}\right)$;

(2) $\tilde{a}_1\cdot\tilde{a}_2=\left([a_1\cdot a_2,b_1\cdot b_2,c_1\cdot c_2,d_1\cdot d_2];\right.$

$\left.\mu_{\tilde{a}_1}\cdot\mu_{\tilde{a}_2},\nu_{\tilde{a}_1}+\nu_{\tilde{a}_2}-\nu_{\tilde{a}_1}\cdot\nu_{\tilde{a}_2}\right)$;

(3) $\lambda\tilde{a}_1=\left([\lambda a_1,\lambda b_1,\lambda c_1,\lambda d_1];1-\left(1-\mu_{\tilde{a}_1}\right)^{\lambda},\nu_{\tilde{a}_1}^{\lambda}\right)$;

(4) $\tilde{a}_1^{\lambda}=\left([a_1^{\lambda},b_1^{\lambda},c_1^{\lambda},d_1^{\lambda}];\mu_{\tilde{a}_1}^{\lambda},1-\left(1-\nu_{\tilde{a}_1}\right)^{\lambda}\right)$

**Definition 3([26]).** Let $\tilde{a}_1=\left([a_1,b_1,c_1,d_1];\mu_{\tilde{a}_1},\nu_{\tilde{a}_1}\right)$ and $\tilde{a}_2=\left([a_2,b_2,c_2,d_2];\mu_{\tilde{a}_2},\nu_{\tilde{a}_2}\right)$ be two intuitionistic trapezoidal fuzzy number, then the normalized Hamming distance between $\tilde{a}_1$ and $\tilde{a}_2$ is defined as follows:

$$\begin{aligned}&d(\tilde{a}_1,\tilde{a}_2)=\\&\frac{1}{8}\Big(\left|\left(1+\mu_{\tilde{a}_1}-\nu_{\tilde{a}_1}\right)a_1-\left(1+\mu_{\tilde{a}_2}-\nu_{\tilde{a}_2}\right)a_2\right|\\&+\left|\left(1+\mu_{\tilde{a}_1}-\nu_{\tilde{a}_1}\right)b_1-\left(1+\mu_{\tilde{a}_2}-\nu_{\tilde{a}_2}\right)b_2\right|\\&+\left|\left(1+\mu_{\tilde{a}_1}-\nu_{\tilde{a}_1}\right)c_1-\left(1+\mu_{\tilde{a}_2}-\nu_{\tilde{a}_2}\right)c_2\right|\\&+\left|\left(1+\mu_{\tilde{a}_1}-\nu_{\tilde{a}_1}\right)d_1-\left(1+\mu_{\tilde{a}_2}-\nu_{\tilde{a}_2}\right)d_2\right|\Big)\end{aligned} \tag{3}$$

**Definition 4.** For a normalized intuitionistic trapezoidal fuzzy decision making matrix $\tilde{R}=\left(\tilde{r}_{ij}\right)_{m\times n}=\left([a_{ij},b_{ij},c_{ij},d_{ij}];\mu_{ij},\nu_{ij}\right)_{m\times n}$, where $0\le a_{ij}\le b_{ij}\le c_{ij}\le d_{ij}\le 1$, $0\le\mu_{ij},\nu_{ij}\le 1, 0\le\mu_{ij}+\nu_{ij}\le 1$, the intuitionistic trapezoidal fuzzy positive ideal solution and intuitionistic trapezoidal fuzzy negative ideal solution are defined as follows:

$$\tilde{r}^{+}=\left([a^{+},b^{+},c^{+},d^{+}];\mu^{+},\nu^{+}\right)=\left([1,1,1,1];1,0\right)$$

$$\tilde{r}^{-}=\left([a^{-},b^{-},c^{-},d^{-}];\mu^{-},\nu^{-}\right)=\left([0,0,0,0];0,1\right).$$

**Definition 5.** Let $\tilde{a}_1=\left([a_1,b_1,c_1,d_1];\mu_{\tilde{a}_1},\nu_{\tilde{a}_1}\right)$ and $\tilde{a}_2=\left([a_2,b_2,c_2,d_2];\mu_{\tilde{a}_2},\nu_{\tilde{a}_2}\right)$ be two intuitionistic trapezoidal fuzzy number, then the distance between $\tilde{a}_1=\left([a_1,b_1,c_1,d_1];\mu_{\tilde{a}_1},\nu_{\tilde{a}_1}\right)$, $\tilde{a}_2=\left([a_2,b_2,c_2,d_2];\mu_{\tilde{a}_2},\nu_{\tilde{a}_2}\right)$ an intuitionistic trapezoidal fuzzy positive ideal solution are denoted as $d\left(\tilde{a}_1,\tilde{r}^{+}\right)$ and $d\left(\tilde{a}_2,\tilde{r}^{+}\right)$, if $d\left(\tilde{a}_1,\tilde{r}^{+}\right)<d\left(\tilde{a}_2,\tilde{r}^{+}\right)$, then $\tilde{a}_1>\tilde{a}_2$.

## III. SOME ARITHMETIC AGGREGATION OPERATORS WITH INTUITIONISTIC TRAPEZOIDAL FUZZY NUMBERS

In the following, some arithmetic aggregation operators with intuitionistic trapezoidal fuzzy numbers are developed as follows:

**Definition 6([26]).** Let $\tilde{a}_j\left(j=1,2,\cdots,n\right)$ be a collection of intuitionistic trapezoidal fuzzy numbers, and let ITFWAA: $Q^n \to Q$, if

$$\mathrm{ITFWAA}_\omega\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\sum_{j=1}^{n}\tilde{a}_j\omega_j$$

$$\left(\left[\sum_{j=1}^{n}a_j\omega_j,\sum_{j=1}^{n}b_j\omega_j,\sum_{j=1}^{n}c_j\omega_j,\sum_{j=1}^{n}d_j\omega_j\right];1-\prod_{j=1}^{n}\left(1-\mu_{\tilde{a}_j}\right)^{\omega_j},\prod_{j=1}^{n}\left(\nu_{\tilde{a}_j}\right)^{\omega_j}\right) \quad (4)$$

where $\omega=\left(\omega_1,\omega_2,\cdots,\omega_n\right)^T$ be the weight vector of $\tilde{a}_j\left(j=1,2,\cdots,n\right)$, and $\omega_j>0$, $\sum_{j=1}^{n}\omega_j=1$, then ITFWAA is called the intuitionistic trapezoidal fuzzy weighted arithmetic averaging(ITFWAA) operator. Especially, if $\omega=\left(1/n,1/n,\cdots,1/n\right)$, then ITFWAA operator is reduced to a intuitionistic trapezoidal fuzzy arithmetic averaging (ITFAA) operator:

$$\mathrm{ITFAA}\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\sum_{j=1}^{n}\tilde{a}_j\Big/n \quad (5)$$

**Definition 7.** Let $\tilde{a}_j\left(j=1,2,\cdots,n\right)$ $\left(j=1,2,\cdots,n\right)$ be a collection of intuitionistic trapezoidal fuzzy numbers. An intuitionistic trapezoidal fuzzy ordered weighted averaging (ITFOWA) operator of dimension $n$ is a mapping ITFOWA: $Q^n \to Q$, that has an associated vector $w=\left(w_1,w_2,\cdots,w_n\right)^T$ such that $w_j>0$ and $\sum_{j=1}^{n}w_j=1$. Furthermore,

$$\mathrm{ITFOWA}_w\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\sum_{j=1}^{n}\tilde{a}_{\sigma(j)}w_j$$

$$\left(\left[\sum_{j=1}^{n}a_{\sigma(j)}w_j,\sum_{j=1}^{n}b_{\sigma(j)}w_j,\sum_{j=1}^{n}c_{\sigma(j)}w_j,\sum_{j=1}^{n}d_{\sigma(j)}w_j\right];\right.$$

$$\left.1-\prod_{j=1}^{n}\left(1-\mu_{\tilde{a}_{\sigma(j)}}\right)^{w_j},\prod_{j=1}^{n}\left(\nu_{\tilde{a}_{\sigma(j)}}\right)^{w_j}\right) \quad (6)$$

where $\left(\sigma(1),\sigma(2),\cdots,\sigma(n)\right)$ is a permutation of $\left(1,2,\cdots,n\right)$, such that $\tilde{\alpha}_{\sigma(j-1)}\geq\tilde{\alpha}_{\sigma(j)}$ for all $j=2,\cdots,n$.

The ITFOWA operator has the following properties.

**Theorem 1.** (Commutativity).

$$\mathrm{ITFOWA}_w\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\mathrm{ITFOWA}_w\left(\tilde{a}_1^*,\tilde{a}_2^*,\cdots,\tilde{a}_n^*\right)$$

where $\tilde{a}_j^*\left(j=1,2,\cdots,n\right)$ is any permutation of $\tilde{a}_j\left(j=1,2,\cdots,n\right)$.

**Theorem 2.** (Idempotency) If $\tilde{a}_j\left(j=1,2,\cdots,n\right)=\tilde{a}$ for all $j$, then

$$ITFOWA_w\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\tilde{a}$$

From Definitions 6 and 7, we know that the ITFWAA operator weights only the intuitionistic trapezoidal fuzzy numbers, while the ITFOWA operator weights only the ordered positions of the intuitionistic trapezoidal fuzzy numbers instead of weighting the intuitionistic trapezoidal fuzzy numbers themselves. Therefore, weights represent different aspects in both the ITFWAA and ITFOWA operators. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose an intuitionistic trapezoidal fuzzy hybrid aggregation (ITFHA) operator.

**Definition 8.** An intuitionistic trapezoidal fuzzy hybrid aggregation (ITFHA) operator of dimension $n$ is a mapping ITFHA: $Q^n \to Q$, that has an associated vector $w=\left(w_1,w_2,\cdots,w_n\right)^T$ such that $w_j>0$ and $\sum_{j=1}^{n}w_j=1$. Furthermore,

$$\mathrm{ITFHA}_{\omega,w}\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\sum_{j=1}^{n}\dot{\tilde{a}}_{\sigma(j)}w_j$$

$$\left(\left[\sum_{j=1}^{n}\dot{a}_{\sigma(j)}w_j,\sum_{j=1}^{n}\dot{b}_{\sigma(j)}w_j,\sum_{j=1}^{n}\dot{c}_{\sigma(j)}w_j,\sum_{j=1}^{n}\dot{d}_{\sigma(j)}w_j\right];\right.$$

$$\left.1-\prod_{j=1}^{n}\left(1-\dot{\mu}_{\tilde{a}_{\sigma(j)}}\right)^{w_j},\prod_{j=1}^{n}\left(\dot{\nu}_{\tilde{a}_{\sigma(j)}}\right)^{w_j}\right) \quad (7)$$

where $\dot{\tilde{a}}_{\sigma(j)}$ is the $j$ th largest of the weighted intuitionistic trapezoidal fuzzy numbers $\dot{\tilde{a}}_j\left(\dot{\tilde{a}}_j=\tilde{a}_j^{n\omega_j},\ j=1,2,\cdots,n\right)$, $\omega=\left(\omega_1,\omega_2,\cdots,\omega_n\right)^T$ be the weight vector of $\tilde{a}_j\left(j=1,2,\cdots,n\right)$, and $\omega_j>0$, $\sum_{j=1}^{n}\omega_j=1$, and $n$ is the balancing coefficient.

**Theorem 3**.The ITFWAA operator is a special case of the ITFHA operator.

Proof. Let $w=\left(1/n,1/n,\cdots,1/n\right)$, then

$$ITFHA_{\omega,w}\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\sum_{j=1}^{n}\dot{\tilde{a}}_{\sigma(j)}w_j=\frac{1}{n}\sum_{j=1}^{n}\dot{\tilde{a}}_{\sigma(j)}$$

$$=\sum_{j=1}^{n}\tilde{a}_j\omega_j=ITFWAA_{\omega}\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)$$

Which completes the proof of Theorem 3.

**Theorem 4** .The ITFOWA operator is a special case of the ITFHA operator.

Proof. Let $\omega=\left(1/n,1/n,\cdots,1/n\right)$, then $\dot{\tilde{a}}_j=\tilde{a}_j, i=1,2,\cdots,n$.

$$ITFHA_{\omega,w}\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\sum_{j=1}^{n}\dot{\tilde{a}}_{\sigma(j)}w_j$$

$$=\sum_{j=1}^{n}\tilde{a}_{\sigma(j)}w_j=ITFOWA_{w}\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)$$

This completes the proof of Theorem 4.

So we know that the ITFHA operator generalizes both the ITFWAA and ITFOWA operators, and reflects the importance degrees of both the given arguments and their ordered positions.

## IV. An Approach to Group Decision Making with Intuitionistic Trapezoidal Fuzzy Information

In this section, we shall investigate the multiple attribute group decision making (MAGDM) problems based on the ITFWAA and ITFHA operator in which both the attribute weights and the expert weights take the form of real numbers, attribute values take the form of intuitionistic trapezoidal fuzzy numbers.

Let $A=\left\{A_1,A_2,\cdots,A_m\right\}$ be a discrete set of alternatives, and $G=\left\{G_1,G_2,\cdots,G_n\right\}$ be the set of attributes, $\omega=\left(\omega_1,\omega_2,\cdots,\omega_n\right)$ is the weighting vector of the attribute $G_j\left(j=1,2,\cdots,n\right)$, where $\omega_j\in\left[0,1\right]$, $\sum_{j=1}^{n}\omega_j=1$. Let $D=\left\{D_1,D_2,\cdots,D_t\right\}$ be the set of decision makers, $\nu=\left(\nu_1,\nu_2,\cdots,\nu_n\right)$ be the weighting vector of decision makers, with $\nu_k\in\left[0,1\right]$, $\sum_{k=1}^{t}\nu_k=1$. Suppose that $\tilde{R}_k=\left(\tilde{r}_{ij}^{(k)}\right)_{m\times n}=\left(\left[a_{ij}^{(k)},b_{ij}^{(k)},c_{ij}^{(k)},d_{ij}^{(k)}\right];\mu_{ij}^{(k)},\nu_{ij}^{(k)}\right)_{m\times n}$ is the intuitionistic trapezoidal fuzzy decision matrix, $\mu_{ij}^{(k)}\in\left[0,1\right]$, $\nu_{ij}^{(k)}\in\left[0,1\right]$, $\mu_{ij}^{(k)}+\nu_{ij}^{(k)}\le 1$, $i=1,2,\cdots,m$, $j=1,2,\cdots,n$, $k=1,2,\cdots,t$.

In the following, we apply the ITFWAA and ITFHA operator to multiple attribute group decision making based on intuitionistic trapezoidal fuzzy information. The method involves the following steps:

**Step 1.** Utilize the decision information given in the intuitionistic trapezoidal fuzzy decision matrix $\tilde{R}_k$, and the ITFWAA operator

$$\tilde{r}_i^{(k)}=\left(\left[a_i^{(k)},b_i^{(k)},c_i^{(k)},d_i^{(k)}\right];\mu_i^{(k)},\nu_i^{(k)}\right)$$
$$=ITFWAA_{\omega}\left(\tilde{r}_{i1}^{(k)},\tilde{r}_{i2}^{(k)},\cdots,\tilde{r}_{in}^{(k)}\right), i=1,2,\cdots,m, k=1,2,\cdots,t.$$

to derive the individual overall preference intuitionistic trapezoidal fuzzy values $\tilde{r}_i^{(k)}$ of the alternative $A_i$.

**Step 2.** Utilize the ITFHA operator:

$$\tilde{r}_i=\left(\left[a_i,b_i,c_i,d_i\right];\mu_i,\nu_i\right)$$
$$=ITFHA_{\nu,w}\left(\tilde{r}_i^{(1)},\tilde{r}_i^{(2)},\cdots,\tilde{r}_i^{(t)}\right), i=1,2,\cdots,m$$

to derive the collective overall preference intuitionistic trapezoidal fuzzy values $\tilde{r}_i\left(i=1,2,\cdots,m\right)$ of the alternative $A_i$ ,where $\nu=\left(\nu_1,\nu_2,\cdots,\nu_n\right)$ be the weighting vector of decision makers, with $\nu_k\in\left[0,1\right]$, $\sum_{k=1}^{t}\nu_k=1$ ; $w=\left(w_1,w_2,\cdots,w_n\right)$ is the associated weighting vector of the ITFHA operator, with $w_j\in\left[0,1\right]$, $\sum_{j=1}^{n}w_j=1$.

**Step 3.** Calculate the distances between collective overall values $\tilde{r}_i=\left(\left[a_i,b_i,c_i,d_i\right];\mu_i,\nu_i\right)$ and intuitionistic trapezoidal fuzzy positive ideal solution.

$$d\left(\tilde{r}_i,\tilde{r}^+\right)=$$
$$\frac{1}{8}\Big(\left|\left(1+\mu_i-\nu_i\right)a_i-\left(1+\mu^+-\nu^+\right)a^+\right|$$
$$+\left|\left(1+\mu_i-\nu_i\right)b_i-\left(1+\mu^+-\nu^+\right)b^+\right|$$
$$+\left|\left(1+\mu_i-\nu_i\right)d_i-\left(1+\mu^+-\nu^+\right)d^+\right|$$
$$+\left|\left(1+\mu_i-\nu_i\right)d_i-\left(1+\mu^+-\nu^+\right)d^+\right|\Big)$$

**Step 4.** Rank all the alternatives $A_i\left(i=1,2,\cdots,m\right)$ and select the best one(s) in accordance with $d\left(\tilde{r}_i,\tilde{r}^+\right)\left(i=1,2,\cdots,m\right)$. The smaller $d\left(\tilde{r}_i,\tilde{r}^+\right)$, the better the alternatives $A_i$.

**Step 5.** End.

## V. Illustrative Example

Let us suppose there is a risk investment company, which wants to invest a sum of money in the best option. There is a panel with five possible alternatives (engineer construction projects) to invest the money. The risk investment company must take a decision according to the following four attributes: ①$G_1$ is the risk analysis; ②$G_2$ is the growth analysis; ③$G_3$ is the social-political impact analysis; ④$G_4$ is the environmental impact analysis. The five possible alternatives $A_i(i=1,2,\cdots,5)$ are to be evaluated using the intuitionistic trapezoidal fuzzy numbers by the three decision makers (whose weighting vector $\nu=(0.35,0.40,0.25)^T$ ) under the above four attributes (whose weighting vector $\omega=(0.2,0.1,0.3,0.4)^T$ ), and construct, respectively, the decision matrices as listed in the following matrices $\tilde{R}_k=\left(\tilde{r}_{ij}^{(k)}\right)_{5\times 4}\left(k=1,2,3\right)$ as follows:

$$\tilde{R}_1=\begin{bmatrix} ([0.5,0.6,0.7,0.8];0.5,0.4) & ([0.1,0.2,0.3,0.4];0.6,0.3) & ([0.5,0.6,0.8,0.9];0.3,0.6) & ([0.4,0.5,0.6,0.7];0.2,0.7) \\ ([0.6,0.7,0.8,0.9];0.7,0.3) & ([0.5,0.6,0.7,0.8];0.7,0.2) & ([0.4,0.5,0.7,0.8];0.7,0.2) & ([0.5,0.6,0.7,0.9];0.4,0.5) \\ ([0.1,0.2,0.4,0.5];0.6,0.4) & ([0.2,0.3,0.5,0.6];0.5,0.4) & ([0.5,0.6,0.7,0.8];0.5,0.3) & ([0.3,0.5,0.7,0.9];0.2,0.3) \\ ([0.3,0.4,0.5,0.6];0.8,0.1) & ([0.1,0.3,0.4,0.5];0.6,0.3) & ([0.1,0.3,0.5,0.7];0.3,0.4) & ([0.6,0.7,0.8,0.9];0.2,0.6) \\ ([0.2,0.3,0.4,0.5];0.6,0.2) & ([0.3,0.4,0.5,0.6];0.4,0.3) & ([0.2,0.3,0.4,0.5];0.7,0.1) & ([0.5,0.6,0.7,0.8];0.1,0.3) \end{bmatrix}$$

$$\tilde{R}_2=\begin{bmatrix} ([0.4,0.5,0.6,0.7];0.4,0.3) & ([0.1,0.2,0.3,0.4];0.5,0.2) & ([0.4,0.5,0.7,0.8];0.2,0.5) & ([0.3,0.4,0.5,0.6];0.1,0.6) \\ ([0.5,0.6,0.7,0.8];0.6,0.2) & ([0.4,0.5,0.6,0.7];0.6,0.1) & ([0.3,0.4,0.6,0.7];0.6,0.1) & ([0.4,0.5,0.6,0.8];0.3,0.4) \\ ([0.1,0.2,0.3,0.4];0.5,0.3) & ([0.1,0.2,0.4,0.5];0.4,0.3) & ([0.4,0.5,0.6,0.7];0.4,0.2) & ([0.2,0.4,0.6,0.8];0.5,0.2) \\ ([0.2,0.3,0.4,0.5];0.7,0.1) & ([0.1,0.2,0.3,0.5];0.5,0.2) & ([0.1,0.2,0.4,0.6];0.2,0.3) & ([0.5,0.6,0.7,0.8];0.1,0.5) \\ ([0.1,0.2,0.3,0.4];0.5,0.1) & ([0.2,0.3,0.4,0.5];0.3,0.2) & ([0.1,0.2,0.3,0.4];0.6,0.2) & ([0.4,0.5,0.6,0.7];0.4,0.2) \end{bmatrix}$$

$$\tilde{R}_3=\begin{bmatrix} ([0.6,0.7,0.8,0.9];0.4,0.5) & ([0.2,0.3,0.4,0.5];0.5,0.4) & ([0.6,0.7,0.9,1.0];0.2,0.7) & ([0.5,0.6,0.7,0.8];0.1,0.8) \\ ([0.7,0.8,0.9,1.0];0.6,0.4) & ([0.6,0.7,0.8,0.9];0.6,0.3) & ([0.5,0.6,0.8,0.9];0.6,0.3) & ([0.6,0.7,0.8,1.0];0.3,0.6) \\ ([0.2,0.3,0.5,0.6];0.5,0.5) & ([0.3,0.4,0.6,0.7];0.4,0.5) & ([0.6,0.7,0.8,0.9];0.4,0.4) & ([0.4,0.6,0.8,1.0];0.5,0.4) \\ ([0.4,0.5,0.6,0.7];0.7,0.2) & ([0.2,0.4,0.5,0.6];0.5,0.4) & ([0.2,0.4,0.6,0.8];0.2,0.5) & ([0.7,0.8,0.9,1.0];0.1,0.7) \\ ([0.3,0.4,0.5,0.6];0.5,0.3) & ([0.4,0.5,0.6,0.7];0.3,0.4) & ([0.3,0.4,0.5,0.6];0.6,0.2) & ([0.6,0.7,0.8,0.9];0.4,0.4) \end{bmatrix}$$

Then, we utilize the proposed procedure to get the most desirable alternative(s).

**Step 1.** Utilize the decision information given in the intuitionistic trapezoidal fuzzy decision matrix $\tilde{R}_k$ , and the ITFWAA operator to derive the individual overall preference intuitionistic trapezoidal fuzzy values $\tilde{r}_i^{(k)}$ of the alternative $A_i$ .

$$\tilde{r}_1^{(1)}=([0.42,0.52,0.65,0.75];0.3472,0.5490)$$
$$\tilde{r}_2^{(1)}=([0.49,0.59,0.72,0.86];0.6041,0.3129)$$
$$\tilde{r}_3^{(1)}=([0.31,0.45,0.62,0.76];0.4229,0.3270)$$
$$\tilde{r}_4^{(1)}=([0.34,0.48,0.61,0.74];0.4565,0.3464)$$
$$\tilde{r}_5^{(1)}=([0.33,0.43,0.53,0.63];0.4715,0.1990)$$
$$\tilde{r}_1^{(2)}=([0.33,0.43,0.56,0.66];0.2446,0.4431)$$
$$\tilde{r}_2^{(2)}=([0.39,0.49,0.62,0.76];0.4996,0.2000)$$
$$\tilde{r}_3^{(2)}=([0.23,0.37,0.52,0.66];0.4622,0.2259)$$
$$\tilde{r}_4^{(2)}=([0.28,0.38,0.51,0.65];0.3424,0.2837)$$
$$\tilde{r}_5^{(2)}=([0.23,0.33,0.43,0.53];0.4798,0.1741)$$
$$\tilde{r}_1^{(3)}=([0.52,0.62,0.75,0.85];0.3472,0.5490)$$
$$\tilde{r}_2^{(3)}=([0.59,0.69,0.82,0.96];0.6041,0.3129)$$
$$\tilde{r}_3^{(3)}=([0.41,0.55,0.72,0.86];0.4229,0.3270)$$
$$\tilde{r}_4^{(3)}=([0.44,0.58,0.71,0.84];0.4565,0.3464)$$
$$\tilde{r}_5^{(3)}=([0.43,0.53,0.63,0.73];0.4715,0.1990)$$

**Step 2.** Utilize the ITFHA operator to derive the collective overall preference intuitionistic trapezoidal fuzzy values $\tilde{r}_i\left(i=1,2,\cdots,m\right)$ of the alternative $A_i$ (Let $w=\left(0.20,0.50,0.30\right)^T$ ).

$$\tilde{r}_1=\left(\left[0.4167,0.5157,0.6444,0.7434\right];0.2984,0.5455\right)$$
$$\tilde{r}_2=\left(\left[0.4836,0.5826,0.7113,0.8499\right];0.5545,0.3037\right)$$
$$\tilde{r}_3=\left(\left[0.3102,0.4488,0.6123,0.7509\right];0.4384,0.3214\right)$$
$$\tilde{r}_4=\left(\left[0.3384,0.4593,0.5939,0.7344\right];0.3774,0.3165\right)$$
$$\tilde{r}_5=\left(\left[0.3041,0.4076,0.5111,0.6146\right];0.4899,0.1913\right)$$

**Step 3.** Calculate the distances between collective overall values $\tilde{r}_i=\left(\left[a_i,b_i,c_i,d_i\right];\mu_i,\nu_i\right)$ and intuitionistic trapezoidal fuzzy positive ideal solution.

$$d\left(\tilde{r}_1,\tilde{r}^+\right)=0.7816, d\left(\tilde{r}_2,\tilde{r}^+\right)=0.5892, d\left(\tilde{r}_3,\tilde{r}^+\right)=0.7037$$
$$d\left(\tilde{r}_4,\tilde{r}^+\right)=0.7181, d\left(\tilde{r}_5,\tilde{r}^+\right)=0.7018$$

**Step 4.** Rank all the alternatives $A_i\left(i=1,2,3,4,5\right)$ in accordance with the distances $d\left(\tilde{r}_i,\tilde{r}^+\right)$ between collective overall values $\tilde{r}_i=\left(\left[a_i,b_i,c_i,d_i\right];\mu_i,\nu_i\right)$ and intuitionistic trapezoidal fuzzy positive ideal solution: $A_2\succ A_5\succ A_3\succ A_4\succ A_1$, and thus the most desirable alternative is $A_2$ .

## VI. CONCLUSION

In this paper, with respect to multiple attribute group decision making (MAGDM) problems in which both the attribute weights and the expert weights take the form of real numbers, attribute values take the form of intuitionistic trapezoidal fuzzy numbers, a new group decision making analysis methods are developed. Firstly, some operational laws of intuitionistic trapezoidal fuzzy numbers are introduced. Then, we have developed intuitionistic trapezoidal fuzzy weighted arithmetic averaging (ITFWAA) operator, intuitionistic trapezoidal fuzzy ordered weighted averaging (ITFOWA) operator and intuitionistic trapezoidal fuzzy hybrid aggregation (ITFHA) operator. The ITFHA operator first weights the given arguments, and then reorders the weighted arguments in descending order and weights these ordered arguments by the ITFHA weights, and finally aggregates all the weighted arguments into a collective one. Obviously, the ITFHA operator generalizes both the ITFWAA and ITFOWA operators, and reflects the importance degrees of both the given argument and the ordered position of the argument. Furthermore, the ITFHA operator can relieve the influence of unfair arguments on the decision results by using the ITFHA weights to assign low weights to those “false” or “biased” ones. We have studied some desirable properties of these operators and applied the ITFWAA and ITFHA operators to group decision making with intuitionistic trapezoidal fuzzy information. Furthermore, we have developed an ITFWAA and ITFHA operators-based approach to solve the MAGDM problems in which both the attribute weights and the expert weights take the form of real numbers, attribute values take the form of intuitionistic trapezoidal fuzzy numbers. Finally, an illustrative example are given to verify the developed approach and to demonstrate its practicality and effectiveness. In the future, we shall continue working in the extension and application of the developed operators to other domains.

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