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ON THE AVERAGE NUMBER OF REBALANCING OPERATIONS IN WEIGHT-BALANCED TREES

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Abstract. It is shown that the average number of rebalancing operations (rotations and double rotations) in weight-balanced trees is constant.

1. Introduction

Balanced trees are a popular method of maintaining sets in a digital computer. The basic set operations MEMBER, INSERT, DELETE have O(log n) processing time for a set of n elements.

Balanced trees come in two kinds. The balance criterion is either on the height (AVL-trees [2], 2-3 trees [1], brother trees [9], ...) or on the weight (weight-balanced trees) of the subtrees. In the first kind of trees one either allows subtrees to have only small differences in height (AVL-trees) or one allows nodes of different arity (2-3 trees, brother trees). Here we deal with weight-balanced trees. Weight-balanced trees were introduced by Nievergelt and Reingold [8].

A node in a binary tree either has two sons or no son at all. Nodes with no sons are called leaves.

Definition 1. Let T be a binary tree. If T is a single leaf, then the *root-balance* $\rho(T)$ is $\frac{1}{2}$, otherwise we define $\rho(T)=|T_l|/|T|$, where $|T_l|$ is the number of leaves in the left subtree of T and $|T|$ is the number of leaves in tree T.

Definition 2. A binary tree T is said *to be of bounded balance* α, or in the set BB[α], for $0 \leqslant \alpha \leqslant \frac{1}{2}$, if and only if

(1) $\alpha \leqslant \rho(T) \leqslant 1-\alpha$;

(2) T is a single leaf or both subtrees are of bounded balance α.

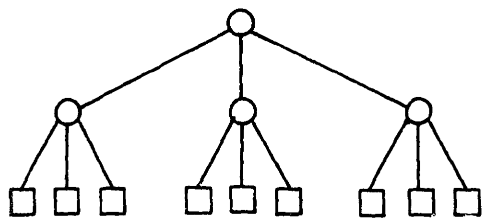
Remark. Note that $|T_r|/|T| = 1-\rho(T)$. By interchanging left and right we may therefore assume w.l.o.g. that $\rho(T) \leqslant \frac{1}{2}$.

Balanced trees share many common properties:

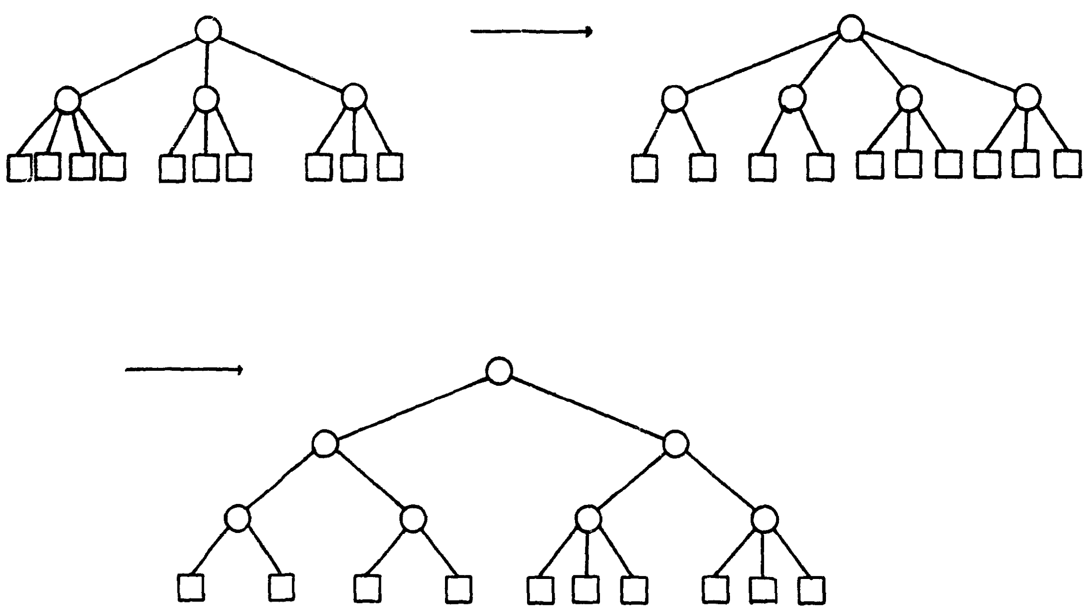
(1) the depth is bounded by $O(\log |T|)$;

(2) upon insertion or deletion of a leaf at most $O(\log |T|)$ rebalancing operations (rotations and double rotations in the case of AVL-trees and BB[α]-trees, node splittings and combinations in the case of 2-3 trees and brother trees) are required to rebalance the tree. The rebalancing operations are limited to the path of search. In all known examples of balanced trees it is easy to construct examples which require each node on the path of search to be rebalanced (usually after the deletion of a leaf).

Example 1. Consider the following tree:



Inserting a new leaf in front of the left-most leaf gives rise to the following sequence of rebalancing operations:



Note however, that inserting yet another leaf will require at most one rebalancing operation. This suggests that on the average (averaged over a random sequence of insertions and deletions) a smaller number of rebalancing operations suffices. Note also, that deleting the left-most leaf will reverse the sequence above and recreate the original tree.

(3) Simulation results show that on the average (random sequence of insertions and deletions) a constant number of rebalancing operations suffices. Karlton et al. [5] report that on the average 0.46 (0.23) rebalancing operations ($\triangleq$ rotations and double rotations) are required to rebalance an AVL-tree upon the insertion (deletion) of a leaf. There are plausibility arguments which support the empirical evidence [4; 6, p. 462; 8].

The plausibility arguments are based on the *unjustified* assumption that node balances (the height difference between left and right subtrees in AVL-trees, the root balance in BB[α]-trees) are independent random variables. The plausibility arguments yield constants which are in close agreement with empirical evidence.

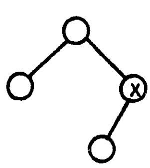
Here we give a rigorous proof that the average number of rebalancing operations in BB[α]-trees is bounded by a constant. Actually we prove a stronger result.

There is a constant c (depending on α) such that: The total number of rebalancing operations required for executing an arbitrary sequence of n insertions and deletions on an initially empty BB[α]-tree is bounded by $c \cdot n$.

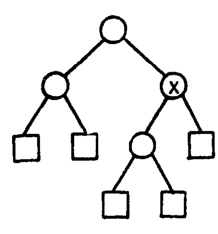
This contrasts with simulation results which by their nature consider random sequences of insertion and deletions. We do not average over *many* sequences of insertions and deletions but only over the elements of a *single* sequence of insertions and deletions. (This implies for every heuristic that a constant number of rebalancing operations will suffice on the average. For this reason we do not give any particular heuristic (for heuristics cf. [6, 10]).) However, our constant is much larger than empirical evidence suggests. (About 27 for $\alpha = \frac{1}{4}$.) We do not claim that our constant is best possible.

We also correct a serious mistake in the original paper of Nievergelt and Reingold on BB[α]-trees.

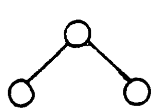
In our drawings we will not draw leaves, i.e. the tree



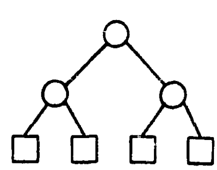
stands for



Inserting a leaf means replacing a leaf by a tree consisting of one node and two leaves and deleting a leaf means replacing the father of the leaf by the other subtree. Deletion of the right son of node x gives



which stands for



Note that the balance $\rho(x)$ of node x is the quotient of the number of leaves in the left subtree and the total number of leaves. Hence $\rho(x) = \frac{2}{3}$.

The motivation for this paper is twofold. Firstly it treats an interesting theoretical question in tree searching and narrows the gap between theory and practice. Secondly it treats an important question of practical relevance. The updating behavior of a tree structure is the bottleneck in time-shared tree manipulation (see [3]).

2. The effect of rotations and double-rotations in weight-balanced trees

BB[α]-trees are balanced by rotations and double rotations. Fig. 1 is taken from [8]. Squares represent nodes, triangles represent subtrees, the root-balance is given beside each node. Symmetrical variants of the operations exist.

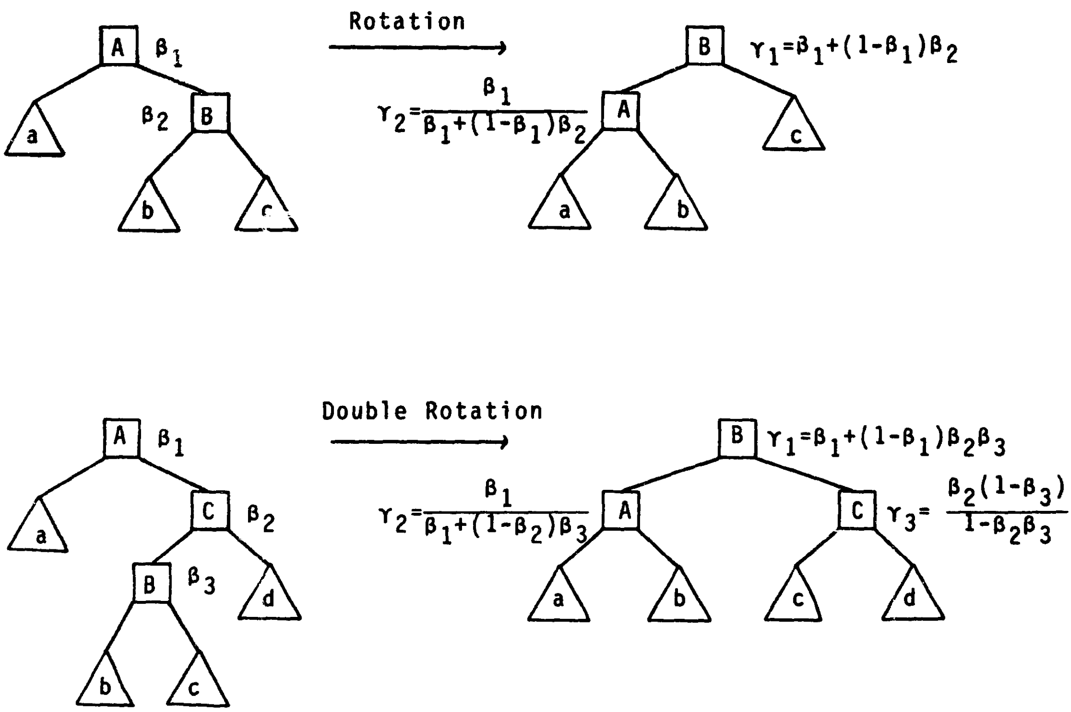


Fig. 1.

Nievergelt and Reingold state the following theorem in [8] without proof.

Fact (Nievergelt and Reingold): If $\alpha \leq 1-\frac{1}{2}\sqrt{2}$ and the insertion or deletion of a node in a tree in BB[α] causes a subtree T of that tree to have root-balance less than α, T can be rebalanced by performing one of the two transformations shown above. More precisely, let β_2 denote the balance of the right subtree of T after the insertion or deletion has been done. If $\beta_2 < (1-2\alpha)/(1-\alpha)$, then a rotation will rebalance T, otherwise a double rotation will rebalance T.

This theorem is false. Consider the example in Fig. 2: $\alpha = \frac{2}{11}$. The root v of this tree has balance $\rho(v) = \frac{2}{11}$. Deleting one of the leaves with father x requires the root to be

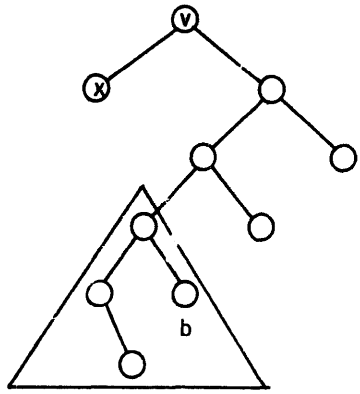


Fig. 2.

rebalanced; it has balance $\frac{1}{10}$. A double rotation gives Fig. 3. Node v has balance $\frac{1}{6} < \frac{2}{11}$. A rotation makes the balance of v even worse. This example shows that the 'theorem' of [8] is wrong for $\frac{1}{6} < \alpha \leqslant \frac{2}{11}$. The same counter example works for any α with $0 < \alpha \leqslant \frac{2}{11}$. More precisely, replace the triangle in Fig. 2 by a BB[α] tree with b leaves such that

$$\frac{1}{b+1} < \alpha \leqslant \frac{2}{b+6}.$$

Such b exists for all α, $0 < \alpha \leqslant \frac{2}{11}$. This follows from the observation that

$$\frac{1}{b+1} \leqslant \frac{2}{(b+1)+6}$$

for $b \geqslant 5$ and that $2/(b+6) = \frac{2}{11}$ for $b = 5$.

We show in this section that a stronger version of the above theorem is indeed true for $\frac{2}{11} < \alpha \leqslant 1 - \frac{1}{2}\sqrt{2}$. Before doing so we want to show that Fig. 1 correctly gives the root-balances of all nodes. Also we state a lemma about the effect of an insertion or deletion on root-balances.

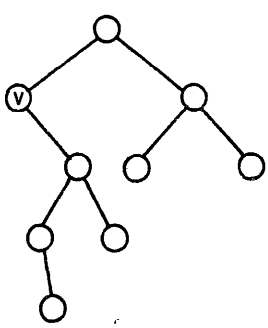


Fig. 3.

Let a, b, c, d be the number of leaves in the subtrees shown in Fig. 1. We treat the case of rotation and leave the case of double rotation to the reader. By the definition of root-balance

$$\beta_1 = a/(a+b+c) \quad \text{and} \quad \beta_2 = b(b+c).$$

Then

$$\begin{aligned}(a+b) &= \beta_1(a+b+c)+\beta_2(b+c)\\ &= \beta_1(a+b+c)+\beta_2((a+b+c)-a)\\ &= (\beta_1+\beta_2(1-\beta_1))(a+b+c).\end{aligned}$$

Hence the root-balance of node A after the rotation is

$$a/(a+b) = \beta_1/(\beta_1+\beta_2(1-\beta_1))$$

and the root-balance of node B is

$$(a+b)/(a+b+c) = \beta_1+\beta_2(1-\beta_1).$$

Next we study the effect of insertion and deletions on root-balances.

Lemma 1. *Let T be a tree in* $\mathrm{BB}[\alpha]$ *with root v.*

(1) *If we inset a leaf into the right subtree of T, then $\rho(v) \geq \alpha/(1+\alpha)$ after the insertion.*

(2) *If we delete a leaf from the left subtree of T, then*

$$\rho(v) \geq \frac{\alpha l}{l+1-\alpha} \geq \frac{\alpha}{2-\alpha}$$

after the deletion, where l is the number of leaves in the left subtree after the deletion.

Proof. Let T have $n+1$ leaves, it left subtree have $l+1$ leaves. Then

$$\alpha \leq \frac{l+1}{n+1} \quad \text{and hence} \quad n \leq \frac{l+1}{\alpha} - 1.$$

(1) After inserting a leaf into T's right subtree we have

$$\begin{aligned}\rho(v) = \frac{l+1}{n+2} &\geq \frac{l+1}{(l+1)/\alpha+1}\\ &\geq \frac{1}{1/\alpha+1} \quad \text{since } (l+1)/((l+1)/\alpha+1) \text{ is monotonically increasing in } l \text{ and } l \geq 0\\ &= \frac{\alpha}{1+\alpha}.\end{aligned}$$

(2) After deleting a leaf from T's left subtree we have

$$\rho(v) = \frac{l}{n} \geq \frac{l}{(l+1)/\alpha - 1}$$

$$\geq \frac{1}{2/\alpha - 1} = \frac{\alpha}{2-\alpha} \quad \text{since } l/((l+1)/\alpha + 1) \text{ is monotonically increasing in } l \text{ and } l \geq 1.$$

Remark. $\alpha/(1+\alpha) \geq \alpha/(2-\alpha)$ for $0 \leq \alpha \leq \frac{1}{2}$.

We are now able to state the correct version of Nievergelt and Reingold's theorem. We will actually prove more. We not only show that rotations and double rotations suffice to rebalance the tree but, moreover, that they suffice to move all root-balances into the interval $[(1+\delta)\alpha, 1-(1+\delta)\alpha]$ for some small δ. This observation will allow us to show in the next section that the average number of rebalancing steps per insertion and deletion is constant.

Let

$$g(\alpha, \delta) = \frac{\delta}{[1+(1+\delta)(1-\alpha)](2-\alpha)} \,.$$

Then $g(\alpha, 0) = 0$, g is increasing in α and $\delta (0 \leq \alpha \leq 1$ and $\delta \geq 0)$. Also $g(\alpha, \delta) \leq \delta/(2-\alpha)$ and $g(\alpha, \delta) \geq \frac{1}{2} \cdot \delta/(2-\alpha)$ for $\frac{2}{11} \leq \alpha \leq 1$ and $0 \leq \delta \leq 0.01$. (We only need that range later on.)

Theorem 1. *There is a continuous, increasing function $c: [0, 0.01] \to R$ with $c(0) = 0$, $c(0.01) = 0.0043$ such that: for $\alpha \in R$, $\frac{2}{11} < \alpha \leq 1 - \frac{1}{2}\sqrt{2} - c(\delta)$, and for T a binary tree with subtrees T_l and T_r such that:*

(1) *T_l and T_r are in* $\mathrm{BB}[\alpha]$;

(2) *$|T_l|/|T| < \alpha$ and either*

(2.1) *$|T_l|/(|T|-1) \geq \alpha$ (T is obtained by insertion of a leaf into the right subtree of T) or*

(2.2) *$(|T_l|+1)/(|T|+1) \geq \alpha$ (T is obtained by deletion of a leaf from the left subtree of T).*

(3) *β_2 is the root-balance of T_r;*

we have:

(i) *if $\beta_2 \leq 1/(2-\alpha) + g(\alpha, \delta)$, then a rotation rebalances the tree, more precisely*

$$\gamma_1, \gamma_2 \in [(1+\delta)\alpha, 1-(1+\delta)\alpha]$$

after the rotation, where γ_1 and γ_2 are as shown in Fig. 1;

(ii) *if $\beta_2 > 1/(2-\alpha) + g(\alpha, \delta)$, then a double rotation rebalances the tree, more precisely*

$$\gamma_1, \gamma_2, \gamma_3 \in [(1+\delta)\alpha, 1-(1+\delta)\alpha]$$

where $\gamma_1, \gamma_2, \gamma_3$ are shown as in Fig. 1;

(iii) *if* $\frac{2}{11} < \alpha \leq \frac{1}{4}$ *and* $|T| \leq 10$, *then we only claim*

$$\gamma_1, \gamma_2 \text{ (resp. } \gamma_1, \gamma_2, \gamma_3) \in [\alpha, 1-\alpha].$$

Remark 1. For $\delta = 0$, this corrects the theorem of Nievergelt and Reingold.

Remark 2. A larger range of values for δ is possible. However, subsequent computations become more complicated and no additional insights are gained.

Proof of Theorem 1. We need to show that nodes A, B (A, B, C) have balances in the interval $[(1+\delta)\alpha, 1-(1+\delta)\alpha]$ after a rotation (double rotation). This is done by tedious but simple calculations.

By Lemma 1 we may assume $\beta_1 \geq \alpha/(1+\alpha)$ in Case 2.1 and

$$\beta_1 \geq \alpha|T_l|/(|T_l|+1-\alpha) \geq \alpha/(2-\alpha)]$$

in Case 2.2; in any case $\beta_1 \geq \alpha/(2-\alpha)$. Also $\beta_2, \beta_3 \in [\alpha, 1-\alpha]$ since T_r is in $\mathrm{BB}[\alpha]$.

Case I: $\beta_2 \leq 1/(2-\alpha) + g(\alpha, \delta)$, i.e. rotation is applied.

(I.1) We have to show $(1+\delta)\alpha \leq \gamma_1 \leq 1-(1+\delta)\alpha$:

$$\gamma_1 = \beta_1 + (1-\beta_1)\beta_2 \quad \text{the RHS is increasing in } \beta_2 \text{ and increasing in } \beta_1.$$

Hence,

$$\gamma_1 \geq \frac{\alpha}{2-\alpha} + \left(1 - \frac{\alpha}{2-\alpha}\right)\alpha = \alpha\left[\frac{1}{2-\alpha} + \frac{2-2\alpha}{2-\alpha}\right]$$

$$\geq 1.4\alpha \quad \text{since [ ] is decreasing in } \alpha \text{ and } \alpha \leq \tfrac{1}{3}.$$

Also

$$\gamma_1 \leq \alpha + (1-\alpha)\left(\frac{1}{2-\alpha} + g(\alpha, \delta)\right).$$

Consider

$$h(\alpha, \delta) = 1-(1+\delta)\alpha - \left[\alpha + (1-\alpha)\left(\frac{1}{2-\alpha} + g(\alpha, \delta)\right)\right].$$

We have to show

$$h(\alpha, \delta) \geq 0 \quad \text{for } 0 \leq \delta \leq 0.01 \text{ and } \tfrac{2}{11} < \alpha \leq 1 - \tfrac{1}{2}\sqrt{2} - c(\delta),$$

$$h(\alpha, \delta) = 1 - 2\alpha - \frac{1-\alpha}{2-\alpha} - \delta\alpha - (1-\alpha)g(\alpha, \delta)$$

$$= \frac{2\alpha^2 - 4\alpha + 1}{2-\alpha} - \delta\alpha - (1-\alpha)g(\alpha, \delta).$$

Since $2\alpha^2 - 4\alpha + 1 = 2(\alpha^2 - 2\alpha + \frac{1}{2})$ has zeroes $1 - \frac{1}{2}\sqrt{2}$ and $1 + \frac{1}{2}\sqrt{2}$ we conclude

$$h(\alpha, 0) \geq 0 \quad \text{for } \alpha \leq 1 - \tfrac{1}{2}\sqrt{2}.$$

Furthermore $\delta\alpha+(1-\alpha)g(\alpha,\delta)\leqslant\delta[\alpha+(1-\alpha)/(2-\alpha)]$ and hence

$$h(\alpha,\delta)\geqslant\frac{2\alpha^2-4\alpha+1-\delta[\alpha(2-\alpha)+(1-\alpha)]}{2-\alpha}.$$

The numerator is a quadratic expression in α and is decreasing in δ. Hence $h(\alpha,\delta)\geqslant 0$ for $\alpha\leqslant 1-\frac{1}{2}\sqrt{2}-c_1(\delta)$, where $c_1(0)=0$, c_1 continuous and increasing and $c_1(0.01)\approx 0.0043$.

(I.2) We have to show $(1+\delta)\alpha\leqslant\gamma_2\leqslant 1-(1+\delta)\alpha$:

$$\gamma_2=\frac{\beta_1}{\beta_1+(1-\beta_1)\beta_2}$$ the RHS is decreasing in β_2 and increasing in β_1.

We want to show $\gamma_2\geqslant(1+\delta)\alpha$ first. We have $\beta_1\geqslant\alpha/(1+\alpha)$ in Case 2.1 and $\beta_1\geqslant\alpha|T_l|/(|T_l|+1-\alpha)$ in Case 2.2. We treat the case $|T_l|=1$ separately.

If $|T_l|\geqslant 2$ or if Case 2.1 applies, then we have

$$\beta_1\geqslant\min(\alpha/(1+\alpha),2\alpha/(3-\alpha))=2\alpha/(3-\alpha)\quad\text{for }\alpha\leqslant\tfrac{1}{3}.$$

Hence,

$$\gamma_2\geqslant\frac{2\alpha/(3-\alpha)}{2\alpha/(3-\alpha)+[1-2\alpha/(3-\alpha)][1/(2-\alpha)+g(\alpha,\delta)]}$$

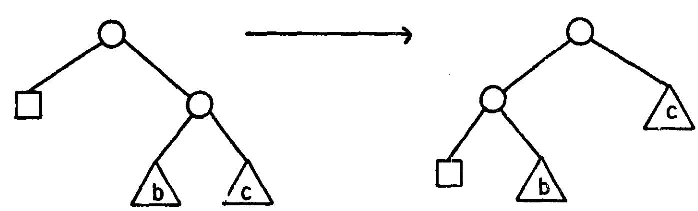
$$\geqslant\frac{1}{\alpha+\dfrac{3-3\alpha}{2}\left[\dfrac{1+\delta}{2-\alpha}\right]}\cdot\alpha\quad\text{since } g(\alpha,\delta)\leqslant\delta/(2-\alpha)$$

$$\geqslant\frac{1}{\alpha+\dfrac{3\cdot 1.01}{4}(1-\alpha)}\cdot\alpha\quad\text{since }\alpha\geqslant 0\text{ and }\delta\leqslant 0.01$$

$$\geqslant 1.2\alpha\geqslant(1+\delta)\alpha$$

since $f(\alpha)=\alpha+3.03\cdot(1-\alpha)/4$ is increasing in α and $\alpha\geqslant 1-\frac{1}{2}\sqrt{2}<0.3$.

It remains to consider the Case 2.2 and $|T_l|=1$. Since $\alpha>\frac{2}{11}$ and $(|T_l|+1)/(|T|+1)\geqslant\alpha>\frac{2}{11}$ we have $|T|\leqslant 9$. Our tree $|T|$ has the following form:



γ_2 is smallest when b is as large as possible. Hence we only have to consider the case that $b/(b+c)\leqslant 1/(2-\alpha)+g(\alpha,\delta)\leqslant 1.01/(2-\alpha)$ and b as large as possible. Furthermore $b+c=|T|-1$.

If $\frac{2}{11}<\alpha\leqslant\frac{1}{4}$, then we have to show $\gamma_2\geqslant\alpha$, otherwise we have to show $\gamma_2\geqslant(1+\delta)\alpha$.
$|T|=9$: then $\alpha\leqslant\frac{2}{10}$ and hence $b\leqslant[1.01/1.8]\cdot 8\approx 4.49$ and, hence, $b=4$ and $\gamma_2=1/(b+1)=\frac{1}{5}\geqslant\alpha$;
$|T|=8$: then $\alpha\leqslant\frac{2}{9}$ and hence $b\leqslant[1.01/(2-\frac{2}{9})]\cdot 7\approx 3.97$ and, hence, $b=3$ is maximal and $\gamma_2=1/(b+1)=\frac{1}{4}\geqslant\alpha$;
$|T|=7$: then $\alpha\leqslant\frac{2}{8}$ and hence $b\leqslant[1.01/(2-\frac{2}{8})]\cdot 6\approx 3.46$ and, hence, $b=3$ is maximal and $\gamma_2=1/(b+1)=\frac{1}{4}\geqslant\alpha$;
$|T|=6$: then $\alpha\leqslant\frac{2}{7}$ and hence $b\leqslant[1.01/(2-\frac{2}{7})]\cdot 5\approx 2.94$ and, hence, $b=2$ is maximal and $\gamma_2=1/(b+1)=\frac{1}{3}\geqslant(1+\delta)\alpha$;
$|T|=5$: then $\alpha\leqslant 1-\frac{1}{2}\sqrt{2}$ and hence $b\leqslant[1.01/(2-1+\frac{1}{2}\sqrt{2})]\cdot 4\approx 2.37$ and, hence, $b=2$ is maximal and $\gamma_2=1/(b+1)=\frac{1}{3}\geqslant(1+\delta)\alpha$;
$|T|=4$: then $\alpha\leqslant 1-\frac{1}{2}\sqrt{2}$ and hence $b\leqslant[1.01/(2-1+\frac{1}{2}\sqrt{2})]\cdot 3\approx 1.78$ and, hence, $b=1$ is maximal and $\gamma_2=1/(b+1)=\frac{1}{2}\geqslant(1+\delta)\alpha$.
$|T|\leqslant 3$ is impossible since $|T_l|/|T|\leqslant\alpha$ is one of the hypotheses of the theorem.

Next we have to show $\gamma_2\leqslant 1-(1+\delta)\alpha$:

$$\gamma_2\leqslant\frac{\alpha}{\alpha+(1-\alpha)\cdot\alpha}=\frac{1}{2-\alpha}\leqslant 1-1.01\alpha$$

if $1\leqslant 2-\alpha-2.02\alpha+1.01\alpha^2$, if $1\leqslant 2-3.02\alpha$, if $\alpha\leqslant 1/3.02$. This finishes Case I of the proof.

Case II: $\beta_2>1/(2-\alpha)+g(\alpha,\delta)\geqslant 1/(2-\alpha)$, i.e. a double rotation is applied.

(II.1) We have to show $(1+\delta)\alpha\leqslant\gamma_1\leqslant 1-(1+\delta)\alpha$:

$$\gamma_1=\beta_1+(1-\beta_1)\beta_2\beta_3 \quad \text{the RHS is increasing in } \beta_1,\ \beta_2 \text{ and } \beta_3.$$

Hence,

$$\begin{aligned}
\gamma_1&\geqslant\alpha/(2-\alpha)+(1-\alpha/(2-\alpha))(1/(2-\alpha))\cdot\alpha\\
&\geqslant\frac{\alpha+\frac{2-2\alpha}{2-\alpha}\cdot\alpha}{2-\alpha}\geqslant\alpha\cdot\frac{4-3\alpha}{(2-\alpha)^2}\\
&\geqslant\alpha\cdot\frac{4-3\cdot\frac{2}{11}}{(2-\frac{2}{11})^2}\quad\text{since } f(\alpha)=(4-3\alpha)/(2-\alpha)^2 \text{ is increasing for } \alpha\leqslant\tfrac{2}{3}\\
&=1.045\geqslant(1+\delta)\alpha,\\
\gamma_1&\leqslant\alpha+(1-\alpha)(1-\alpha)(1-\alpha)\\
&=1-2\alpha+3\alpha^2-\alpha^3=(1-\alpha)-(\alpha^2-3\alpha+1)\alpha\\
&\leqslant(1-\alpha)-(0.3^2-3\cdot 0.3+1)\alpha\quad\text{since } \alpha^3-3\alpha+1 \text{ is decreasing in } \alpha \text{ for } \alpha\leqslant\tfrac{3}{2} \text{ and } \alpha\leqslant 1-\tfrac{1}{2}\sqrt{2}\leqslant 0.3\\
&\leqslant 1-1.1\alpha\leqslant 1-(1+\delta)\alpha.
\end{aligned}$$

(II.2) We have to show $(1+\delta)\alpha \leqslant \gamma_2 \leqslant 1-(1+\delta)\alpha$. We show $\gamma_2 \geqslant (1+\delta)\alpha$ first. By Lemma 1 $\beta_2 \geqslant \alpha/(1+\alpha)$ in Case 2.1 and $\beta_2 \geqslant \alpha l/(l+1-\alpha)$ in Case 2.2, where l is the number of leaves in the left subtree of T. Furthermore

$$\gamma_2 = \frac{\beta_1}{\beta_1+(1-\beta_1)\beta_2\beta_3} = \frac{1}{1+(1/\beta_1-1)\beta_2\beta_3}$$

is increasing in β_1 and decreasing in β_2 and β_3.

Case 2.1: $\beta_1 \geqslant \alpha/(1+\alpha)$. Then

$$\gamma_2 \geqslant \frac{\alpha/(1+\alpha)}{\alpha/(1+\alpha)+(1-\alpha/(1+\alpha))(1-\alpha)^2}$$

$$= \frac{1}{\alpha+(1-\alpha)^2}\cdot\alpha$$

$$\geqslant \frac{1}{1-\frac{2}{11}+(\frac{2}{11})^2}\cdot\alpha = \tfrac{121}{103}\alpha \geqslant (1+\delta)\alpha$$

since $1-\alpha+\alpha^2$ is decreasing in α for $\alpha \leqslant \frac{1}{2}$ and $\alpha \geqslant \frac{2}{11}$.

Case 2.2: $\beta_1 \geqslant \alpha l/(l+1-\alpha)$. We treat the case $|T_l| \geqslant 3$ analytically and the case $|T_l| \leqslant 2$ by explicit consideration. If $|T_l| \geqslant 3$, then $\beta_1 \geqslant 3\alpha/(4-\alpha)$. Hence

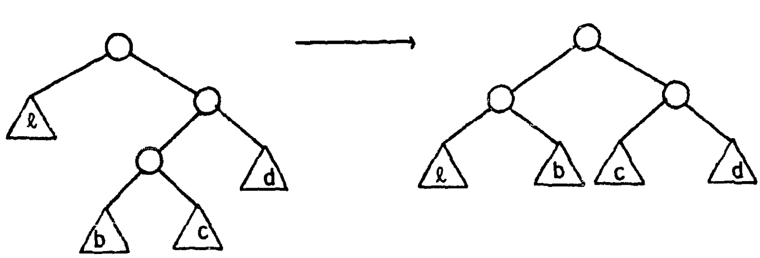
$$\gamma_2 \geqslant \frac{3\alpha/(4-\alpha)}{3\alpha/(4-\alpha)+(1-3\alpha/(4-\alpha))(1-\alpha)^2}$$

$$= \frac{3}{3\alpha+4\cdot(1-\alpha)^3}\cdot\alpha$$

$$\geqslant \frac{3}{3\cdot\frac{2}{11}+4\cdot(\frac{9}{11})^3}\cdot\alpha \quad \text{since } 3\alpha+4\cdot(1-\alpha)^3 \text{ is decreasing for } \alpha \leqslant \tfrac{1}{2} \text{ and } \tfrac{2}{11} < \alpha \leqslant 1-\tfrac{1}{2}\sqrt{2}$$

$$\geqslant 1.09\alpha \geqslant (1+\delta)\alpha.$$

It remains to consider Case 2.2 and $|T_l| \leqslant 2$. Since $\alpha > \frac{2}{11}$ and $(|T_l|+1)/(|T|+1) \geqslant \alpha > \frac{2}{11}$ we have $|T| \leqslant 15$. The tree T has the following form:



γ_2 is smallest when b is as large as possible. Hence we only have to consider the case that $b/(b+c) \leq 1-\alpha$ and $(b+c)/(b+c+d) \leq 1-\alpha$ and b maximal. Hence

$$\begin{aligned} b &\leq \lfloor (1-\alpha)(b+c) \rfloor \leq \lfloor (1-\alpha) \lfloor (1-\alpha)(b+c+d) \rfloor \rfloor \\ &= \lfloor (1-\alpha) \cdot \lfloor (1-\alpha)(|T|-|T_l|) \rfloor \rfloor \end{aligned}$$

$|T_l| = 2$:
$|T| = 15$: then $\alpha \leq (|T_l|+1)/(|T|+1) = \frac{3}{16}$ and $b \leq \lfloor (\frac{9}{11} \lfloor \frac{9}{11} \cdot 13 \rfloor) \rfloor = 8$ and hence $\gamma_2 = 2/(b+2) \geq \frac{2}{10} \geq (1+\delta)\alpha$;
$|T| = 14$: then $\alpha \leq \frac{3}{15}$ and $b \leq 7$ and, hence, $\gamma_2 = 2/(b+2) \geq \frac{2}{9} \geq (1+\delta)\alpha$;
$|T| = 13$: then $\alpha \leq \frac{3}{14}$ and $b \leq 7$ and, hence, $\gamma_2 \geq \frac{2}{9} \geq (1+\delta)\alpha$;
$|T| = 12$: then $\alpha \leq \frac{3}{13}$ and $b \leq 6$ and, hence, $\gamma_2 \geq \frac{2}{8} \geq (1+\delta)\alpha$;
$|T| = 11$: then $\alpha \leq \frac{3}{12}$ and $b \leq 5$ and, hence, $\gamma_2 \geq \frac{2}{7} \geq (1+\delta)\alpha$;
$|T| \leq 10$: then $b \leq 4$ and, hence, $\gamma_2 \geq \frac{2}{6} \geq (1+\delta)(1-\frac{1}{2}\sqrt{2}) \geq (1+\delta)\alpha$;
$|T_l| = 1$: since $(|T_l|+1)/(|T|+1) \geq \alpha > \frac{2}{11}$ we have $|T| \leq 9$;
$|T| = 9$: then $\alpha \leq \frac{2}{10}$ and $b \leq 4$ and, hence, $\gamma_2 = 1/(b+1) = \frac{1}{5} \geq \alpha$;
$|T| = 8$: then $\alpha \leq \frac{2}{9}$. if $b \leq 3$, then $\gamma_2 = \frac{1}{4} \geq \alpha$. If $b = 4$, then $c+d \leq 3$ and hence either $c \leq 1$ or $d \leq 1$. Hence $\alpha \leq \frac{1}{5}$ and $\gamma_2 \geq \frac{1}{5} \geq \alpha$;
$|T| = 7$: then $\alpha \leq \frac{2}{8}$ and $b \leq 3$ and, hence, $\gamma_2 \geq \frac{1}{4} \geq \alpha$;
$|T| = 6$: then $\alpha \leq \frac{2}{7}$. If $b \leq 2$, then $\gamma_2 \geq \frac{1}{3} \geq (1+\delta)\alpha$. If $b = 3$, then $c = d = 1$ and hence $\alpha \leq \frac{1}{5}$. Thus $\gamma_2 = \frac{1}{4} \geq \alpha$;
$|T| \leq 5$: then $\alpha \leq -\frac{1}{2}\sqrt{2}$ and $b \leq 2$ since $c \geq 1$ and $d \geq 1$ and, hence, $\gamma_2 = \frac{1}{3} \geq (1+\delta)(1-\frac{1}{2}\sqrt{2}) \geq (1+\delta)\alpha$.

Next we show $\gamma_2 \leq 1-(1+\delta)\alpha$:

$$\gamma_2 \leq \frac{\alpha}{\alpha + (1-\alpha)\left(\dfrac{1}{2-\alpha} + g(\alpha, \delta)\right)\alpha} \leq \frac{2-\alpha}{2-\alpha+(1-\alpha)(1+\frac{1}{2}\delta)}.$$

Consider

$$\begin{aligned} h(\alpha, \delta) &= 1-(1+\delta)\alpha - \frac{(2-\alpha)}{(2-\alpha)+(1-\alpha)(1+\frac{1}{2}\delta)} \\ &= \frac{(2\alpha^2-4\alpha+1) - \delta\alpha(3-2\alpha) + (1-\alpha)^2\delta/2 - \alpha(\frac{1}{2}\delta^2)(a-\alpha)}{(3-2\alpha)+(1-\alpha)\cdot\frac{1}{2}\delta}. \end{aligned}$$

The numerator of this expression is a quadratic equation in α and is decreasing in $\delta (\delta \leq 0.01, \alpha < \frac{1}{2})$. For $\delta = 0$ its zeroes are $1-\frac{1}{2}\sqrt{2}, 1+\frac{1}{2}\sqrt{2}$. Hence $h(\alpha, \delta) \geq 0$ for $\alpha \leq 1-\frac{1}{2}\sqrt{2}-c_2(\delta)$ with $c_2(0) = 0$, c_2 continuous and increasing and $c_2(0.01) \approx 0.0016$.

(II.3) We have to show $(1+\delta)\alpha \leq \gamma_3 \leq 1-(1+\delta)\alpha$:

$$\gamma_3 = \frac{\beta_2(1-\beta_3)}{1-\beta_2\beta_3} \quad \text{is increasing in } \beta_2 \text{ and decreasing in } \beta_3;$$

$$\gamma_3 \geq \frac{1-(1-\alpha)}{1/(1/(2-\alpha)+g(\alpha, \delta))-(1-\alpha)} \geq (1+\delta)\alpha$$

iff

$$\frac{1}{1/(1/(2-\alpha)+g(\alpha,\delta))-(1-\alpha)} \geq (1+\delta)$$

iff

$$1+(1-\alpha)(1+\delta) \geq \frac{(1+\delta)(2-\alpha)}{1+(2-\alpha)g(\alpha,\delta)}$$

iff

$$g(\alpha,\delta)\cdot(2-\alpha) \geq \frac{(1+\delta)(2-\alpha)}{1+(1-\alpha)(1+\delta)} - 1 = \frac{\delta}{1+(1-\alpha)(1+\delta)}.$$

This is true by definition of $g(\alpha, \delta)$.

Also

$$\gamma_3 \leq \frac{(1-\alpha)(1-\alpha)}{1-(1-\alpha)\alpha}$$

$$= \frac{1-2\alpha+\alpha^2}{1-\alpha+\alpha^2} = 1 - \frac{1}{1-\alpha+\alpha^2}\alpha$$

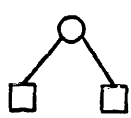
$$\leq 1 - \frac{1}{1-\frac{2}{11}+(\frac{2}{11})^2} \leq 1-(1+\delta)\alpha$$

since $1-\alpha+\alpha^2$ is decreasing for $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{2}{11}$.

Finally taking $c(\delta) = \max[c_1(\delta), c_2(\delta)]$ finishes the proof of the theorem.

Corollary 1. *If $\frac{2}{11} < \alpha \leq 1 - \frac{1}{2}\sqrt{2}$, then rotation and double rotation along the path of search suffice to rebalance the tree after insertion or deletion of a leaf.*

Proof. Inserting a leaf creates a subtree of the form



It has root balance $\frac{1}{2}$. Deleting a leaf means replacing a tree by one of its direct subtrees. In either case the new subtree is in BB[α]. Theorem 1 implies that we can walk back to the root and rebalance the tree by rotations and double rotations.

The corollary above is the correct version of the 'theorem' stated in [8]. In the next section we use Theorem 1 to prove an upper bound on the average number of rebalancing operations.

3. The average number of rebalancing operations

In this section we will prove our main theorem: the average number of rebalancing operations is constant. We need some notation first.

A *transaction* is either an insertion or a deletion. A *transaction goes through a node* v if v is on the path of search to the leaf to be inserted or deleted or (alternatively) if the leaf (to be inserted or deleted) is a descendant of v. A *node* v *takes part in a rebalancing operation*, if it is one of the nodes explicitly shown in Fig. 1. A *node causes a rebalancing operation* if it is the root of one of trees shown on the left side in Fig. 1. Furthermore nodes retain the identity as shown in Fig. 1, i.e. if a rotation to the left is applied to a tree with root A, then node A has subtrees of weight a and b respectively after the rotation. Note also that new nodes are created by insertions and that nodes are destroyed by deletions. Finally consider any sequence of transactions. We start with a tree T_0 and apply the first transaction to it. Then the tree is rebalanced as described at the end of the previous section, resulting in tree T_1. The next transaction is applied to T_1, T_1 is rebalanced, ... Let $T_0, T_1, T_2, \ldots, T_m, \ldots$ be any such sequence of BB[α]-trees.

Lemma 2. *Let* $0 \leq \delta \leq 0.01$, $\frac{2}{11} < \alpha \leq 1 - \frac{1}{2}\sqrt{2} - c(\delta)$ *and let* v *be a node. If*

(1) v *causes a rebalancing operation in* T_m *(after the transaction was applied to* T_m*) and*

(2) *either* v *took part in a rebalancing operation before or* v *was not a node of the initial tree* T_0 *and never took part in a rebalancing operation before and*

(3) n *is the number of leaves in the subtree with root* v *in* T_m *and* $n \geq 11$ *if* $\alpha \leq \frac{1}{4}$, *then at least* $\lceil \delta\alpha n \rceil$ *transactions went through* v *since* v *took part in a rebalancing operation for the last time or* v *was created.*

Proof. Let $j < m$ be such that: v took part in a rebalancing operation in T_j, but not in $T_{j+1}, \ldots, T_{m-1}$ or v did not exist in T_j but existed in $T_{j+1}, \ldots, T_{m-1}$ and never took part in a balancing operation. In the second case the balance $\rho(v)$ of node v in T_{j+1} is $\frac{1}{2}$. In the first case the balance $\rho(v) = t'/n'$ of node v in T_{j+1} is in $[(1+\delta)\alpha, 1-(1+\delta)\alpha]$ or $\alpha \leq \frac{1}{4}$ and $n' \leq 10$. 10. This is an immediate consequence of Theorem 1. Also the balance $\rho(v) = t/n$ of node v in T_m is outside the interval $[\alpha, 1-\alpha]$, say $t/n < \alpha$.

Node v did not take part in a rebalancing operation in trees $T_{j+1}, \ldots, T_{m-1}$. In these trees d_l (i_l) deletions (insertions) were performed in the left subtree of v and d_r (i_r) deletions (insertions) were performed in the right subtree of v. Hence

$$t = t' - d_l + i_l,$$

$$n = n' - d_l - d_r + i_l + i_r.$$

The number of transactions which went through v is $d_l + d_r + i_l + i_r$. We need a lower bound on that number. Certainly abs$(n - n')$ is a lower bound. Hence we are done if $n' \leq 10$ and $\alpha \leq \frac{1}{4}$. Suppose $n' > 10$ or $\alpha > \frac{1}{4}$ and hence $t'/n' \in [(1+\delta)\alpha, 1-(1+\delta)\alpha]$.

Assume to the contrary that $d_l+d_r+i_l+i_r<\delta\alpha n$. Then

$$\begin{aligned}\frac{t'}{n'}-(1+\delta)\alpha &= \frac{t+d_l-i_l}{n+d_l+d_r-i_l-i_r}-(1+\delta)\alpha\\ &\leqslant \frac{t+d_l}{n+d_l-i_r}-(1+\delta)\alpha \leqslant \frac{t+\delta\alpha n}{n+\delta\alpha n}-(1+\delta)\alpha\\ &\leqslant \frac{\alpha n+\delta\alpha n}{n+\delta\alpha n}-(1+\delta)\alpha = -\frac{(1+\delta)\delta\alpha^2}{1+\delta\alpha}<0, \quad \text{a contradiction.}\end{aligned}$$

Lemma 2 shows that many transactions go through a node v between rebalancing operations involving v. In order to finish off the proof all we need is a clever way of counting transactions and rebalancing operations.

With every node v we associate accounts: the transaction accounts $\mathrm{TA}_i(v)$ and the balancing operation accounts $\mathrm{BO}_i(v)$, $0\leqslant i<\infty$. Furthermore there is a special account S. All accounts have initial value zero.

Let T_0 be a tree in BB[α] and let $T_0, T_1, \ldots, T_m, \ldots$ be as above a transaction sequence of BB[α]-trees. Let v be any node of T_m, and let n be the number of leaves in the subtree of T_m with root v. Let i be such that $(1/(1-\alpha))^i\leqslant n<(1/(1-\alpha))^{i+1}$. Note that $n\geqslant 2$ and, hence, $i\geqslant 1$.

If the transaction applied to T_m goes through node v, then we charge one unit to transaction accounts $\mathrm{TA}_{i-1}(v)$, $\mathrm{TA}_i(v)$ and $\mathrm{TA}_{i+1}(v)$.

If v causes a rebalancing operation in T_m, then if v took part in a rebalancing operation before or was not a node of initial tree T_0, then we charge one unit to account $\mathrm{BO}_i(v)$ otherwise we charge one unit to special account S.

Note that for every node v of the initial tree T_0 at most one unit is charged to account S and, hence, $S\leqslant|T_0|-1$. It remains to sum the contents of the balancing operation accounts $\mathrm{BO}_i(v)$.

Whenever one unit is charged to account $\mathrm{BO}_i(v)$ we are in a situation to which Lemma 2 applies: if $n\geqslant 11$ or $\alpha>\frac{1}{4}$, then at least $\delta\alpha n$ transactions went through v since v took part in a rebalancing operation (if it ever did) or v was created. Since (cf. Fig. 4)

$$n-[1/(1-\alpha)]^{i-1}\geqslant\delta\alpha n \quad\text{and}\quad [1/(1-\alpha)]^{i+2}-n\geqslant\delta\alpha n$$

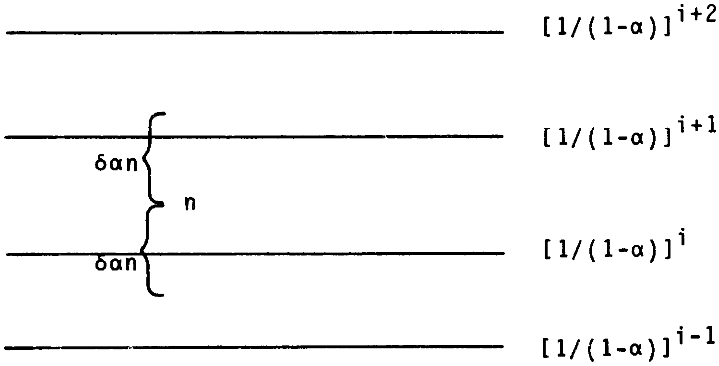


Fig. 4.

we even know that $\delta\alpha n$ units were charged to $\mathrm{TA}_i(v)$ since v took part in a rebalancing operation (if it ever did) or v was created. Hence,

$$\mathrm{BO}_i(v) \leq \frac{1}{\delta\alpha n}\mathrm{TA}_i(v) \leq \frac{(1-\alpha)^i}{\delta\alpha}\mathrm{TA}_i(v)$$

if $\alpha > \frac{1}{4}$ or $(1/(1-\alpha))^i \geq 11$. Since $\alpha > \frac{2}{11}$ this is certainly the case for $i \geq 12$.

We are now ready to estimate the total number A of rebalancing operations required to perform the first m transactions. Let k be any integer ≥ 12. Then

$$\begin{aligned} A &= S + \sum_v \sum_i \mathrm{BO}_i(v) \\ &= S + \sum_v \sum_{i<k} \mathrm{BO}_i(v) + \sum_v \sum_{i \geq k} \mathrm{BO}_i(v). \end{aligned}$$

Now $S \leq |T_0| - 1$ and

$$\sum_v \sum_{i \geq k} \mathrm{BO}_i(v) \leq$$

$$\leq \frac{1}{\delta\alpha} \cdot \sum_v \sum_{i \geq k} (1-\alpha)^i\, \mathrm{TA}_i(v)$$

$$\leq \frac{1}{\delta\alpha} \cdot \sum_v \sum_{i \geq k} \sum_{j=0}^{m-1}$$ [if the jth transaction goes through v and for the number n of leaves in the subtree of T_j with root v $[1/(1-\alpha)]^{i-1} \leq n < [1/(1-\alpha)]^{i+2}$ **then** $(1-\alpha)^i$ **else** 0]

$$\leq \frac{1}{\delta\alpha} \cdot \sum_{j=0}^{m-1} \sum_{i \geq k} \sum_v [\cdots]$$

$$\leq \frac{1}{\delta\alpha} \cdot \sum_{j=0}^{m-1} \sum_{i \geq k} 3 \cdot (1-\alpha)^i$$

since T_j is a tree in $\mathrm{BB}[\alpha]$ and hence for fixed i a transaction goes through at most one node v with $[1/(1-\alpha)]^i \leq n < [1/(1-\alpha)]^{i+1}$

$$\leq [3(1-\alpha)^k/\delta\alpha^2] \cdot m$$

and

$$\sum_v \sum_{i<k} \mathrm{BO}_i(v) \leq$$

$$\leq \sum_{j=0}^{m-1} \sum_{i<k} \sum_v$$ [if v causes a rebalancing operation in T_j and for the number n of leaves in the subtree of T_j with root v

$$[1/(1-\alpha)]^i \leq n < [1/(1-\alpha)]^{i+1}$$

then 1 **else** 0]

$$\leq \sum_{j=0}^{m-1} \text{maximal depth of a BB}[\alpha] \text{ tree with } [1/(1-\alpha)]^k \text{ leaves}$$

$$\leq [\text{max. depth of a BB}[\alpha]\text{-tree with } [1/(1-\alpha)]^k \text{ leaves}] \cdot m$$

$$\leq (k-1) \cdot m.$$

Altogether we have shown

$$|T_0| - 1 + \min_{\substack{k \in N \\ k \geq 12}} [k - 1 + 3(1-\alpha)^k/\delta\alpha^2] \cdot m$$

rebalancing operations suffice to perform an arbitrary sequence of m transactions with initial tree T_0 and $\frac{2}{11} < \alpha \leq 1 - \frac{1}{2}\sqrt{2} - c(\delta)$, i.e. we have

Theorem 2. *Let* $0 < \delta \leq 0.01$ *and* $\frac{2}{11} < \alpha \leq 1 - \frac{1}{2}\sqrt{2} - c(\delta)$ *where c is defined as in Theorem 1. Then there is a constant d such that: for* T_0 *any tree in* BB[α], *at most* $|T_0| - 1 + d \cdot m$ *balancing operations are required to perform an arbitrary sequence of m insertions and deletions with initial tree* T_0.

Corollary 2. *There is a constant d such that* $d \cdot m$ *balancing operations suffice to perform m insertions and deletions on an initially empty tree.*

It is worth to estimate the constant d for a specific example: $\alpha = \frac{1}{4}$ and $\delta = 0.01$.

Let $k = 25$, then $3(1-\alpha)^K/\delta\alpha^2 \approx 3.61$. A BB[$\frac{1}{4}$] tree with $\leq (\frac{4}{3})^{25} \approx 1329$ leaves has depth at most 23.87 (cf. [7, 8]). Hence $d \leq 27.48$.

Remark. 27 is a rather pessimistic estimate. This constant could be improved by a more careful version of Theorem 1 combined with a detailed analysis of trees of small depth (cf. [10]).

References

[1] Aho, Hopcroft and Ullman, *The Design and Analysis of Computer Algorithms* (Addison-Wesley, Reading, MA, 1974).

[2] Adel'son-Velskii and Laudis, An algorithm for the organization of information, *Soviet. Math. Dokl.* **3** (1962) 1259–1262.

[3] Bayer and Schkolnik, Concurrency of operations on B-trees, *Acta Informat.* **9** (1) (1977) 1–22.

[4] Foster, Information storage and retrieval using AVL-trees, *Proc. ACM Nat. Conf.* **20** (1965) 192–205.

[5] Karlton, Fuller, Scroggs and Kaehler, Performance of height balanced trees, *J. ACM* **19** (1) (1976) 23–28.

[6] Knuth, *The Art of Computer Programming, Vol. III: Sorting and Searching* (Addison-Wesley, Reading, MA, 1974).
[7] Mehlhorn, *Effiziente Algorithmen* (Teubner Verlag, 1977).
[8] Nievergelt and Reingold, Binary search trees of bounded balance, *SIAM J. Comput.* 2 (1) (1973).
[9] Ottman and Six, Eine neue Klasse von ausgeglichenen Bäumen, *Angewandte Informatik* (1976) 395–400.
[10] Yao, On random 2-3 trees, *Acta Informat.* **9** (2) (1978) 159–170.