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Iterative and Recursive Matrix Theories*

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TO THE MEMORY OF CALVIN C. ELGOT

Matrix theories are algebraic theories (in the sense of Lawvere) in which each morphism $p: [m] \to [n]$ is an $m \times n$ matrix of morphisms $p_{ij}: [1] \to [1]$. We develop the iterative matrix theories in which systems of linear equations have solutions and the recursive matrix theories in which systems of polynomial equations have solutions. There are intimate connections with formal power series as developed in formal language theory: the iterative matrix theories are based on rational sets of formal power series and the recursive matrix theories are based on algebraic sets of formal power series. The motivation is the applications of these matrix theories to the study of computer program behavior.

1. INTRODUCTION

Matrix theories naturally arise in the study of computer programming languages which have nondeterministic behavior [13]. The structure of a program with m input control lines and n output control lines is a morphism $p: [m] \to [n]$ [7]. Breadth-first nondeterministic behavior means that control may enter on several lines simultaneously and may exit on several lines simultaneously [11]. These considerations lead to treating p as an $m \times n$ matrix of morphisms $p_{ij}: [1] \to [1]$, each of which is the structure of the program with control entering the ith input line and leaving the jth output line. The behavior of the program is a coproduct preserving functor [5, 7, 13, 14]. As coproducts are biproducts in matrix theories, the program behaviors are algebras in the standard algebraic theory sense. This contrasts sharply with the situation for ordinary deterministic programs, where free behaviors do not exist [13].

The standard result in models of computer program behavior is the

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construction of a "space" in which systems of simultaneous equations have solutions, e.g., [1, 7, 9]. The papers [4, 9] establish such a space, the iterative algebraic theories, for linear equations. As matrix theories provide an algebraically robust model for the structure of nondeterministic programs, it is valuable for considerations in theoretical computer science to find those matrix theories in which systems of equations have solutions. The relationship of studies such as this to programming languages is given in [10].

Elgot has essentially done this in Section 12 of [6] for the case of matricial theories and linear equations. Since matricial theories generalize matrix theories, some of the structures we desire have already been implicitly found. However, by considering only the case of matrix theories, we demonstrate connections to older work in automata and formal language theory via the use of formal power series. These connections substantially reinforce our position that matrix theories are a fruitful mathematical model for the structure of nondeterministic programming languages [13].

Specifically, in Section 2 we develop the necessary formal power series results, following the terminology used by formal language theorists [2, 8, 12, 15, 16]. We then proceed to apply these results to finding the matrix theories in which linear systems of equations have solutions. These matrix theories are equivalent to rational power series and in the idempotent case, to their supports, the regular languages. This connection is implicit in [6]; we merely changed notation to develop the connection. In the last section, we develop the matrix theories in which systems of polynomial equations have solutions. These matrix theories are equivalent to algebraic power series and in the idempotent case to their supports, the context-free languages.

2. Formal Power Series

We give only the notation and results required here. For proofs and additional details, see [15].

Let R be a commutative semiring. R is said to be positive if $a+b=0$ implies $a=b=0$, for all $a, b \in R$. Let X^* be the free monoid generated by X. Functions $s: X^* \to R$ are called formal power series with noncommuting variables. A formal power series s is denoted by the formal sum

$$s = \sum_{w \in X^*} (s, w)w$$

with coefficients $(s, w) \in R$. The semiring of all formal power series is denoted $R\langle\langle X^*\rangle\rangle$. For each formal power series s, the support of s is the set $\text{supp}(s) = \{w \mid (s, w) \neq 0\}$. The support of a formal power series is a language over the alphabet X. The formal power series s is a polynomial if its support is finite. The semiring of polynomials is denoted $R\langle X^*\rangle$.

Let λ denote the identity of X^*. The series s is said to be quasiregular if $(s, \lambda) = 0$. If s is quasiregular, then $\lim_{n\to\infty} \sum_{k=1}^{n} s^k$ exists, is denoted s^+ and is called the quasi-inverse of s. A subsemiring, $A \subseteq R\langle\langle X^*\rangle\rangle$, is said to be rationally closed if it contains the quasi-inverse of quasiregular series in A. The subsemiring of rational series, $R^{\text{rat}}\langle\langle X^*\rangle\rangle$, is the smallest rationally closed semiring containing the polynomials.

Let $z = (z_1, ..., z_n)^t$ be a vector of n variables. A proper linear system is a system of n equations

$$z = P + Qz,$$

where P is a $1 \times n$ matrix over $R\langle\langle X^*\rangle\rangle$ and Q is an $n \times n$ matrix of quasiregular elements from $R\langle\langle X^*\rangle\rangle$.

PROPOSITION 2.1. *$s \in R^{\text{rat}}\langle\langle X^*\rangle\rangle$ iff there exists a proper linear system $z = P + Qz$, where P and Q have rational entries, such that s is the first component of the unique solution. P and Q can be chosen to be polynomials. If moreover R is positive, then the support of any $s \in R^{\text{rat}}\langle\langle X^*\rangle\rangle$ is a regular language over alphabet X.* ∎

Our terminology in the above is drawn from [15]. Quasiregular elements correspond to Elgot's "positive" sequences. Proper linear systems correspond to Elgot's semipositive morphisms. Our use of positive, as in positive semiring, is the natural extension of the usage in the traditional mathematics, e.g., the Perron–Frobenius theory of positive matrices.

Let Z be a set of n variables disjoint from the generators X. An algebraic system of equations is the collection of equations

$$z_i = p_i, \qquad i = 1, ..., n, \quad z_i \in Z,$$

where each p_i is a polynomial over X and Z, i.e., $p_i \in R\langle(X \cup Z)^*\rangle$. An algebraic system of equatons is proper iff $\forall i, j$: $(p_i, \lambda) = (p_i, z_j) = 0$. An algebraic system of equations is nonsingular iff $\forall i$, $\forall w \in Z^* - \{\lambda\}$: $(p_i, w) = 0$. Notice that the standard terminology is awkward here. A proper linear system of polynomial equations is a nonsingular algebraic system but is not necessarily a proper algebraic system.

DEFINITION 2.2. A normal algebraic system of equations is either a proper algebraic system of equations or a nonsingular algebraic system of equations.

This definition is not entirely satisfying. The intent of normality is to describe all algebraic systems of equations for which solutions may be obtained starting from the approximation $z^0 = (0, 0, \ldots, 0)^t$. Definition 2.2 fails to capture all such situations. We know of no work providing the full generalization desired. However, the notion described here as the nonsingular algebraic systems of equations satisfactorily abstracts the situations of interest in theoretical studies of programming languages. The term "nonsingular" is a generalization from [3] to the nondeterministic case. Greibach schemes are both proper and nonsingular [1, 10]. The converses are false. The algebraic system $\{z = z^2 + x\}$ is proper but not Greibach, while the algebraic system $\{z = \lambda + xz\}$ is nonsingular but not Greibach.

PROPOSITION 2.3. *Each normal algebraic system of equations has a unique solution $s = (s_1, \ldots, s_n)$ such that $(s_i, \lambda) = (p_i, \lambda)$ for $i = 1, \ldots, n$.*

Proof sketch. See [15] for the case of proper algebraic systems of equations, where $(s_i, \lambda) = (p_i, \lambda) = 0$, $i = 1, \ldots, n$. The nonsingular case has essentially the same proof, where only it must be noted that any solution s must satisfy $(s_i, \lambda) = (p_i, \lambda)$, $i = 1, \ldots, n$. Uniqueness is shown by induction on the approximations of s. ∎

We generalize the terminology of [15] in the following:

DEFINITION 2.4. A vector of formal power series, $s = (s_1, \ldots, s_n)$, is said to be algebraic, and each element of the vector is said to be an algebraic formal power series, if s is the unique solution of a normal algebraic system of equations, $z_i = p_i$, $i = 1, \ldots, n$, such that $(s_i, \lambda) = (p_i, \lambda)$, $i = 1, \ldots, n$.

From [2, 15] we have: the support of an algebraic formal power series over a positive semiring R is a context-free language. That language is λ-free in case the formal power series is quasiregular. The family of algebraic formal power series is denoted $R^{\mathrm{alg}}\langle\langle X^*\rangle\rangle$. Finally, a formal power series is algebraic iff it is the solution of a normal algebraic system of equations, the coefficients thereof being at most algebraic. Hence $R^{\mathrm{alg}}\langle\langle X^*\rangle\rangle$ is closed under substitution as well as being a subsemiring of $R\langle\langle X^*\rangle\rangle$.

3. MATRIX THEORIES

Matrix theories are defined in [6]. We give an equivalent formulation:

> A matrix theory is a finitary algebraic theory with objects $[n]$ for each natural number n. The object $[n]$ is both the coproduct and the product of $[1]$ with itself n times.

Therefore each morphism $p: [m] \to [n]$ is uniquely representable as an $m \times n$ matrix of morphisms $p_{ij}: [1] \to [1]$ in the matrix theory. Morphism composition is matrix multiplication since there is, by the existence of biproducts, a unique semiadditive structure on the category.

Further, Elgot showed in [6] that the category of matrix theories is equivalent to the category of semirings. One may view a matrix theory T as a collection of matrices over the semiring $T([1],[1])$. The algebras of T are semimodules, i.e., semiring R-modules, whenever there is a semiring homomorphism $T([1],[1]) \to R$ [13]. A morphism of matrix theories, $f: T \to T$, is an additive functor from T to T' leaving each $[n]$ fixed.

The free semiring generated by X is the semiring $\mathbb{N}\langle X^*\rangle$ of polynomials in the noncommuting variables X with coefficients in the set of natural numbers $\mathbb{N}$ [6, 15]. $\mathbb{N}\langle X^*\rangle$ is a positive semiring, hence it is also the free positive semiring over generators X. Since a matrix theory T is determined by the semiring $T([1],[1])$, say that a matrix theory is positive whenever the semiring $T([1],[1])$ is positive. If T is a positive matrix theory, for all morphisms f, g: $[m] \to [n]$, $f+g=0$: $[m] \to [n]$ implies f and g are the $m \times n$ 0 matrix.

It is then immediate from [6] that the free matrix theory, MX, generated by X is the collection of all matrices with elements in $\mathbb{N}\langle X^*\rangle$. MX is also the free positive matrix theory. From [13], the algebras of MX are $\mathbb{N}\langle X^*\rangle$-modules. The interpretation of this data is that each member of X is a single-entrance, single-exit atomic program; monoid multiplication is program composition, and the semiring sum represents nondeterministic alternatives between program segments with multiplicities given by the base semiring $\mathbb{N}$. An $\mathbb{N}\langle X^*\rangle$-module S is the collection of states upon which the program acts; the right action $S \times \mathbb{N}\langle X^*\rangle \to S$ defining the module describes the action of programs upon the states.

A semiring R is said to be idempotent if $r+r=r$ for all $r \in R$. A matrix theory T is said to be idempotent if the semiring $T([1],[1])$ is idempotent. If T is an idempotent matrix theory, then for all morphisms f: $[m] \to [n]$, $f+f=f$. Every idempotent matrix theory is a positive matrix theory since for all morphisms f, g: $[m] \to [n]$, $f+g=0$ implies $f=f+f+g=f+g=f+g+g=g=0$. Let $\mathbb{B}$ be the free semiring over the empty set of generators; the set $\{0,1\}$ is the carrier of $\mathbb{B}$. The natural choice of terminology is to call $\mathbb{B}$ the boolean semiring. It is not the two-element Boolean ring as $1+1=1$ in $\mathbb{B}$. The free idempotent semiring generated by X is the semiring $\mathbb{B}\langle X^*\rangle$ of polynomials in noncommuting variables X over the boolean semiring. It is immediate that $\mathbb{B}\langle X^*\rangle$ is isomorphic to the set of supports of the polynomials in $\mathbb{N}\langle X^*\rangle$, which set is the collection of finite subsets of X^*.

It is now clear that the free idempotent matrix theory generated by X, BX, is the collection of all matrices with coefficients in $\mathbb{B}\langle X^*\rangle$. The algebras may receive an interpretation for computer science similar to the one above. However, one also considers an element of $\mathbb{B}\langle X^*\rangle$ as a set of possible programs, exactly one of which is nondeterministically activated to perform the state transition formalized by the module action $S \times \mathbb{B}\langle X^* \rangle \to S$.

Elgot defined ideal morphisms and ideal algebraic theories in [7].

PROPOSITION 3.1. *No matrix theory is ideal.*

Proof. Let δ: $[1] \to [2]$ be the matrix $[id, id]$. The morphism δ is not distinguished in the sense of [6], i.e., not an injection such as $[id, 0]$. Let π: $[2] \to [1]$ be the projection $[id, 0]^t$. Then $\delta \cdot \pi = id$: $[1] \to [1]$, which is distinguished. ∎

In [7], Elgot defined iterative algebraic theories to be ideal algebraic theories in which the iteration equations can be solved for ideal morphisms. In matrix theories the iteration equations become linear equations. Since the notion of iteration is fundamental to applications, we retain that term in the next section, despite the fact that an iterative matrix theory is not an iterative algebraic theory. The idea in both cases is to capture the behavior of the looping or iterative construct of programming languages.

4. ITERATIVE MATRIX THEORIES

The definition of an iterative matrix theory requires two preliminary concepts. The first extends the notion of quasiregular element to arbitrary semirings.

DEFINITION 4.1. An element a of a semiring R is said to be quasiregular iff it has a quasi-inverse, i.e., there exists an a^+ such that $a^+ = a + aa^+$. A matrix over R is said to be regular if all of its coefficients are quasiregular.

Note that quasiregularity is preserved by semiring homomorphisms and that 0 is quasiregular.

To follow Elgot's notation as closely as possible, we use the following:

DEFINITION 4.2. A proper iteration in a matrix theory T is a matrix equation

$$Z = [Q \,\vdots\, P] \begin{bmatrix} Z \\ \cdots \\ I \end{bmatrix},$$

where Z is an $n \times p$ matrix, Q is a regular $n \times n$ matrix over $T([1], [1])$, P is an $n \times p$ matrix over $T([1], [1])$, and I is the $p \times p$ identity matrix.

The above is the matrix theory correspondent of the iteration equation $\xi = f(\xi, 1_p)$ from [6, 7]. Note that in [6], Elgot used the term positive element where we have used quasiregular element and the term positive morphism where we have used regular matrix. In the sequel, we write iterations in the equivalent form $Z = P + QZ$.

DEFINITION 4.3. A subhemiring of semiring R is a multiplicatively closed submonoid of R. It fails to be a subsemiring only in that it may lack the multiplicative unit. If the set of quasiregular elements of R form a subhemiring, it is said to be the quasiregular subhemiring of R.

EXAMPLES. Let R be a commutative semiring. In each of $R\langle\langle X^*\rangle\rangle$, $R^{\mathrm{alg}}\langle\langle X^*\rangle\rangle$, $R^{\mathrm{rat}}\langle\langle X^*\rangle\rangle$, $R\langle X^*\rangle$ the set of all quasiregular elements forms the quasiregular subhemiring.

DEFINITION 4.4. A matrix theory T is iterative if (i) the quasiregular elements of $T([1], [1])$ form a subhemiring, and (ii) every proper iteration has a unique solution in T.

Every morphism of matrix theories between iterative matrix theories, $f: T \to T'$, is a morphism of iterative matrix theories since f preserves regularity and hence the unique solutions of proper iterations.

The following lemmas develop the connection between our formulation and that of [6].

LEMMA 4.5. *Let a: $[m] \to [n]$ be a morphism in MX. If a is regular then a is ideal, but the converse is false.*

Proof. For $\alpha \in \mathbb{N}$, α is quasiregular iff $\alpha = 0$. Hence the quasiregularity of Definition 4.1 coincides with the formal power series definition in $\mathbb{N}\langle\langle X^*\rangle\rangle$. As a is regular, $(a_{ij}, \lambda) = 0$ for all i and j. Therefore no row of a, a_i, is distinguished. For each morphism f: $[n] \to [p]$, $a_i \cdot f$ is not distinguished as MX is free. Therefore a is ideal. However, the polynomial $s = 2\lambda$, i.e., $(s, \lambda) = 2$, $(s, w) = 0$ for $w \neq \lambda$, is ideal but not quasiregular. ∎

LEMMA 4.6. *The regular morphisms of MX form a right ideal in the sense of Elgot* [6, 7].

Proof. Let a: $[k] \to [m]$, b: $[m] \to [n]$ be MX morphisms with a regular. Then $(a_{ik}, \lambda) = 0$ for all i, k implies $((ab)_{ij}, \lambda) = (\sum_{k=1}^{m} a_{ik} \cdot b_{kj}, \lambda) = \sum_{k=1}^{m} (a_{ik}, \lambda)(b_{kj}, \lambda) = 0$. ∎

THEOREM 4.7. *The free (positive) iterative matrix theory generated by X, IMX, is the collection of all matrices with elements in $\mathbb{N}^{\mathrm{rat}}\langle\langle X^*\rangle\rangle$.*

Proof. Positivity follows immediately from the positivity of $\mathbb{N}\langle\langle X^*\rangle\rangle$. The semiring $\mathbb{N}^{\mathrm{rat}}\langle\langle X^*\rangle\rangle$ and the matrix theory IMX are iterative by

Proposition 2.1 and the fact that the quasiregular elements of $\mathbb{N}^{\text{rat}}\langle\langle X^*\rangle\rangle$ are closed under the semiring operations. Let T be an iterative matrix theory and let $f: X \to T([1], [1])$ be a function such that $f(x)$ is quasiregular for each $x \in X$. As MX is the free matrix theory, f has a unique extension to the morphism of matrix theories $\bar{f}: MX \to T$; by condition (i) of the definition of an iterative matrix theory, $\bar{f}(s)$ is quasiregular for any quasiregular polynominal s. By Proposition 2.1, every morphism s in $IMX([1], [1]) = \mathbb{N}^{\text{rat}}\langle\langle X^*\rangle\rangle$ is the first component of some proper linear system $z = P + Qz$, where P and Q are morphisms of MX. Define f': $\mathbb{N}^{\text{rat}}\langle\langle X^*\rangle\rangle \to T([1], [1])$ by sending s to the first component of the unique solution to the iteration $z = \bar{f}(P) + \bar{f}(Q)z$ in T, a proper iteration since $\bar{f}$ preserves regularity. To show that f' is a semiring homomorphism, let σ_i be the solution of the proper linear system $z_i = P_i + Q_i z_i$ and let s_i be the first component of σ_i, $i = 1, 2$. The system

$$\begin{bmatrix} z \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \bar{P}_1 + \bar{P}_2 \\ P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} 0 & \bar{Q}_1 & \bar{Q}_2 \\ 0 & Q_1 & 0 \\ 0 & 0 & Q_2 \end{bmatrix} \begin{bmatrix} z \\ z_1 \\ z_2 \end{bmatrix},$$

where bar denotes the first row and z is a single variable, has the solution $[s_1 + s_2, \sigma_1 + \sigma_2]^t$. Transforming this system by $\bar{f}$ gives the solution $[f'(s_1) + f'(s_2), f'(\sigma_1), f'(\sigma_2)]^t$ in T, hence $f'(s_1 + s_2) = f'(s_1) + f'(s_2)$. The system

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} P_1 \bar{P}_2 \\ P_2 \end{bmatrix} + \begin{bmatrix} Q_1 & P_1 \bar{Q}_2 \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

where bar denotes the first row, has the solution $[\sigma_1 s_2, \sigma_2]^t$. To show that $f'(\sigma_1) f'(s_2)$ is a solution for z_1 in the $\bar{f}$ transform of this system, calculate:

$$\begin{aligned}
&\bar{f}(P_1 \bar{P}_2) + \bar{f}(Q_1) f(\sigma_1) f'(s_2) + \bar{f}(P_1 \bar{Q}_2) f'(s_2) \\
&\quad = \bar{f}(P_1)(\bar{f}(\bar{P}_2) + \bar{f}(\bar{Q}_2) f'(s_2)) + \bar{f}(Q_1) f'(\sigma_1) f'(s_2) \\
&\quad = \bar{f}(P_1) f'(s_2) + \bar{f}(Q_1) f'(\sigma_1) f'(s_2) \\
&\quad = (\bar{f}(P_1) + f'(Q_1) f'(\sigma_1)) f'(s_2) \\
&\quad = f'(\sigma_1) f'(s_2),
\end{aligned}$$

hence $f'(s_1 s_2) = f'(s_1) f'(s_2)$. For each m and n define the family of morphisms δ_{ij}: $[m] \to [n]$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, by

$$(\delta_{ij})_{kl} = \begin{cases} id: [1] \to [1] & \text{if } i = k, j = l, \\ 0 & \text{if not.} \end{cases}$$

Any morphism $p: [m] \to [n]$ in a matrix theory has the representation $\sum_{i,j} p_{ij} \delta_{ij}$. Every morphism of matrix theories preserves the δ_{ij}. Define $\bar{f}': IMX \to T$ by $\bar{f}'(p) = \sum_{i,j} f'(p_{ij}) \delta_{ij}$. Let $i: MX \to IMX$ be the insertion. Then $i \cdot \bar{f}' = \bar{f}$ as each morphism g in MX is the solution to the proper iteration $Z = g + 0Z$. $\bar{f}'$ is the unique extension of $\bar{f}$ as solutions are unique in iterative matrix theories. ∎

In the previous proof, the hypothesis that f maps the generators X into the quasiregular elements of T is necessary. It is similar to the hypothesis in the case of free iterative theories that f maps generators into ideal morphisms [9, 4]. The proof that f' is a semiring homomorphism is taken directly from the proof of Theorem II.1.4 in [15].

In [6] Elgot considered the free matrix theory MX and then P, the set of regular matrices over the semiring $\mathbb{N}\langle\langle X^*\rangle\rangle$ of all formal power series. The ideal algebraic theory generated by P then consists of P together with the distinguished morphisms of MX. This theory is iterative in Elgot's sense [4, 3], hence cannot be a matrix theory. Moreover, it is too large to be a free iterative theory since it contains formal power series which cannot be obtained as solutions to proper iterations.

COROLLARY 4.8. *The free iterative idempotent matrix theory generated by X is the semiring of all matrices with elements in* $\mathbb{B}^{\mathrm{rat}}\langle\langle X^*\rangle\rangle = \operatorname{supp}(\mathbb{N}^{\mathrm{rat}}\langle\langle X^*\rangle\rangle)$, *the semiring of regular languages over alphabet X.* ∎

5. Recursive Matrix Theories

LEMMA 5.1. *Let R be a semiring. The multiples of the identity $U = \{0, 1, 1 + 1, ...\}$ form a subsemiring of R. Let Q be any subhemiring of R containing only quasiregular elements. Then $QU = UQ = Q$.* ∎

DEFINITION 5.2. Let T be a matrix theory with quasiregular subhemiring $Q \subseteq T([1], [1])$ and subsemiring $U \subseteq T([1], [1])$ of multiples of the identity. An n-term over T is a nonempty finite sequence alternating between morphisms $f_i: [n] \to [n]$ and natural numbers, $\langle f_1, n_1, ..., f_p, n_p\rangle$, $p > 0$, subject to the constraint that only the last natural number, n_p, may be zero. The evaluation of an n-term with respect to $n \times n$ matrix Z is the matrix $f_1 \cdot Z^{n_1} \cdot \cdots \cdot f_p \cdot Z^{n_p}$. A recursion equation in T is a formal polynomial equation

$$Z = \sum_t u_t t,$$

where t varies over n-terms, $u_t \in U$ and only finitely many u_t are nonzero. A solution to a recursion equation, if it exists, is a morphism $s: [n] \to [n]$ of T

such that s is the weighted sum, by weights u_t, of the evaluations of the n-terms with respect to s.

All the n-terms of the form $\langle f_i, 0\rangle$ which appear in a recursion equation with nonzero weights u_i may clearly be combined into the single n-term $\langle \sum u_i f_i, 0\rangle$ which, with weight 1, replaces the $\langle f_i, 0\rangle$ in the recursion equation without changing the solutions. We consider only recursion equations in this form so that the recursion equations have only at most a single constant term, $\langle g, 0\rangle$.

We restrict our attention to recursion equations $Z = \sum_t u_t t$ in which the morphisms $f\colon [n] \to [n]$ appearing in the n-terms t with $u_t \neq 0$ are matrices over the set $Q \cup U$, where Q is the quasiregular subhemiring of $T([1], [1])$ and U is the subsemiring of multiples of 1 in $T([1], [1])$. Now any recursion may be viewed as an algebraic system of equations [15, p. 118],

$$z_{ij} = p_{ij},$$

where p_{ij} is a polynomial in $U\langle (Q \cup \zeta)^* \rangle$, $\zeta = \{z_{ij}\}$. This is accomplished by evaluating the recursion equation with respect to Z formally in the semiring $U\langle (Q \cup U \cup \zeta)^* \rangle$, and then multiplying the $Q \cup U$-words to find elements of Q.

A recursion equation is said to be proper, nonsingular, or normal as the algebraic system of equations obtained by the above process is proper, nonsingular, or normal. These three cases must be separately treated.

Definition 5.3. A matrix theory T in which the quasiregular elements of $T([1], [1])$ form a subhemiring is said to be nonsingular recursive if every nonsingular recursion equation has a unique solution in T. Every morphism of matrix theories between nonsingular recursive matrix theories $f\colon T \to T'$ is a morphism of nonsingular matrix theories.

Theorem 5.4. *The free (positive) nonsingular matrix theory RMX is the collection of all matrices with elements in $\mathbb{N}^{\mathrm{alg}}\langle\langle X^* \rangle\rangle$.*

Proof. RMX is a nonsingular recursive matrix theory by Proposition 2.3. For T any nonsingular recursive matrix theory, let Q be the quasiregular subhemiring of $T([1], [1])$. Let $f\colon X \to Q$; as in the proof of Theorem 4.7, f has a unique extension $\bar{f}$ to MX which clearly maps $\mathbb{N}$ on U and preserves quasiregularity. Hence f preserves nonsingular systems. We can thus define $f'\colon \mathbb{N}^{\mathrm{alg}}\langle\langle X^* \rangle\rangle \to T([1], [1])$ as in Theorem 4.7. The only nontrivial point is to check that f' is a semiring homomorphism. Let s_i, $i = 1, 2$, be the first component of the unique solution of the nonsingular systems, $i = 1, 2$:

$$S_i\colon z_k^i = p_k^i, \qquad k = 1,\ldots, n_i.$$

Then clearly $s = s_1 s_2$ is the unique solution of the nonsingular system:

$$\left| \begin{array}{l} z = p_1^1 p_2^1 \\ S_1 \\ S_2 \end{array} \right.$$

Nonsingularity being preserved by $\bar{f}$ and T being nonsingular recursive, we deduce:

$$f'(s_1 s_2) = f'(s_1) f'(s_2).$$

Similarly, $s = s_1 + s_2$ is the unique solution of

$$\left| \begin{array}{l} z = p_1^1 + p_2^1 \\ S_1 \\ S_2 \end{array} \right.$$

(which is nonsingular since $\mathbb{N}^{\mathrm{alg}}\langle\langle X^* \rangle\rangle$ is positive).

The proof then goes on exactly as in Theorem 4.7. ∎

As every proper iteration is a nonsingular recursion equation, every nonsingular recursive matrix theory is an iterative matrix theory. The converse is clearly false since the set $\mathbb{N}^{\mathrm{rat}}\langle\langle X^* \rangle\rangle$ is not a nonsingular recursive matrix theory: not every recursion has a rational solution.

In the case of proper recursion equations it is necessary to specify the desired solution since proper algebraic systems of equations do not, in general, have unique solutions. For example, $\{z = zz\}$ is proper and any multiplicative idempotent is a solution. In particular 0 and 1 are solutions. Our choice is guided by the principle that solutions should be obtained by approximation from the $n \times n$ 0-matrix.

Definition 5.5. A matrix theory T is said to be proper recursive if every proper recursive equation has a unique quasiregular solution in T. Every morphism of matrix theories between proper recursive matrix theories, $f: T \to T'$, is a morphism of proper recursive matrix theories.

If T is a proper recursive matrix theory, then the only quasiregular multiplicative idempotent in $T([1], [1])$ is 0, as 0 must be the unique quasiregular solution to $\{z = zz\}$.

Theorem 5.6. *The free (positive) proper recursive matrix theory is RMX.*

Proof. The previous proof comes over, using the unique quasiregular solutions to lift $f: X \to Q$ to all of RMX. ∎

The category of nonsingular recursive matrix theories and the category of proper recursive matrix theories are subcategories of the category of matrix theories which have a nonnull intersection. The Greibach scheme matrix theories lie in the intersection.

The category of normal recursive matrix theories is defined to be the union of the two categories of Definition 5.3 and 5.5. All normal recursion equations have unique solutions in this category, where for proper equations unique must be read as unique quasiregular. By Theorems 5.4 and 5.6, *RMX* is the free normal recursive matrix theory generated by *X*. The normal recursive matrix theories are algebraically closed—complete in the terminology of [10]—with respect to normal recursions.

Arguments similar to the above apply in the (additively) idempotent case. The free normal recursive matrix theory generated by X is the semiring of all matrices over $\mathbb{B}^{\mathrm{alg}}\langle\langle X^*\rangle\rangle = \mathrm{supp}(\mathbb{N}^{\mathrm{alg}}\langle\langle X^*\rangle\rangle)$. $\mathbb{B}^{\mathrm{alg}}\langle\langle X^*\rangle\rangle$ is the set of standard context-free languages [2, 15]. Algebraic closure in this situation is the well-known closure of the context-free languages under language substitution.

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