Journal of Mathematical Analysis and Applications **240,** 290–293 (1999)
Article ID jmaa.1999.6583, available online at http://www.idealibrary.com on IDEAL®

NOTE

A Strengthened Carleman's Inequality

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Submitted by L. Debnath

Received January 25, 1999

In this paper, the results given in [2] have been generalized and a new simpler proof is given. © 1999 Academic Press

Key Words: Carleman's inequality, Monotonicity.

In [2], the Carleman's inequality was generalized. In this note, the results given in [2] can be further generalized and a new much simpler proof can be given.

The following Carleman's inequality is well known (see [1, Chapt. 9.12]).

THEOREM A. *Let* $a_n \geq 0, n = 1, 2, \ldots,$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$. *Then*

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n. \tag{1}$$

Recently, [2] gave an improvement of Theorem A, and the following result was proved.

THEOREM B (see [2, Theorem 3.1]). *Let* $a_n \geq 0$, $n = 1, 2, \ldots,$ *and* $0 < \sum_{n=1}^{\infty} a_n < \infty$. *Then*

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \frac{1}{2(n+1)}\right) a_n. \tag{2}$$

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In this note, we shall prove the following theorem.

THEOREM 1. *Let* $a_n \geq 0$, $n = 1, 2, \dots,$ *and* $0 < \sum_{n=1}^{\infty} a_n < \infty$. *Then*

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 + \frac{1}{n + \frac{1}{5}}\right)^{-1/2} a_n. \tag{3}$$

To prove Theorem 1, we first prove the following Lemma.

LEMMA 1. *Let* $x_n = [1 + (1/n)]^n$, *then*

$$x_n \left(1 + \frac{1}{n + \frac{1}{5}}\right)^{1/2} < e < x_n \left(1 + \frac{1}{n + \frac{1}{6}}\right)^{1/2} \tag{4}$$

for every positive integer n.

Proof. We make the following auxiliary function

$$f(x) = x \ln\left(1 + \frac{1}{x}\right) + \frac{1}{2} \ln\left(1 + \frac{1}{x + \frac{1}{5}}\right), x \in [1, \infty). \tag{5}$$

It is easy to see that

$$f'(x) = -\frac{1}{x+1} + \ln\left(1 + \frac{1}{x}\right) - \frac{1}{2} \frac{1}{\left(x + \frac{6}{5}\right)\left(x + \frac{1}{5}\right)}$$

and for $x \in [1, +\infty)$, it can be shown that

$$\begin{aligned} f''(x) &= \frac{1}{(x+1)^2} - \frac{1}{x(x+1)} + \frac{1}{2\left(x + \frac{1}{5}\right)^2} - \frac{1}{2\left(x + \frac{6}{5}\right)^2} \\ &= \frac{-5x(25x^2 + 10x - 7) - 72}{1250x(x+1)^2\left(x + \frac{1}{5}\right)^2\left(x + \frac{6}{5}\right)^2} < 0. \end{aligned}$$

Therefore, $f'(x)$ is decreasing on $[1, +\infty)$. Then for any $x \in [1, +\infty)$, we have $f'(x) > \lim_{x \to +\infty} f'(x) = 0$, thus, $f'(x)$ is increasing on $[1, +\infty)$, and $f(x) < \lim_{x \to +\infty} f(x) = 1$ for $x \in [1, +\infty)$. By the definition of $f(x)$, it turns out $x_n[1 + 1/(n + (1/5))]^{1/2} < e$.

Similarly we make the following auxiliary function

$$f_1(x) = x \ln\left(1 + \frac{1}{x}\right) + \frac{1}{2} \ln\left(1 + \frac{1}{x + \frac{1}{6}}\right), x \in [1, +\infty). \tag{6}$$

A direct calculation shows that $f_1''(x) > 0$ for $x \in [1, +\infty)$. Thus, $f_1'(x)$ is increasing on $[1, +\infty)$. Then for any $x \in [1, +\infty)$, we have $f_1'(x) < \lim_{x \to +\infty} f_1'(x) = 0$, therefore, $f_1'(x)$ is decreasing on $[1, +\infty)$, and $f_1(x) > \lim_{x \to +\infty} f_1(x) = 1$ for $x \in [1, +\infty)$. Obviously, the definition of $f_1(x)$ implies that $x_n[1 + 1/(n + (1/6))]^{1/2} > e$. Hence (4) is true for every positive integer n. This completes the proof of Lemma 1. ∎

Remark 1. By a direct calculation, we have

$$\frac{6n+2}{6n+5} < \left(1 + \frac{1}{n + \frac{1}{6}}\right)^{-1/2} < \left(1 + \frac{1}{n + \frac{1}{5}}\right)^{-1/2} < \frac{2n+1}{2n+2} \qquad (7)$$

for every positive integer n. Thus, Theorem 2.1 in [2] is contained in Lemma 1.

Proof of Theorem 1. Assume that $c_m > 0$ for $m = 1, 2, \ldots$. By the arithmetic–geometric average inequality, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} &= \sum_{n=1}^{\infty} \left(\frac{c_1 a_1 \cdot c_2 a_2 \cdots c_n a_n}{c_1 c_2 \cdots c_n}\right)^{1/n} \\
&= \sum_{n=1}^{\infty} (c_1 c_2 \cdots c_n)^{-1/a} (c_1 a_1 \cdot c_2 a_2 \cdots c_n a_n)^{1/n} \\
&\le \sum_{n=1}^{\infty} (c_1 c_2 \cdots c_n)^{1/n} \cdot \frac{1}{n} \sum_{m=1}^{n} c_m a_m \\
&= \sum_{m=1}^{\infty} c_m a_m \sum_{n=m}^{\infty} \frac{1}{n} (c_1 c_2 \cdots c_n)^{-1/n} \\
&= \sum_{m=1}^{\infty} c_m a_m \cdot \sum_{n=m}^{\infty} \frac{1}{n(n+1)} \\
&= \sum_{m=1}^{\infty} \frac{1}{m} c_m a_m = \sum_{m=1}^{\infty} \left(1 + \frac{1}{m}\right)^{m} a_m.
\end{aligned}$$

By Lemma 1, we obtain

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{m=1}^{\infty} \left(1 + \frac{1}{m + \frac{1}{5}}\right)^{-1/2} a_m$$

Thus, inequality (3) is proved. ∎

Remark 2. With the inequality (7), Theorem 1 implies Theorem 3.1 in [2].

Finally, we point out that (3.5) in [2] should be

$$(c_1 a_1 \cdot c_2 a_2 \cdots c_n a_n)^{1/n} \le \frac{1}{n} \sum_{m=1}^{n} c_m a_m .$$

Otherwise, the equality (3.6) in [2] is not true.

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