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European Journal of Combinatorics 28 (2007) 916–920
European Journal of Combinatorics
www.elsevier.com/locate/ejc

Characterizing almost-median graphs

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Received 27 June 2005; accepted 25 October 2005
Available online 23 November 2005

Abstract

Almost-median and semi-median graphs are two natural generalizations of the well-known class of median graphs. In this note we prove that a semi-median graph is almost-median if and only if it does not contain any convex cycle of length greater than four.

MSC: 05C12; 05C75

1. Introduction

Classes of graphs that admit isometric embeddings into hypercubes have been investigated for many years now. A hierarchy of these classes was proposed by Imrich and Klavžar [7], and was further studied in [4]. The hierarchy was extended also to the nonbipartite case of classes of isometric subgraphs of Hamming graphs; see [2,5].

Median graphs are undoubtedly the most well known and well studied class of isometric subgraphs of hypercubes (see a survey [10] and the references therein). Two natural generalizations of median graphs, almost-median and semi-median graphs, were introduced [7] with the intention of shedding more light on the recognition complexities of some classes in the hierarchy. The results regarding the recognition of these classes that followed [3,9] gave some partial improvements (for instance, almost-linear algorithms for some special classes of almost-median graphs were found). However, the best known recognition complexity for isometric subgraphs of hypercubes remains $O(mn)$ [8], and the same holds for the newly introduced classes in general. It is an open question whether this is optimal (for each of these classes), and an answer will likely require further knowledge about their structure.

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0195-6698/$ - see front matter
doi:10.1016/j.ejc.2005.10.009

Almost-median and semi-median graphs have been characterized in [3], using a certain expansion procedure by means of which one can inductively describe these two classes. In this note we present a new characterization of almost-median graphs of different flavor: among semi-median graphs they are precisely the graphs that possess no convex cycles C_{2n}, for $n \geq 3$. This result was conjectured by Peterin in [12].

In the next section we fix the notation and state some necessary preliminary results, and in the last section we prove the characterization of almost-median graphs.

2. Notation and preliminaries

For a graph G, the *distance* $d_G(u, v)$, or briefly $d(u, v)$, between vertices u and v is defined as the number of edges on a shortest u, v-path. A subgraph H of G is called *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. A subgraph H of G is *convex* if for any $u, v \in V(H)$, all shortest u, v-paths belong to H.

The *Cartesian product* $G \Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which the vertex (a, x) is adjacent to the vertex (b, y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The graphs $K_2 \Box P_n$ are called *ladders*. The Cartesian product of k copies of K_2 is a *hypercube* or *k-cube* Q_k. Isometric subgraphs of hypercubes are called *partial cubes*. An important subclass of partial cubes is the class of *median graphs* [11]. These are the graphs in which there exists a unique vertex x to every triple of vertices u, v, and w such that x lies simultaneously on a shortest u, v-path, a shortest u, w-path, and a shortest w, v-path.

For partial cubes, the following vertex sets play a crucial role. Let ab be an edge of a connected, bipartite graph $G = (V, E)$. Then

$$W_{ab} = \{w \in V \mid d_G(a, w) < d_G(b, w)\},$$
$$U_{ab} = \{w \in W_{ab} \mid w \text{ has a neighbor in } W_{ba}\}.$$

By abuse of language we shall use the same notation for the sets W_{ab}, U_{ab} and the subgraphs induced by them.

Clearly, W_{ab} and W_{ba} are disjoint. Moreover, as all graphs considered are bipartite, $V(G) = W_{ab} \cup W_{ba}$. Djoković [6] proved that a graph G is a partial cube if and only if it is bipartite and if for any edge ab of G the subgraph W_{ab} is convex. It follows from results in [1] that median graphs are precisely the bipartite graphs in which all U_{ab}'s are convex. By this result, the following notions make sense.

A bipartite graph is a *semi-median* graph if it is a partial cube in which for any edge ab the subgraph induced by U_{ab} is connected. Similarly, a bipartite graph is an *almost-median* graph if it is a partial cube such that for any edge ab the subgraph induced by U_{ab} is isometric.

Two edges $e = xy$ and $f = uv$ of G are in the Djoković–Winkler [6,13] relation Θ if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u).$$

Clearly, Θ is reflexive and symmetric. Winkler [13] proved that a bipartite graph is a partial cube if and only if $\Theta = \Theta^*$, where Θ^* denotes the transitive closure of Θ. We will need the following basic property of Θ; see [8]:

Lemma 1. *Suppose P is a walk connecting the endpoints of an edge e. Then P contains an edge f with $e \Theta f$.*

The following lemma also follows from the definition of the relation Θ.

Lemma 2. *Suppose P is a shortest path in a graph. Then no two edges of P are in relation Θ.*

Let G be a bipartite graph, and let $e = xy$ and $f = ab$ be two edges such that $d(x, a) < d(x, b)$. It is easy to see that e and f are in relation Θ precisely when $d(x, a) = d(y, b) = d(x, b) - 1 = d(y, b) - 1$. The following lemma is then easy to derive.

Lemma 3. *Let G be a bipartite graph and C an isometric cycle in G. Two edges of C are in relation Θ precisely when they are antipodal in C.*

Another relevant relation defined on the edge set of a graph is δ. We say an edge e is in relation δ to an edge f if e and f are opposite edges of a 4-cycle without diagonals in G or if $e = f$. Clearly δ is reflexive and symmetric. Moreover, it is contained in Θ. Thus its transitive closure δ^* is contained in Θ^*. In [7] it is shown that a bipartite graph is a semi-median graph if and only if $\Theta = \delta^*$, in analogy with Winkler's characterization of partial cubes.

Suppose $e = e_1 \delta e_2 \delta \ldots \delta e_k = f$ is a sequence of edges by virtue of which e and f are in relation δ^*. If, in addition, the endpoints of these edges induce two paths, then the union of the squares that contain e_i, e_{i+1} for $i = 1, 2, \ldots, k-1$ forms a ladder, that is, the Cartesian product of a path of length $k-1$ by an edge. In such a case we shall frequently use the expression that two edges e, f in relation δ^* are connected by a "ladder" (or by "isometric ladder", if the two paths, sides of the ladder, are isometric). In this language, a partial cube is an almost-median graph if and only if every two edges in relation Θ are connected by an isometric ladder.

3. The characterization

Theorem 4. *A graph G is almost-median if and only if G is a semi-median graph that contains no convex cycle C_{2n}, for $n > 2$.*

Proof. Let G be an almost-median graph. Then G is also semi-median. Suppose G has a convex cycle C on at least 6 vertices. Then any two antipodal edges in C are in relation Θ, yet they are not connected by an isometric ladder, since it would contradict the convexity of C. This is in contradiction with G being almost-median which proves one direction of the theorem.

For the converse, let G be a semi-median graph that contains no convex cycle C_{2n}, for $n > 2$. The assumption that G is not almost-median will lead us to a contradiction. If G is not almost-median then there are two edges ab and xy which are in relation Θ but no ladder between them is isometric. Among all such pairs of edges in G let them be chosen so that their distance $n \geq 2$ is as small as possible. Let C be a cycle formed by a shortest path $a = u_0, u_1, u_2, \ldots, u_n = x$, edge xy, a shortest path $y = v_n, v_n - 1, \ldots, v_1, v_0 = b$ and edge ba. We shall prove that C is convex which provides the desired contradiction. The proof is by induction on n.

First assume $n = 2$. Then C is isomorphic to C_6, and it is clear that C is induced (otherwise either $ab\Theta xy$ is violated or an isometric ladder between ab and xy exists contrary to our assumption). We now derive that C is isometric because G is bipartite. To see that C is convex is now easy and is left to the reader (the proof is very similar to, but simpler, than those in cases 2 and 3 below).

In what follows assume that $n \geq 3$, that is, C has more than 6 vertices. The assumption that there exists a path P that violates convexity of C will lead us to a contradiction. We may assume that P connects two vertices of C so that internal vertices of P are not on C, and that P is a shortest path between its end-vertices. We distinguish three cases.

Case 1. Suppose P is a shortest path between u_k and v_m that is shorter than at least one of the paths on C between u_k and v_m. (Note that in this case $0 < k < n$ and $0 < m < n$.) Since

$u_k \in W_{ab}$, $v_m \in W_{ba}$ there exists an edge $e = uv$ on P that is in relation Θ with ab. Observe that P creates two cycles together with C of which at least one is shorter than C. Since C is a shortest cycle in G with respect to having two edges in relation Θ with no isometric ladder between them, we infer that there is an isometric ladder between ab and e or between xy and e (to prevent the shorter cycle contradicting the minimality of C). Without loss of generality we may assume that there is an isometric ladder between ab and $e = uv$. Let wz be the edge on this ladder, forming a square together with ab. We claim that $w \neq u_1$. Indeed, if $w = u_1$, then $d(w, x) = d(a, x) - 1$, and since $wz \Theta xy \Theta ab$ we have $d(w, x) = d(a, x) - 1 = d(z, y) = d(b, y) - 1$. Thus the cycle formed by wz, xy, and the shortest paths between w and x, and z and y would be shorter than C. However, wz and xy are also not connected by an isometric ladder (because such a ladder would imply the existence of an isometric ladder between ab and xy), so we get in contradiction with the minimality of C. This proves that $w \neq u_1$, and by symmetry $z \neq v_1$. Next we claim that aw is not in relation Θ with any edge on C between u_0 and u_k. Suppose $aw \Theta u_t u_{t+1}$ Then we infer that $d(w, u_{t+1}) = d(a, u_{t+1}) - 1$; hence

$$d(w, x) = d(a, x) - 1.$$

Since $wz \Theta ab \Theta xy$ we get

$$d(w, x) = d(a, x) - 1 = d(z, y) = d(b, y) - 1$$

which leads us to the same contradiction with minimality as earlier (namely, $wz \Theta xy$ and the distance between wz and xy is smaller than n, yet there is no isometric ladder between them, because the ladder obtained from such an isometric ladder by adding ab would be an isometric ladder between ab and xy).

Now consider a walk (which is in fact a path) between a and w that first traverses the path $u_1, \ldots, u_k$, then goes along P between u_k and u, and then traverses one side (in W_{ab}) of the isometric ladder that connects uv with wz. By Lemma 1 there exists an edge on this walk that is in relation Θ with aw. We already proved that such an edge is not on C, and it is also not on the ladder because the ladder is isometric (by Lemma 2 no two vertices on a shortest path can be in relation Θ). We derive that aw is in relation Θ with an edge of P. By applying the same reasoning in the subgraph W_{ba} we infer that bz is in relation Θ with some edge on P. Since $aw \Theta bz$, we derive that two edges of P are in relation Θ. So by Lemma 2, P is not a shortest path, and this case is concluded.

Case 2. P is a shortest path between u_k and v_m that is of the same length as both paths on C between u_k and v_m. Let C_1 and C_2 be the cycles that P creates with C; note that they are of the same length as C. Since P is a shortest path, all its edges are pairwise not in relation Θ. Hence, combining Lemma 1 and a simple counting argument, for every edge e on P exactly one edge on C_1 (respectively C_2) exists that is in relation Θ with e. Hence all edges of C_1 (respectively C_2) on C between u_k and v_m are pairwise not in relation Θ, and thus they form a shortest path between u_k and v_m on C_1 (respectively C_2). We infer that $u_{k-1}u_k \Theta v_m w$, where w is a neighbor of v_m on P, and also $u_{k+1}u_k \Theta v_m w$ (if $k = 0$ then replace u_{k-1} by v_0, and if $k = n$ then replace u_{k+1} by v_n). By transitivity of the relation Θ this implies $u_{k-1}u_k \Theta u_{k+1}u_k$ which is clearly a contradiction.

Case 3. P is a shortest path connecting two vertices of C in W_{ab} (or W_{ba}). Since the path between u_0 and u_n (as well as the path between v_0 and v_n) is a shortest path, we infer that P is of the same length as the path between the corresponding vertices on C. This observation combined with Case 1 implies that C is isometric. We may assume without loss of generality that $u_k, w_{k+1}, \ldots, w_{m-1}, u_m$ is a path of length $m - k$ in W_{ab} whose internal vertices are not in C. Note that the cycle C' (obtained by replacing in C the path $u_k, u_{k+1}, \ldots, u_{m-1}, u_m$ with the path $u_k, w_{k+1}, \ldots, w_{m-1}, u_m$) is also isometric for the same reason as C is. We infer that edges $u_k u_{k+1}$ and $u_k w_{k+1}$ are in relation Θ with the same (the antipodal) edge of C in W_{ba}. This is in contradiction with the transitivity of the relation Θ, and so this case is also complete.

We conclude that C is convex, a desired contradiction. $\square$

Acknowledgments

I wish to thank Sandi Klavžar and Iztok Peterin for careful reading of the manuscript. The author was supported by the Ministry of Science and Technology of Slovenia under the grant P1-0297.

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