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On the complexity of bandwidth allocation in radio networks[☆]

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ABSTRACT

We define and study an optimization problem that is motivated by bandwidth allocation in radio networks. Because radio transmissions are subject to interference constraints in radio networks, physical space is a common resource that the nodes have to share in such a way, that concurrent transmissions do not interfere. The bandwidth allocation problem we study under these constraints is the following. Given bandwidth (traffic) demands between the nodes of the network, the objective is to schedule the radio transmissions in such a way that the traffic demands are satisfied. The problem is similar to a multicommodity flow problem, where the capacity constraints are replaced by the more complex notion of non-interfering transmissions. We provide a formal specification of the problem that we call *round weighting*. By modeling non-interfering radio transmissions as independent sets, we relate the complexity of round weighting to the complexity of various independent set problems (e.g. maximum weight independent set, vertex coloring, fractional coloring). From this relation, we deduce that in general, round weighting is hard to approximate within $n^{1-\varepsilon}$ (n being the size of the radio network). We also provide polynomial (exact or approximation) algorithms e.g. in the following two cases: (a) when the interference constraints are specific (for instance for a network whose vertices belong to the Euclidean space), or (b) when the traffic demands are directed towards a unique node in the network (also called *gathering*, analogous to single commodity flow).

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1. Introduction

The goal of this paper is to study how to allocate bandwidth efficiently to connections in a radio network. We address the static or off-line version of the problem, in which a fixed traffic demand $f(u, v)$ is given for any two devices u and v in the network, and the topology is known. The objective is to schedule radio transmissions in order to satisfy these demands, in such a way that, on average, about $f(u, v)dt$ units of traffic travel from u to v during a time interval dt.

In classical wired networks, this corresponds to the multi-commodity flow problem, which has been extensively studied in the literature [23,25,28]. In this problem, there is an undirected graph $G = (V, E)$, a capacity function $c : E \to \mathbb{Q}^+$ and a traffic demand function $f : V \times V \to \mathbb{Q}^+$, such that $f(u, v)$ is the amount of traffic that has to be routed from vertex u to vertex v. A solution for this problem is a positive function ϕ, such that for any path P in the graph, $\phi(P) \in \mathbb{Q}^+$ is the amount of traffic routed through P. Given ϕ, the load (use) of an edge $e \in E$ is simply calculated as $\text{load}(e) = \sum_{P:e\in P} \phi(P)$. Therefore, ϕ is feasible if and only if $\text{load}(e) \le c(e)$. The goal is to satisfy the demand constraints: $\sum_{P\in\mathcal{P}, P \text{ connecting } u \text{ to } v} \phi(P) = f(u, v)$, where $\mathcal{P}$ denotes the set of all non-trivial paths in G.

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Unfortunately, radio signals are subject to signal attenuation, and to interference constraints: a device cannot understand a radio transmission if the quality of the signal is not good enough, with respect to media noise, and also to other radio signals. This means that, in radio networks, transmissions must be performed in communication steps, such that interfering transmissions do not happen at the same time. In other words, the communication resource that has to be shared, is not the set of links as in classical networks, but physical space. We model the attenuation and interference constraints of the problem, assuming that two relations are given: a *transmission relation*, defined over the devices in the network, and determining if a transmission from u to v is possible; and an *interference model*, defined over the potential transmissions, and determining which can be performed concurrently. We assume that the nodes are synchronized, and that the traffic pattern is fixed, and known in advance, or steady enough so that it can be estimated. During a communication step, some radio transmissions take place. We assume that these transmissions are successful when they do not interfere with each other. We also restrict ourselves to the *fixed power model*, where each node uses the same transmission power, and in each communication step a node either transmits or it does not. The more general case, in which the devices can change their transmission power, can be addressed with techniques that are similar to the ones presented here.

1.1. Definitions and notation

Given a set S, we denote as $[S, S]$ the set of (unordered) pairs of S and $[x, y]$ (or simply xy) the elements of $[S, S]$ (so $xy = [x, y] = [y, x] = yx$). In a similar fashion, (S, S) is the set of ordered pairs of S and (x, y) the elements of (S, S) (so $(x, y) \neq (y, x)$).

We are given a (vertex) set V, $|V| = n$, a set of *feasible transmissions* $E_T \subset (V, V)$ whose elements we refer to as *transmission-arcs* (or simply *calls*, as $(u, v) \in E_T$ represents a device u transmitting to – calling – v). The *transmission digraph* is defined as $G_T = (V, E_T)$. The interference rules are captured by an *interference model*, which determines if a set of calls $R \subseteq E_T$ interferes or not. Hence, the interference model is the family of the sets of non-interfering calls, that we denote as $\mathcal{R}$ (notice that $|\mathcal{R}|$ may be exponentially large). An element of $\mathcal{R}$ is called a *round*.

We are also given a (directed) bandwidth demand function $f : (V, V) \to \mathbb{Q}^+$, $(u, v) \mapsto f(u, v)$ expressing a desired average bandwidth from vertex u to vertex v. We study the case of a general bandwidth demand f, but also pay special attention to *gathering* instances in which demands are directed towards a unique sink, i.e., $f(u, v) = 0$ if $v \neq t$, for some fixed $t \in V$. This is because these instances have particular applications (see [2,4,5]). As usual in flow problems (see, for instance, [28]), given $u, v \in V$, we define $\mathcal{P}_{uv}$ as the set of paths in G_T from u to v, and $\mathcal{P}_e$ as the set of paths in G_T containing the call $e \in E_T$. Finally, $\mathcal{P}$ will denote the set of all non-trivial paths in G_T.

Interference models

Even though we provide some general results that are valid for any interference model (see Section 2), most of our results depend on the interference model. The emphasis of this paper is on the *binary interference model*, which is given by a set $E_I \subset [E_T, E_T]$ that defines a binary *interference relation*. Two calls $e, f \in E_T$, $e \neq f$ *interfere* if $[e, f] \in E_I$. If two calls do not interfere, we say that they are *compatible*. The interference model is then obtained by defining a round $R \subseteq E_T$ as any set of compatible calls. The *interference graph* is defined as $G_I = (E_T, E_I)$ and the set of rounds $\mathcal{R}$ corresponds to the set of all independent sets of G_I.

We assume that the transmission and interference relations are derived as follows: for each $u \in V$, two sets $E_T(u)$, $E_I(u)$ are given that satisfy $u \in E_T(u) \subseteq E_I(u)$. Informally speaking, u can *transmit* to any $v \in E_T(u)$, but, when transmitting, it interferes *reception* at any $w \in E_I(u)$. Formally, the set of feasible transmissions $E_T = \{(u, v) : u \in V, v \in E_T(u)\}$, and two transmissions $(u, v) \neq (u', v') \in E_T$ interfere, i.e. $[(u, v), (u', v')] \in E_I$, if $v \in E_I(u') \vee v' \in E_I(u)$. Indeed, in this paper we focus on the *metric* case, where for some numbers $d_I, d_T, d_I \geq d_T > 0$ ($d_I \geq d_T$ because a radio signal normally interferes in an area that is larger than the area where it can be properly received), and a metric d defined on the vertex set V, the set $E_T(u)$ (respectively $E_I(u)$) is the set of vertices at a distance at most d_T (respectively d_I) from u.[1] Moreover, we are mainly interested in two cases: the *graph-metric* case, where d is the usual distance when the vertices belong to an underlying graph G (i.e. the distance between u and v is the length of a shortest path in G); and the *Euclidean-metric* case (or simply Euclidean case), where the nodes correspond to points in $\mathbb{Q}^2$. When restricted to the graph-metric case, with distances d_I, d_T, we denote the interference graph as $\mathcal{I}(G, d_I, d_T)$.

1.2. The round weighting problem

The input of the problem is given by the transmission relation, the interference model, and the demand function f. A solution for the problem is a weight function $w : \mathcal{R} \to \mathbb{Q}^+$ that allows to establish the traffic demands.

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[1] Notice that in the metric case the transmission digraph G_T is symmetric. For convenience, G_T can be considered as undirected in this case.

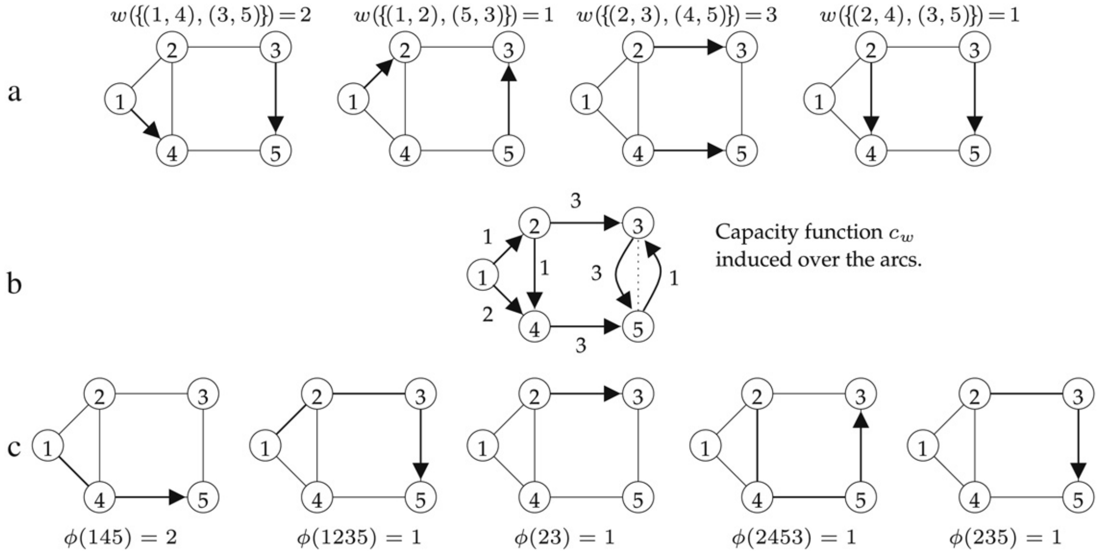

Fig. 1. An example of round weighting in the graph-metric case with $d_I = d_T = 1$. Traffic demands are $f(1,5) = 3, f(2,3) = 2, f(2,5) = 1$, and $f(x,y) = 0$ otherwise. (a) A round weight function w with cost $W(w) = 7$; (b) the induced capacity c_w; and (c) a flow function ϕ. Because ϕ satisfies the traffic demand f, w is valid.

For $e \in E_T$, let $\mathcal{R}_e$ be the set of all rounds containing the call e. The *capacity induced by* w is the function $c_w : E_T \to \mathbb{Q}^+$, $c_w(e) = \sum_{R \in \mathcal{R}_e} w(R)$.[2] The weight function w is *feasible* (or *valid*) if there exists a solution to the following associated multi-commodity flow instance. More precisely, there must exist a flow ϕ in the transmission graph, such that

– ϕ satisfies the traffic demand f:

$$\forall u, v \in V : \sum_{P \in \mathcal{P}_{uv}} \phi(P) \geq f(u, v), \tag{1}$$

– and ϕ respects the capacity induced by w:

$$\forall e \in E_T : \sum_{P \in \mathcal{P}_e} \phi(P) \leq c_w(e). \tag{2}$$

The cost of a solution w, denoted as $W(w)$ (or simply W), is defined as $W = \sum_{R \in \mathcal{R}} w(R)$. The objective of the round weighting problem is to minimize this cost over all feasible weight functions w. We write shortly W^* for the optimum value.[3]

An example of round weighting in the graph-metric case is given in Fig. 1.[4]

1.3. Motivation

This work is motivated by a problem raised by France Télécom R & D: an Internet provider wants to design efficient strategies to provide Internet access in a rural area using wireless devices. For this, one needs to define a global *scheduling* of the radio transmissions in order to provide a certain bandwidth between the nodes. This leads to the study of a specific *round scheduling problem*. One of the major difficulties of the round scheduling problem is the level of detail it implies. The round weighting problem considered in this paper aims to avoid these difficulties, by defining a more aggregate model. Indeed, the round weighting problem can be seen as a limit instance of the round scheduling problem. This is discussed in Section 6.

2 The notion of *induced capacity* is mainly introduced to emphasize the relation with the classical flow problem (as a path packing problem). It is just an auxiliary variable in the linear program. One could do without it, but we believe that its use helps to clarify the concept. We recall that, in the classical flow problem, a capacity is a function $c : E_T \to \mathbb{Q}^+$, and a concurrent flow in G_T (or simply flow) that satisfies the capacity c and the traffic demand f is a function $\phi : \mathcal{P} \to \mathbb{Q}^+$ such that (a) $(\forall u, v \in V)\ \sum_{P \in \mathcal{P}_{uv}} \phi(P) \geq f(u, v)$; and (b) $(\forall e \in E_T)\ \sum_{P \in \mathcal{P}_e} \phi(P) \leq c(e)$.

3 In the graph-metric case, the transmission (di)graph G_T and the interference graph G_I are induced by the distances d_T and d_I on an underlying graph G. For convenience, we will refer to the round weighting problem in this case simply as the *round weighting problem on G with distances d_T, d_I*.

4 Notice that Fig. 1 does not represent the transmission (di)graph, but the underlying graph over which distances are measured. As in this example $d_T = 1$, edge $[u, v]$ in this graph corresponds to the two pairs of possible calls (u, v) and (v, u). The figure pictures only the "used" transmissions.

1.4. Results and structure of the paper

In Section 2, we study the round weighting problem under any interference model and show that in order to solve this problem it suffices to solve the following auxiliary problem, that we call the *longest round problem*: given a length function $\ell : E_T \rightarrow \mathbb{Q}^+$, find a round of maximum length, i.e., $R \in \mathcal{R}$ such that $\sum_{e \in R} \ell(e)$ is maximum.

Starting with Section 3, we restrict ourselves to the case where the interference model is given by a binary interference relation, and we study the hardness of round weighting in this case. We show that the problem is related to *fractional coloring*, thus it cannot be approximated within $n^{1-\varepsilon}$ for any $\varepsilon > 0$ unless ZPP = NP.

Section 4 provides some positive results when the network structure is particular. Using the longest round problem, we show that round weighting is polynomial in the path. We give an explicit (small) linear program for this case. In this section, we also study the Euclidean-metric case. We show that the problem remains NP-HARD in this case, but admits a PTAS (if $d_I > d_T$ or if the minimum pairwise distance of nodes at a distance strictly larger than d_I is lower bounded by a constant strictly larger than d_I).

Section 5 is devoted to gathering instances in the graph-metric case. We show that the problem remains NP-HARD in this case, but we also provide a 4-approximation in general graphs. Moreover, we give an explicit formula for the optimum, when the graph is a path and $d_T = 1$.

Finally, Section 6 explains the motivation and applications of the problem, and Section 7 contains the conclusions.

1.5. Related work

Many models for wireless networks have been introduced and investigated, and we refer the reader to the recent survey [30], or the recent book [32]. The model used in this paper is a good approximation of realistic scenarios, when the transmission power is uniform (see e.g. [1]).

The problem of efficient routing in wireless networks has not been studied until recently. A problem related to the longest round problem (see Section 2) was studied by Balakrishnan et al. in [1]. That paper considers the problem of finding the set of transmissions which achieves the maximum global throughput. The authors assume a MAC layer based on collision detection. This implies finding a *maximum distance 2 matching* (D2EMIS), also called a *maximum induced matching* (see [31]). This corresponds to finding a longest round in our model, when $d_T = 1$, $d_I = 2$, but when the calls are undirected and the weights are all equal to 1. D2EMIS is known to be APX-COMPLETE [27] for regular graphs, but admits a PTAS for disk graphs [1].

While we were revising our initial work [22], two closely related papers were published:

- The fact that routing can be solved using a dual approach was observed in [6]. In that paper, the authors replace independent sets by a simpler condition: the weight of a round is taken to be its maximum weighted clique. A Lagrangian relaxation is then proposed, and proven to converge. It consists mainly of placing penalties on geographical areas.
- In [29], the authors generalize the D2EMIS problem introduced in [1] in two directions: edges can be weighted, and $d_I = K, K \geq 1$. The objective is, therefore, to find a set $S \subset E$ with maximum weight such that the distance between the selected edges is larger than K. This problem is called the *maximum weighted K-valid matching problem* (MWKVMP). In the case where the calls are undirected, the authors show that MWKVMP is polynomial when $K = 1$, and NP-HARD if $K > 1$. Moreover, they show that if $K \geq 2$ then the problem is not approximable, and they provide a PTAS for unit disk graphs.

Other related problems are the following:

- Given a traffic demand, calls must be scheduled in order to route the traffic. This problem is addressed e.g. in [2–5,13]. The key difference is that here the traffic is not to be routed continuously, but only once, making the problem harder due to initialization and integrality constraints.
- Given a traffic demand, the calls have to be scheduled in a periodic way, so that the traffic demand is routed each T' steps; the communication pattern is said to be *systolic*. The goal in this case is to minimize T'. This version is equivalent to ours and has been studied for wired networks e.g. in [11,17,18].

Basic communication problems for the dissemination of information (like gathering, broadcasting, gossiping) have been widely studied for wired networks [19].

2. The dual approach

In this section, we study the dual of the round weighting problem. The main result, Proposition 1, is used in Section 4 to obtain algorithms for calculating (or approximating), the round weighting problem, and in Section 5 to calculate a general lower bound for the gathering case.

2.1. Formulation of the Dual

Let us state again the round weighting problem:

$$\min_{w,\phi} \sum_{R\in\mathcal{R}} w(R),$$

$$\forall u, v \in V : -\sum_{P\in\mathcal{P}_{uv}} \phi(P) \le -f(u, v), \tag{3}$$

$$\forall e \in E_T : \sum_{P\in\mathcal{P}_e} \phi(P) - \sum_{R\in\mathcal{R}_e} w(R) \le 0, \tag{4}$$

$$w, \phi \ge 0.$$

In order to calculate the dual, let λ_{uv} be the multiplier associated to (3) and $\ell(e)$ that of (4). We obtain (see, for instance, [7]):

$$\max_{\lambda,\ell} \sum_{u,v\in V} \lambda_{uv} f(u, v),$$

$$\forall u, v \in V, \quad \forall P \in \mathcal{P}_{uv} : \sum_{e\in P} \ell(e) \ge \lambda_{uv}, \tag{5}$$

$$\forall R \in \mathcal{R} : \sum_{e\in R} \ell(e) \le 1,$$

$$\lambda, \ell \ge 0.$$

We interpret the dual as follows: consider $\ell(e)$ as the *length* of call e, and therefore inducing a *metric* d_ℓ on the transmission graph, the distance from u to v being the shortest path in terms of the lengths $\ell(e)$, $e \in E_T$.

Note that the goal of the dual problem is to maximize $\sum_{u,v\in V} \lambda_{uv} f(u, v)$, but because $f(u, v) \ge 0$ it follows that $\lambda_{u,v}$ has to be selected as large as possible. Hence, according to (5), the optimum choice is to take $\lambda_{uv} = d_\ell(u, v)$.

In summary, we obtain the following result.

Property 1. *The dual problem of round weighting consists in finding a metric $\ell : E_T \to \mathbb{Q}^+$ onto the call set maximizing the total distance that the traffic needs to travel ($W = \sum_{u,v\in V} d_\ell(u, v) \cdot f(u, v)$), and such that the maximum length of a round is 1 ($\forall R \in \mathcal{R} : w(R) = \sum_{(u,v)\in R} d_\ell(u, v) \le 1$).*

An example on the path. Let us illustrate an application of the dual approach in a very simple example: gathering in the middle of a path.

Take $d_T = 1$, $d_I = 2$, and consider the path $P_n = \{0, 1, \ldots, n-1\}$. Put $n = 2p+1$, $t = p$, and $f(u, t) = 1$ and $f(u, v) = 0$ if $v \ne t$.

Fig. 2 shows a round weighting solution satisfying these conditions, with a total value of $W = 4p - 6$. The lower part on the same figure shows a possible dual solution: to assign $\ell(e) = 1$ for $e \in \mathcal{E} = \{(p + 1, p), (p + 2, p + 1), (p + 3, p + 2), (p + 4, p + 3)\}$ and zero otherwise. This is a valid dual solution, because all the calls in $\mathcal{E}$ interfere with each other. The value of the dual solution is then $L = \sum_{u,v\in P_n} d_\ell(u, v)f(u, v) = \sum_{u\in P_n} d_\ell(u, t)$ ($f(u, v) = 0$ if $v \ne t$ and $f(u, t) = 1$). But if $u < p$ then $d_\ell(u, t) = 0$, if $u = p + 1, p + 2, p + 3, p + 4$ then $d_\ell(u, t) = u - p$, and if $u > p + 4$ then $d_\ell(u, t) = 4$, so $L = 1 + 2 + 3 + 4 + (p - 4)4 = 4p - 6$. It turns out that $W = L$, and therefore both solutions are optimum.

2.2. Separation of the Dual

Now, we study how to solve the dual problem by means of a simpler one: to find a round of maximum length.

Definition 1. Given a weight function defined over the set of calls $\ell : E_T \to \mathbb{Q}^+$, the *longest round problem* is to find a round R such that $\ell(R) = \sum_{e\in R} \ell(e)$ is maximum.

The next result follows from general theorems on separation and optimization given by Grötschel et al. [14,15] in the exact case and by Jansen [21] in the approximation case. It implies that in order to solve/approximate the round weighting problem, it is enough to solve/approximate the longest round problem.

Proposition 1. *If there exists a polynomial algorithm (respectively, a polynomial ρ-approximation) for the longest round problem, then there exists a polynomial algorithm (respectively, a polynomial ρ-approximation) for round weighting.*

Proof. In order to solve the dual problem, we only need to separate it. So, given a metric ℓ in the dual, we need to decide whether it is feasible, and if not to output a violated constraint. Since to check feasibility means to verify that ℓ is positive, and to compute $\sum_{u,v\in V} d_\ell(u, v)f(u, v)$, we only have to check the constraints

$$\forall R \in \mathcal{R} : \ell(R) \le 1.$$

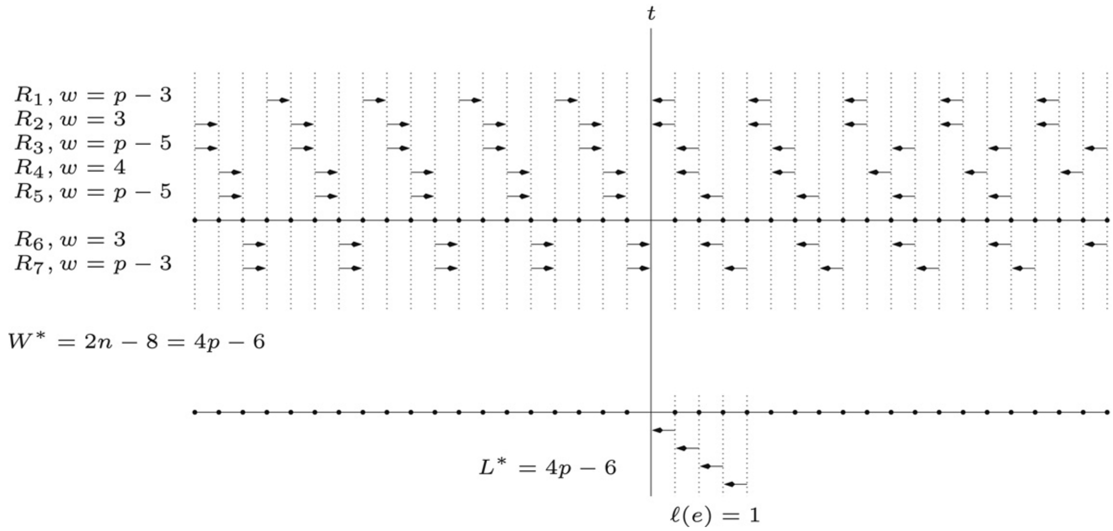

Fig. 2. Example of gathering in the middle of a path with $n = 2p + 1$ vertices. In the upper part, a weight function with value $W^* = 4p - 6$. In the lower part, a dual solution with value $L^* = 4p - 6$. Because the values match, both solutions are optimal.

For this purpose, it is sufficient to find a longest round R_0. If its weight is strictly larger than 1, we output $\ell(R_0) \leq 1$ as violated constraint, and if not then ℓ is feasible. Therefore, if such R_0 can be found efficiently, the result follows (for the exact case).

Let us assume now that we have a ρ-approximation. Then, it provides us with a round R_1 such that $\ell(R_1) > 1$ in polynomial time, or otherwise we know that the metric ℓ/ρ is feasible, and the approximation case follows. □

2.3. Consequences for round weighting in radio networks

In the case of radio networks, the set of feasible rounds $\mathcal{R}$ is given exactly by the set of all independent sets of the interference graph $G_I = (E_T, E_I)$, and a longest round corresponds to a maximum weight independent set of the interference graph whose vertex $e \in E_T$ has weight $\ell(e)$. It follows from Proposition 1 that any ρ-approximation algorithm for the maximum weight independent set on the interference graph induces a ρ-approximation algorithm for the round weighting problem.

However, notice that the result in Proposition 1 relies on an implicit linear program, hence it may not provide practically efficient algorithms.

3. Hardness of round weighting

Now, we study the complexity of finding an optimum solution for round weighting. We address the general problem, for which we show that it cannot be approximated within $|V|^{1-\varepsilon}$, unless ZPP=NP.

For the rest of the paper, we assume the binary interference model, where a set $E_I \subset [E_T, E_T]$ is provided. Given two calls $e, f \in E_T$, either $ef \in E_I$ (and therefore they interfere), or $ef \notin E_I$ (so they are compatible). Therefore $\mathcal{R} = \{S \subset E_T : (\forall e, f \in S)\ ef \notin E_I\}$.

We show that *fractional coloring* (the relaxation of vertex coloring in which the weights associated to the independent sets are not required to be integer) is a specific case of round weighting.

More precisely, given a graph $G = (V, E)$, let $\mathcal{I}(G)$ denote the set of all independent sets of G. The *fractional coloring problem* on G is the following linear program:

$$\text{F.C.(linear program)} \qquad \begin{array}{ll} \min & \displaystyle\sum_{I \in \mathcal{I}(G)} x_I \\ & \displaystyle\sum_{I \in \mathcal{I}(G): v \in I} x_I \geq 1 \quad (\forall v \in V), \\ & x_I \in [0, 1]. \end{array}$$

Proposition 2. *Fractional coloring is a specific case of the round weighting problem in the graph-metric model on a graph G with distances $d_T = 1$, $d_I = 2$. Therefore, the round weighting problem cannot be approximated within $|V|^{1-\varepsilon}$ on a graph G with such distances, for any $\varepsilon > 0$, unless* ZPP = NP.

Proof. We show that for every instance $\mathcal{F}$ of fractional coloring there is an instance $\mathcal{W}$ of round weighting, such that the fractional colorings of $\mathcal{F}$ are in bijection with the round weighting of $\mathcal{W}$, and the value is exactly the same.

Let $G = (V, E)$ be an undirected graph to be fractionally colored. We construct an auxiliary graph $H = (N, A)$ as follows. N is the same as V plus one copy v' for each vertex $v \in V$, and A is the same as E plus arcs (v, v') (that is, we connect each vertex with its copy). We set $f(v, v') = 1$ and zero otherwise (demand 1 unit of traffic from v to its copy v') and set $d_I = 2, d_T = 1$.

As all paths in H from v to v' go through arc (v, v'), we can assume that any round weighting uses only these arcs. Now, any independent set I of G induces a valid round $R(I) = \{(v, v') : v \in I\}$ and, conversely, any round $R = \{(v, v')\}$ induces an independent set $I(R) = \{v : (v, v') \in R\}$ in G.

It follows that any fractional coloring $\bar{w}(I)$ (I an independent set of G) induces a weight function $w(R(I)) = \bar{w}(I)$ such that (a) they have the same value: $\sum_{R\in\mathcal{R}} w(R) = \sum_{I\in\mathcal{I}(G)} \bar{w}(I)$ ($\mathcal{I}(G)$ being the set of all independent sets of G), and (b) it induces a capacity of 1 over arc (v, v') (thus it is feasible for f). Conversely, because any feasible round weight function w must induce a capacity of at least 1 over each arc (v, v'), we obtain that w induces a fractional coloring of G (again, with the same value). Therefore solving fractional coloring in G is equivalent to solving round weighting in H (for $d_I = 2, d_T = 1$ and f as defined before), and the result follows [10]. □

4. Round weighting in some specific cases

In this section, we use the dual approach to find positive results for some specific topologies. For the graph-metric case, we show that the problem is polynomial in paths. For the Euclidean-metric case in the plane, we present a PTAS.

4.1. The path

For the path P_n, with vertices $0, 1, 2, \ldots, n-1$ and edges $(i-1, i)$, $i = 1, \ldots, n-1$, we prove that the round weighting problem is polynomial. We provide a direct approach using an explicit small linear program.

4.1.1. A small linear program for the round weighting problem in the path

In the case of the path, the independent sets can be expressed by simple constraints, in a way that is similar to that used for fractionally coloring circular arc graphs [24,33]. Basically, it suffices to express the compatibility constraints on each small cut. In interval graphs, the convex hull of the independent sets is defined by writing that "the load of each part of the line is at most 1".[5] For circular arc graphs, [33] gives a flow formulation to solve the fractional coloring on these graphs, and remarks that the convex hull of the independent sets can be expressed by simple flow equations.

Proposition 3. *The round weighting problem with distances d_I, d_T in P_n can be solved using a polynomial-size linear program.*

Proof. We recall that the polynomial-size linear program for the multicommodity problem reads as follows. We have a digraph $G = (V, E)$, a capacity function $c : E \to \mathbb{Q}^+$, and a demand function $f : (V, V) \to \mathbb{Q}$. The variables are $x_e^{u,v}$, $u, v \in V, e \in E$, $x_e^{u,v}$ representing the amount of flow that goes from u to v through arc e. The set of constraints consists of (a) a set of flow conservation constraints, and (b) a set of capacity satisfiability constraints.

In the round weighting problem, $G = G_T$ (the transmission graph), the demand function is f, and we can keep the set of constraints regarding flow conservation.

The capacity constraints in the multicommodity problem read

$$(\forall e \in E) \sum_{u,v} x_e^{u,v} \le c(e).$$

As we know, in the case of round weighting, the capacity is induced by the weight function w, so replace these (polynomial number of equations) with

$$(\forall e \in E) \sum_{u,v} x_e^{u,v} \le c_w(e).$$

Therefore, the only thing we have to show is a polynomial way to express $c_w(e)$. For this, we construct an auxiliary flow network $\mathcal{N} = (N, A)$.

First, let us represent a call (u, v) as the directed segment $\overrightarrow{u, v}$ and observe that if $R = \{(u_i, v_i)\}_{i=1}^k$ then the segments $\overrightarrow{u_i, v_i}$, $\overrightarrow{u_j, v_j}$, $1 \le i, j \le k, i \ne j$ are disjoint and can be ordered from left to right: either $u_i, v_i < u_j, v_j$, in which case we say that the call (u_i, v_i) is at the left of (u_j, v_j); or $u_i, v_i > u_j, v_j$, and we say that (u_i, v_i) is at the right of (u_j, v_j). Conversely, provided that a set of calls $\{(u_0, v_0), (u_1, v_1), \ldots, (u_k, v_k)\}$ is ordered, to check that it is a round reduces to verifying that $(\forall i = 1, \ldots, k-1)$ $\overrightarrow{u_i, v_i}$ and $\overrightarrow{u_{i+1}, v_{i+1}}$ are non-adjacent (i.e. adjacency needs to be checked for each two successive segments, only).

With this in mind, we construct the following auxiliary network:

[5] Note that, in our context, we view an independent set as a function on the vertex set, i.e., a (0, 1)-function or a characteristic function. This makes each independent set a vector of $\{0, 1\}^{|V|}$. The convex hull is taken according to this interpretation.

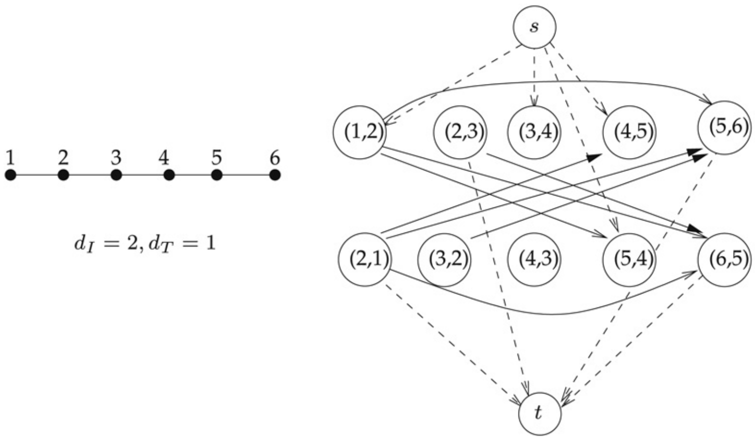

Fig. 3. Sample construction for auxiliary network for a path consisting of 6 vertices in the case $d_I = 2, d_T = 1$. Dashed arcs connect vertex s with every vertex $(u, v) \in E_T$, and each vertex $(u, v) \in E_T$ with vertex t (many of these are omitted for the sake of clarity). Solid arcs connect vertex (u, v) with (u', v') if $u, v < u', v'$ and the corresponding calls are compatible.

- The set of nodes is $N = E_T \cup \{s, t\}$, i.e., the set of calls plus a source s and a sink t.
- s is connected to each $(u, v) \in E_T$, and each $(u, v) \in E_T$ is connected to t. These arcs have a capacity of infinity.
- (u, v), (u', v') are connected if they do not interfere and (u', v') is at the right side of (u, v). The capacity of these arcs is infinity.

Consider now a flow function $\phi : A \to \mathbb{Q}^+$, from s to t (notice that here we consider ϕ as defined over the set of arcs A, so it can be expressed in polynomial-size). We write flow conservation constraints for each vertex $x = (u, v)$, (that is, $x \neq s, t$): $\sum_{(x,y)\in A} \phi(x, y) = \sum_{(y,x)\in A} \phi(y, x)$. Let us now assume that ϕ has value $value(\phi)$ (i.e. $\sum_{v:(v,t)\in A} \phi(v, t) = value(\phi)$). Decompose ϕ into a set of weighted directed paths. Each directed path is a sequence of calls ordered from left to right, and such that they do not interfere, i.e., a round.[6] It follows that a decomposition of ϕ into paths corresponds to a weight function w defined on the calls, and its value $value(\phi)$ corresponds to the weight of w. Moreover, the flow crossing vertex (u, v) in the auxiliary network is $\sum_{((u,v),y)\in A} \phi((u, v), y)$ and is precisely the number of times[7] call (u, v) appears over all the paths transmitting flow in the decomposition of ϕ, that is, the flow crossing vertex (u, v) corresponds to the induced capacity of arc (u, v). Hence, we can write $c_w((u, v)) = \sum_{((u,v),y)\in A} \phi((u, v), y)$ and we are finished, because these are a polynomial number of equations. □

4.2. *Euclidean graphs for fixed d_T, d_I*

4.2.1. *NP-hardness*

We first show that round weighting is NP-hard in the Euclidean case. The reduction is from fractional coloring on unit disk graphs, which is NP-Hard [8].

Proposition 4. *Round weighting is* NP-hard *even if G is a Euclidean graph.*

Proof. The construction is very similar to that of Proposition 2. We make a copy of the original graph, and set demands between corresponding vertices. The hardness follows from the fact that compatibility of calls in the constructed instance is related to independency in the original graph.

We model Euclidean graphs with *Unit Disk Graphs* [8]. Let G be a unit disk graph in $\mathbb{Q}^2$. That is $G = (V, E)$, $V = \{u_i = (x_i, y_i) : i = 1, \ldots, n\}$, and edge $[u_i, u_j] \in E$ if and only if $d(u_i, u_j) \leq D$ for certain $D > 0$.

We construct an instance of round weighting on a Euclidean graph G', such that round weighting on this graph is equivalent to fractionally coloring G. Therefore, we have to define a set of vertices $V' \subset \mathbb{Q}^2$, and distances $0 < d_T \leq d_I$ such that $(u, v) \in E'$ is a call if $d(u, v) \leq d_T$, and two calls (u, v), (u', v') interfere if $d(u, v') \leq d_I$ or $d(u', v) \leq d_I$.

Let $D^+ = \min[d(u, v) : d(u, v) > D]$, i.e., the distance between the closest pair of vertices in G that are not connected. Let also $\delta = \frac{1}{3}(D^+ - D)$, so $D^+ - \delta > D + \delta$.

We set $d_I = D + \delta$, $d_T = \delta$, and construct the round weighting instance as follows:

- For every $u_i \in G$, we make a copy $u_i' = (x_i + \delta, y_i)$.

---

[6] Notice that to calculate a decomposition of ϕ into paths from a flow defined over the arcs is easy (polynomial), and that any decomposition will induce a weight function with the same value W, because $W = value(\phi)$, regardless of the decomposition.

[7] Note that *number of times* does not necessarily need to be an integer.

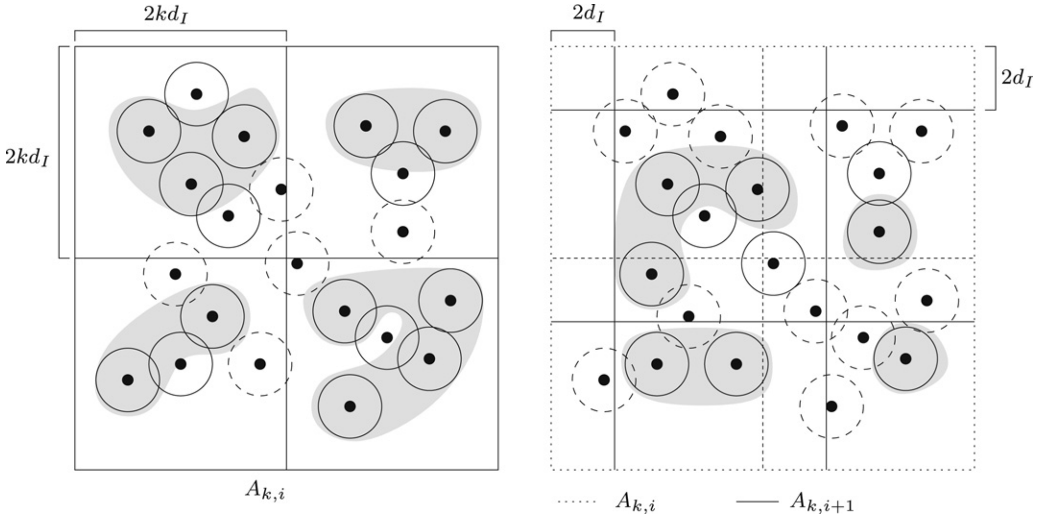

Fig. 4. PTAS in the case of the Euclidean space. On the left, a set $A_{k,i}$: calls that are close to the borders are removed, and a longest round is calculated in each cell separately. On the right, the same procedure is applied for $A_{k,i+1}$ ($A_{k,i}$ is shown also in dotted lines).

- We set $f(u_i, u'_j) = 1$ if $i = j$ and zero otherwise.

If u and v are connected in the original graph, then $d(u, v) \leq D$ and $d(u, v') \leq d(u, v) + d(v, v') \leq D + \delta = d_I$. Thus, the calls (u, u') and (v, v') interfere.

If u and v are not connected in G, then $d(u, v) \geq D^+$ and $D^+ \leq d(u, v) \leq d(u, v') + d(v', v) = d(u, v') + \delta$, thus $d(u, v') \geq D^+ - \delta > D + \delta = d_I$. Thus, the calls (u, u') and (v, v') are compatible.

Hence, two calls (u, u'), (v, v') are compatible if and only if the vertices u and v are independent in the original unit-disk graph. Thus, solving the round weighting problem on G' is equivalent to fractionally coloring G, and therefore it is NP-HARD. □

4.2.2. A polynomial-time approximation scheme

Next, we show that the round weighting problem can be approximated in $\mathbb{Q}^2$ (with Euclidean distances), using a locality idea combined with shifting (see [9,20,16]).

Proposition 5. *Let $V \subset \mathbb{Q}^2$ and $d_I, d_T \in \mathbb{Q}$ such that $d_I \geq d_T > 0$. Assume the metric case, i.e., $E_T = \{(u, v) \in V : d(u, v) \leq d_T\}$ and $E_I = \{[(u, v), (u', v')] : d(u, v') \leq d_I \vee d(u', v) \leq d_I\}$, with $d(x, y)$ being the Euclidean distance between the points $x, y \in \mathbb{Q}^2$. If $d_I > d_T$, then there exists a* PTAS *for the round weighting problem. If $d_I = d_T$, then there exists a* PTAS *for the round weighting problem if the minimum pairwise distance of nodes at a distance strictly larger than d_I is lower bounded by a constant strictly larger than d_I.*

The main idea of the proof is shown in Fig. 4. The Euclidean space is divided into cells, and the calls that are close to the borders of such cells are removed. This allows treating each cell independently, hence, due to their size, a longest round can be found for each of them, and joining the partial solutions provides a global approximation. The approximation ratio comes from shifting this division several times, and showing that there is one of such divisions that does not remove too many calls.

Proof. Let $\varepsilon > 0$ be the desired approximation ratio, i.e., if R^* is a longest round with value $w(R^*) = \sum_{(u,v)\in R^*} \ell(u, v)$, we look for a round R such that its value $w(R) = \sum_{(u,v)\in R} \ell(u, v)$ satisfies $w(R^*)/w(R) \leq 1 + \varepsilon$.

Let us first study the case $d_I > d_T$. Take $k = \max[2, \lceil \frac{1+\varepsilon}{\varepsilon} \rceil]$, thus $\frac{k}{k-1} \leq 1 + \varepsilon$, and consider $A_k = \{(x, y) \in \mathbb{Q}^2 : x = 2 \cdot p \cdot k \cdot d_I \wedge y = 2 \cdot q \cdot k \cdot d_I,\ p, q \in \mathbb{Z}\}$. For $i = 0, \ldots, k-1$, define $A_{k,i} = A_k + (2 \cdot i \cdot d_I, 2 \cdot i \cdot d_I)$ (where $A + a = \{x + a : x \in A\}$, i.e., $A_{k,i}$ is the set A_k shifted a distance $2 \cdot i \cdot d_I$ in the direction of $(1, 1)$, and $+$ is the usual vector addition). Notice that $A_{k,i}$ partitions $\mathbb{Q}^2$ into an infinite number of *cells* whose area is $4 \cdot k^2 \cdot d_I^2$, and that at most $|V|$ of them are non-empty. Also notice, as the area of each cell is $O(k^2)$, in each round, there are at most $p = O(k^2)$ non-interfering calls per cell.

Let us define for $u \in V$, the ball $B_u(d_I) = \{y : d(u, y) \leq d_I\}$. Now, for $0 \leq i \leq k-1$, consider the set $E'_i = E_T \setminus \{(u, v) : B_u(d_I) \cap A_{k,i} \neq \emptyset\}$ (i.e., the calls that are *far* from the borders of the cells). We observe that if $(u, v), (u', v') \in E'_i$, then for any two disjoint cells C, C', no call (u, v) in C interferes with a call (u', v') in C'. Thus, we can calculate a longest round for each cell independently, but since we know that they contain at most p non-interfering calls each, checking every possible round requires time $O(|V|^{2p}) = |V|^{O(k^2)}$ (as there are $O(|V|^2)$ many potential calls). Doing so for each non-empty cell, we obtain that after a time $|V|^{O(k^2)}$ we have calculated a valid round. Let us name w_i the weight associated to such a round and $\bar{w} = \max[w_i : 0 \le i \le k-1]$ the heaviest of them. It follows that $\bar{w}$ can be calculated in time $|V|^{O(k^2)}$, so the time is polynomial for fixed k.

To prove the guarantee, recall that the longest round has a value $w(R^*) = \sum_{(u,v)\in R^*} \ell(u,v)$. To finish the proof, we have to show that $w(R^*)/w(R) \le \frac{k}{k-1}$. However, we observe that $(\forall u : (u,v) \in R^*)\ B_u(d_I)$ intersects at most one $A_{k,i}$ $(0 \le i \le k-1)$. It follows that

$$\sum_{i=0}^{k-1} \sum_{(u,v)\in R^*: B_u(d_I)\cap A_{k,i}\neq\phi} \ell(u,v) \le w(R^*) \Rightarrow (\exists j) \sum_{(u,v)\in R^*: B_u(d_I)\cap A_{k,j}\neq\phi} \ell(u,v) \le \frac{w(R^*)}{k}.$$

Therefore,

$$w(R^*) - \frac{w(R^*)}{k} \le w_j \le \bar{w} \Rightarrow \left(\frac{k-1}{k}\right) w(R^*) \le \bar{w},$$

and we can conclude for the case $d_I > d_T$.

Finally, if $d_I = d_T$, we observe if $\delta < \frac{1}{2}\min[d(u,v) - d_I : u, v \in V, d(u,v) > d_I]$, then any call valid for distances $d_I = d_T$ is valid for the distances $d'_I = d_I + \delta$, $d'_T = d_T$ and vice versa. Therefore, an approximation for this case gives an approximation for the case $d_I = d_T$, and the result follows. □

We observe that the result applies, for instance, for the 2D-grid with the metric distance, because this graph can be embedded in the plane. The only difference is the shape of the neighborhoods of the vertices. Furthermore, the proof can also be extended to dimension q, as the number of calls inside each cell remains polynomial, and not depending on n. Finally, for the general case where the interference and transmission relations are given by sets $E_T(u) \subset E_I(u)$ such that $E_T(u) \subset B_u(r) \subset E_I(u) \subset B_u(r')$, r'/r bounded, we can apply the same technique to obtain a PTAS.

5. Gathering instances

In this section, we study the gathering problem in which all the traffic demands are directed towards a unique vertex $t \in V$, i.e. $f(u,v) = 0, \forall v \neq t$. Gathering instances are quite natural practically. Typically, in one village many houses equipped with wireless devices require access to a gateway, and need to use multi-hop wireless relay routing to do so (see [4] for a more detailed discussion). The problem is how this can be achieved efficiently.

In the graph-metric case, we show that the gathering instances are NP-HARD, even in the uniform case where $f(u,t) = 1$. We also provide a 4-approximation for this uniform case.

5.1. Hardness of gathering

Proposition 6. *The round weighting problem on graphs with distances $d_T = 1$, $d_I = 2$ is* NP-HARD *even when restricted to uniform gathering instances, that is, when there exists $t \in V$ such that $f(u,t) = 1$ $(\forall u \in V, u \neq t)$ and $f(u,v) = 0$ if $v \neq t$.*

Proof. We first give the proof for non-uniform gathering instances, and then extend it to uniform gathering instances. The reduction is from MAXIMUM INDEPENDENT SET.

Let $G = (A, X)$ be a graph for which we want to calculate its maximum independent set. Let α be the size of such independent set. Since checking whether $\alpha = 1$ is polynomial, we assume without loss of generality that the size of the maximum independent set of G is at least 2.

We construct the following auxiliary graph: Take a copy $G' = (A', X')$ of G and connect $x \in A$ with $x' \in A'$. Add an extra vertex t and connect each $x' \in A'$ with t. Add also a source vertex s and connect it with each $x \in A$. Finally, take an integer N (to be fixed later) and set $f(s,t) = N$ and $f(u,v) = 0$ otherwise.

Since $d_I = 2$, $d_T = 1$, the construction ensures that when t is receiving from a vertex $x' \in A'$, no other call of type (y, y'), $y \in A$, $y' \in A'$ is being performed and, moreover, two calls (x, x'), (y, y') are compatible if and only if x and y are independent vertices in G (see Fig. 5 for an example of the construction for $N = 5$).

Now, let W^* denote the optimum weight, and consider the rounds making transmissions of type (x, x'). Each of these rounds induces an independent set in G, as explained above. Let $\alpha(W^*)$ denote the size of the largest of these independent sets. It follows that

$$W^* \ge \frac{N}{\alpha(W^*)} + N. \tag{6}$$

Next, define $I = \{a_i\}_{i=0}^{\alpha-1}$ to be the maximum independent set of G and consider the weight function

$$w(\{(s, a_i), (a'_{(i+1) \bmod \alpha}, t)\}) = \frac{N}{\alpha}, \quad i = 0, \ldots, \alpha - 1 \tag{7}$$

$$w(\{(a_i, a'_i)\}_{i=0}^{\alpha-1}) = \frac{N}{\alpha}. \tag{8}$$

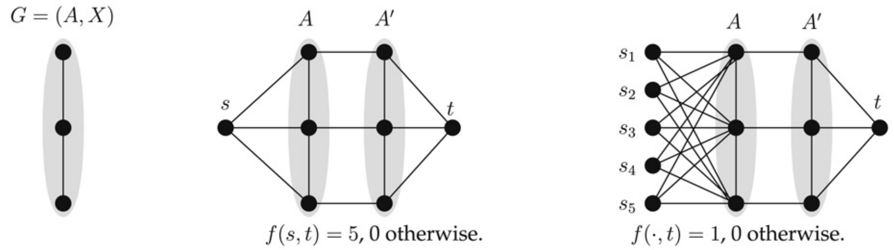

Fig. 5. Hardness of the gathering instance. From left to right: The original graph, the construction for the non-uniform case, and the construction for the uniform case (both for $N = 5$).

(7) defines a valid set of rounds and (8) is a round because I is an independent set in G. Besides, these weights route all the traffic from s to t. The overall weight W satisfies $W = N + N/\alpha$ which together with (6) and due to the optimality of W^* implies that $\alpha(W^*) \geq \alpha$.[8]

To extend the proof for the uniform case, we keep the auxiliary graph, but replace s with vertices s_k, $k = 1, \dots, N$, each of them connected to every $x \in A$ (Fig. 5 shows a sample construction for $N = 5$). Since in the uniform case $f(u, t) = 1$, this means that now the vertices in A and A' also have flow units to be routed, we use the following weight function

$$\begin{aligned} w(\{(x, x')\}) &= 1 && x \in A, \\ w(\{(x', t)\}) &= 2 && x' \in A', \\ w(\{(s_k, a_i), (a'_{(i+1) \bmod \alpha}, t)\}) &= \frac{1}{\alpha} && k = 1, \dots, N, i = 0, \dots, \alpha - 1, \\ w(\{(a_i, a'_i)\}_{i=0}^{\alpha-1}) &= \tfrac{N}{\alpha}. \end{aligned}$$

Now, this gives a weight $W = N + 3n + N/\alpha$. By (6) and the optimality of W^*, we obtain

$$\frac{N}{\alpha^*} \leq \frac{N}{\alpha} + 3n$$

which implies that

$$\frac{\alpha^*}{\alpha} \geq 1 - \frac{3n\alpha^*}{N} \geq 1 - \frac{3n^2}{N}.$$

By setting $N = 3n^3 + 1$, this last inequality yields

$$\alpha^* > \alpha\left(1 - \frac{1}{n}\right) \geq \alpha - 1.$$

Since α^* and α are integers, we obtain $\alpha^* = \alpha$. $\square$

5.2. Constant approximation for uniform gathering

In this subsection, we prove a positive result showing that a simple protocol allows one to approximate the uniform gathering problem up to a factor of 4, for the metric case, in which the distances are measured over the graph.

Proposition 7. *When restricted to uniform gathering instances, the round weighting problem on a graph G with distances d_I, d_T admits a polynomial-time* 4*-approximation.*

Proof. Consider a vertex x at a distance ℓ from t in the graph $G = (V, E)$, and a shortest (in number of edges) path P in the transmission digraph with length $\ell' = \left\lceil \frac{\ell}{d_T} \right\rceil$. Let the vertices of P be numbered $x = 1, 2, \dots, \ell' = t$, and let $\alpha = \left\lceil \frac{d_I + d_T + 1}{d_T} \right\rceil$, so the calls $(i, i+1), (i+\alpha, i+\alpha+1), \dots, (i+a\alpha, i+a\alpha+1), \dots$ do not interfere. It follows that all the calls $(i, i+1)$ can be covered using $\min[\alpha, \ell']$ rounds. Therefore, we can route the traffic unit from x to t using that weight, and doing so for all vertices leads to a simple round weighting with value

$$W^+ = \sum_{x \in V} \min\left\{\alpha, \left\lceil \frac{d(x,t)}{d_T} \right\rceil\right\} f(x, t).$$

Now, consider the ball B of radius $r = \left\lfloor \frac{d_I - d_T}{2} \right\rfloor$ centered at the sink and assign a length $\ell(e) = 1$ to any call ending in this ball and directed towards the sink (i.e., a call (u, v) with $d(v, t) \leq r$ and $d(u, t) > d(v, t)$). Since two calls (u, v), (u', v')

[8] Notice that for the non-uniform construction $N = 1$ may indeed be chosen.

such that $v, v' \in B$ and directed towards the sink interfere ($d(u, v') \le d(u, v) + d(v, v') \le d_T + 2r \le d_T + d_I - d_T \le d_I$), this is a valid dual solution and, moreover, according to this metric we have that

$$d_\ell(x, t) = \min\left\{\left\lceil \frac{d(x, t)}{d_T} \right\rceil, \left\lceil \frac{r+1}{d_T} \right\rceil\right\},$$

and the cost of any round weighting is at least $W^- = \sum_{x\in V} d_\ell(x, t) f(x, t)$.

To obtain the approximation ratio we calculate W^+/W^-. Let $d'(x) = \left\lceil \frac{d(x,t)}{d_T} \right\rceil$, $f'(x) = f(x, t)$ and $\beta = \left\lceil \frac{r+1}{d_T} \right\rceil$. Let also

$$a = \sum_{x\in V: d'(x)\le\beta} d'(x) f'(x), \quad b = \sum_{x\in V:\beta<d'(x)\le\alpha} d'(x) f'(x), \quad b' = \sum_{x\in V:\beta<d'(x)\le\alpha} f'(x), \quad c = \sum_{x\in V:\alpha<d'(x)} f'(x).$$

If follows that $b \le \alpha b'$ and

$$\frac{W^+}{W^-} = \frac{a + b + \alpha c}{a + \beta b' + \beta c} \le \frac{a + \alpha b' + \alpha c}{a + \beta b' + \beta c} \le \frac{a + \alpha(b' + c)}{a + \beta(b' + c)},$$

but since $(\forall \alpha, \beta, p, q \in \mathbb{Q}^+)\ \alpha \ge \beta \Leftrightarrow \frac{p\alpha+q}{p\beta+q} \le \frac{\alpha}{\beta}$, $\frac{W^+}{W^-} \le \frac{\alpha}{\beta}$.

Now, either $d_I \le 3d_T - 1$, in which case $\alpha \le 4$, and we have finished, or $d_I \ge 3d_T$. In this case, we write $d_I = pd_T + q$, $p \ge 3$, $0 \le q < d_T$, and observe that $\alpha \le p + 2$, but since $p \ge 3$, $\beta \ge p/2$, thus $\frac{\alpha}{\beta} \le \frac{p+2}{p/2} \le 4$ and the result follows. □

Notice that the previous result is independent of d_I, d_T, but by doing the same calculations, we obtain that for $d_I = pd_T + q$, $p \ge 4$, the ratio is at most 3. Indeed, the approximation ratio goes to 2 as $d_T/d_I \to 0$.

5.3. *Gathering in the path with $d_T = 1$*

We end this section with further results for gathering in the path. From Section 4, we already know that round weighting is polynomial, but in the case of gathering we can go further, and provide an explicit formula for the optimum.

When $d_T = 1$, the transmission arcs are the usual arcs of the directed path. Then, there is only one simple directed path between two nodes, so the routing is fixed (forced), and the problem reduces to fractional coloring of the associated *interference* graph.

We will use the following notation: we assume that the sink is some node t, and we consider the two parts of the path obtained by removing t, the left (resp. right) part contains q_{left} (resp. q_{right}) nodes. We number the nodes in each directed semi-path (left and right) $0, 1, 2, 3, \ldots$ starting from t. Because the routing is unique, the transmissions are only of the form $(i, i-1)$. Let l_i denote the arc $(i, i-1)$ in the left directed path and r_i the arc $(i, i-1)$ on the right directed path. Let $f(l_i)$ (resp. $f(r_i)$) denote the demand of the left (right) vertex i. To simplify the notation, we introduce a *virtual* vertex l_i for $i \ge q_{\text{left}}$, with no traffic demand, i.e., $f(l_i) = 0$; and we do similarly for the right side. Finally, the flow value on the arc l_i (resp. r_i) will be denoted by $L(i)$ (resp. $R(i)$), we have $L(i) = \sum_{j\ge i} f(l_j)$ and $R(i) = \sum_{j\ge i} f(r_j)$.

In this context, round weighting consists of finding the fractional chromatic number of the interference graph, in which the call l_i (resp. r_i) has weight $L(i)$ (resp. $R(i)$). But because of the particular structure of the interference graph, which turns out to be perfect, this fractional chromatic number equals the maximum weight of a clique [12,26]. This leads to the following result:

Proposition 8. *In the case of gathering in the path with $d_T = 1$, the optimum value for round weighting is*

$$W^* = \max\left[\sum_{i=1}^{d_I+2} L(i), \sum_{i=1}^{d_I+2} R(i), \max_{1\le a\le d_I}\left[\sum_{i=1}^{a} L(i) + \sum_{i=1}^{d_I+1-a} R(i)\right]\right].$$

Moreover, if the traffic demand f is integral, there exists an integral optimal solution.

Proof. First, observe that node l_i (respectively r_i) requires an induced capacity of $L(i)$ (respectively $R(i)$). The edges of the interference graph are then as follows:

- We have one edge between l_i (resp. r_i) and l_j (resp. r_j) when $|i - j| \le d_I + 1$.
- There is an edge between l_i and r_j if and only if $i + j \le d_I + 1$.

It follows that the interference graph is perfect. Indeed, two calls of the form l_i or r_j are in conflict, if and only if, the distance of the senders of the two calls is at most $d_I + 1$. Therefore, the interference graph is an interval graph (even a unit interval graph) and thus a perfect graph.

Now, for any perfect graph G, the fractional chromatic number equals the maximum weight of a clique [12,26]. But the maximum cliques in the interference graph are of type (A) $\{l_i, i \in [j, j+d_I+1], j \geq 1\}$, or (B) $\{r_i, i \in [j, j+d_I+1], j \geq 1\}$, or (C) $\{l_i, i \in [1, a]\} \cup \{r_i, i \in [1, d_I+1-a]\}$, $a = 1, \ldots, d_I$, and we obtain that

$$W^* = \max\left[\max_{j\geq 1}\left[\sum_{i=j}^{j+d_I+1} L(i)\right], \max_{j\geq 1}\left[\sum_{i=j}^{j+d_I+1} R(i)\right], \max_{1\leq a\leq d_I}\left[\sum_{i=1}^{a} L(i) + \sum_{i=1}^{d_I+1-a} R(i)\right]\right].$$

We conclude by observing that, since $L(i)$ and $R(i)$ decrease with i, the first two values in the maximum are attained for $j = 1$. $\square$

In the uniform case, $L(i)$ and $R(i)$ can be stated explicitly (in fact, $L(i) = \max[q_{\text{left}} - i + 1, 0]$, and $R(i) = \max[q_{\text{right}} - i + 1, 0]$), and straightforward computations lead to a more explicit but quite complex formula (see [22]). Note also that a very similar result [2] has appeared recently for the round scheduling problem (for particular locations of the sink).

6. The round scheduling problem

In the round scheduling problem, time is divided into time-slots or steps. During a given step, a round takes place. If a call (u, v) is performed during the round, this is interpreted as vertex u transmitting 1 unit of data to vertex v. The demands are expressed in such a unit. The goal is to choose the right sequence of rounds, in order to route the highest percentage of the traffic demands.

Definition 2. Let $T \in \mathbb{N}$. A *round schedule with time horizon T* (or simply schedule) is a sequence $S = (S_k)_{k=1}^{T}$ where $S_k \in \mathcal{R}$.

Notice that a solution for the round scheduling problem is a *sequence* of rounds, and not a weight function, and therefore, scheduling has integrality constraints.

Given a schedule $S = (S_k)_{k=1}^{T}$, we can construct the following *associated timed flow network*. First, we make $T+1$ copies of each vertex and label them as $u_k : u \in V, k = 0, \ldots, T$. We connect u_k with u_{k+1} with an arc of infinite capacity, and u_{k-1} with v_k with an arc of capacity 1 if and only if $(u, v) \in S_k$, $k = 1, \ldots, T$. Finally, we associate to the traffic demand $f(u, v)$ (in the original network) a traffic demand $\bar{f}$ in the timed flow network with $\bar{f}(u_0, v_T) = f(u, v)$.

Definition 3. Let S be a round schedule with time horizon T. The *throughput of S* is defined as

$$\gamma(S) = \max\{\gamma \in \mathbb{Q}^+ : \text{ the traffic demand } \gamma T\bar{f} \text{ can be satisfied in the associated timed flow network}\}.$$

The optimum throughput for a time horizon T will be denoted by

$$\gamma^*(T) = \max\{\gamma(S) : S \text{ is a round schedule with time horizon T}\}.$$

Observe that $\gamma^*(T)$ models the percentage of each bandwidth demand that is satisfied per time unit.

The goal of the round scheduling problem is to find a sequence of rounds with maximum throughput. Since we are not interested in routing data within a deadline, we are mainly interested in $\lim_{T\to\infty} \gamma^*(T)$.

6.1. The relation between round weighting and round scheduling

We prove here that when the time horizon is large, the scheduling problem can be relaxed to finding how to distribute the rounds, in order to get enough average bandwidth on the arcs to route the traffic. In a sense, when the rounds take place is not essential, what does matter is how often. Henceforth, if the time horizon is long, round scheduling is indeed equivalent to round weighting (Proposition 9).

First, observe that any schedule S induces a weight function $w_S : \mathcal{R} \to \mathbb{Q}^+$, where $w_S(R)$ is the number of times R appears in S, and therefore a capacity function on the transmission graph $c_{w_S}(e) = \sum_{R\in\mathcal{R}:e\in R} w_S(R)$. And the weight function w_S is a round weighting for the bandwidth demand $\gamma f T$.

Proposition 9. *For the round scheduling problem and its relaxation, the round weighting problem, it holds that (a) $(\forall T)\ \gamma^*(T) \leq \frac{1}{W^*}$; and (b) $\gamma^*(T) \to \frac{1}{W^*}$ as $T \to \infty$.*

Proof. For the first part, we observe that for any schedule S with time horizon T, the bandwidth demand $\gamma f T$ is feasible with weight function w_S, so $\frac{w_S}{\gamma T}$ provides a round weighting for f with cost $\frac{\sum_{R\in\mathcal{R}} w_S(R)}{\gamma(S)T} = \frac{1}{\gamma}$. Hence $W^* \leq \frac{1}{\gamma^*(T)}$.

For the second part, we show that given any $\varepsilon > 0$ there exists a time horizon T_0, such that any horizon $T \geq T_0$ admits a scheduling S with throughput $\gamma(S) \geq \frac{1}{W^*} - \varepsilon$.

Consider an optimal round weighting function w with cost W^* that enables satisfying the flow demand f. Because f is integral (or rational) this optimal solution can be chosen rational. Let k be such that kw is integral, and let S be any scheduling having kw as weight function (it suffices to take every round $R \in \mathcal{R}$ exactly $kw(R)$ times, and order the calls arbitrarily). The schedule S lasts kW^* steps, and the bandwidth demand kf is feasible for kw. Let $f_k : E \to \mathbb{Z}$ be an integer flow function satisfying the demand kf feasible for kw.

We look at the schedule obtained by repeating the schedule S m times. Each repetition of S is called a *stage*, so the schedule is made of m stages. Consider a path decomposition of the flow f_k that kw allows one to route, i.e. an integral weight function on the paths. Let η be the weight of the path under consideration in the path decomposition that is being used. During each stage nodes behave as follows:

– If a node has received η packets from its predecessor on a path during the preceding stage, it sends those packets to its successor on that path.
– If a node is the initial node of a path, it sends η packets to its successor.

Note that when this scheme is performed, a node u needs to transmit at most $f_k(u, v)$, packets to node v, and this is possible since during each stage, the inherited capacity of the arc (u, v) is by construction $c_{kw}(u, v) \geq f_k(u, v)$.

Now, on a path with weight γ, during any stage greater than i, γ packets are transmitted on any arc at distance i from the initial node. So, for a path with length L, at least $(m - L)\gamma$ packets are transmitted. If $D < n$ is the maximum length of a path, we are sure to route the bandwidth demand $(m - D)kf$, and the schedule lasts kmW^* steps.

Hence, $\gamma(kmW^*) = \frac{m-D}{m} \cdot \frac{1}{W^*}$, and $\gamma(kmW^*) \to \frac{1}{W^*}$ for $m \to \infty$. □

7. Conclusions

We have characterized the complexity of the round weighting problem in the general case. For the most practical topologies, we have also studied the complexity, and provided polynomial algorithms or approximations.

In addition to the small linear program for the round weighting problem in paths (presented in Section 4.1), we have also developed a dynamic program for the path, as well as a linear program and a dynamic program for trees (for details, see [22]). Moreover, the techniques can also be extended to graphs of bounded treewidth.

Some questions remain open:

- Whereas the algorithms presented for the path (and for trees) work well in practice, the PTAS given for the 2-dimensional grid is purely theoretical. Is it possible to derive simple and efficient algorithms for the 2-dimensional grid (and grid-like networks)?
- Is the problem NP-HARD in the 2-dimensional grid in the case of gathering?
- Is it possible to get simple good approximations for the gathering problem in the general case? Is it possible to get a PTAS for the gathering problem? In fact, we conjecture that such an approximation exists.
- Is it possible to give purely combinatorial approximation algorithms that would not use linear programming?
- Is it possible to implement distributed versions of the algorithms? (That is, using only local knowledge of the vertices.)

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