The 2-extendability of 5-connected graphs on the Klein bottle

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ABSTRACT

A graph is said to be *k-extendable* if any independent set of *k* edges extends to a perfect matching. In this paper, we shall characterize the forbidden structures for 5-connected graphs on the Klein bottle to be 2-extendable. This fact also gives us a sharp lower bound of representativity of 5-connected graphs embedded on the Klein bottle to have such a property, which was considered in Kawarabayashi et al. (submitted for publication) [4].

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0. Introduction

A graph in this paper is a simple graph, that is, one with no loops and no multiple edges. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The number of vertices of G is often called the *order* of G.

A set M of edges in a graph G is said to be a *matching* (or o members of M share a vertex. A matching M is *perfect* if every vertex of G is covered by an edge of M. A graph G with $|V(G)| \geq 2k+2$ is said to be *k-extendable* if every matching $M \subseteq E(G)$ with $|M| = k$, extends to a perfect matching in G.

Plummer [8,9,7] has introduced this notion of *k*-extendability of graphs and discussed it, combining topological properties. For example in [9], he has proved that every 5-connected planar graph of even order is 2-extendable.

Let G be a graph and $\{e_1, e_2\}$ an independent pair of edges $e_1 = u_1 v_1$ and $e_2 = u_2 v_2$. If $G - \{u_1, v_1, u_2, v_2\}$ has an *odd component*, that is, a connected component consisting of an odd number of vertices, then the subgraph in G induced by the odd component and $\{u_1, v_1, u_2, v_2\}$ is called a *generalized butterfly*. It is clear that if G contains a generalized butterfly, then G is not 2-extendable; any matching containing the two edges e_1 and e_2 cannot cover all vertices in the odd component. By these facts, Plummer proved the following theorem:

Theorem 1 ([9]). *Every 4-connected maximal planar graph of even order is 2-extendable unless it contains a generalized butterfly.*

More generally, Plummer [7] has shown that for a given closed surface, there exists an upper bound for a natural number k such that the surface admits embeddings of *k*-extendable graphs and Dean [3] has determined the precise value of the bound. Recently, Aldred et al. [1] have proven that a triangulation of even order on a closed surface of positive genus is 2-extendable if it has sufficiently large representativity. (The representativity of G on a closed surface F^2 denoted by $\gamma(G)$ is defined as follows: $\gamma(G) := \min\{|G \cap \ell| : \ell$ is an essential simple closed curve on $F^2\}$. A graph G on F^2 is said to be *r-representative* if $\gamma(G) \geq r$.) Furthermore in [6], Mizukai et al. have discussed the 2-extendability of 5-connected graphs on the torus

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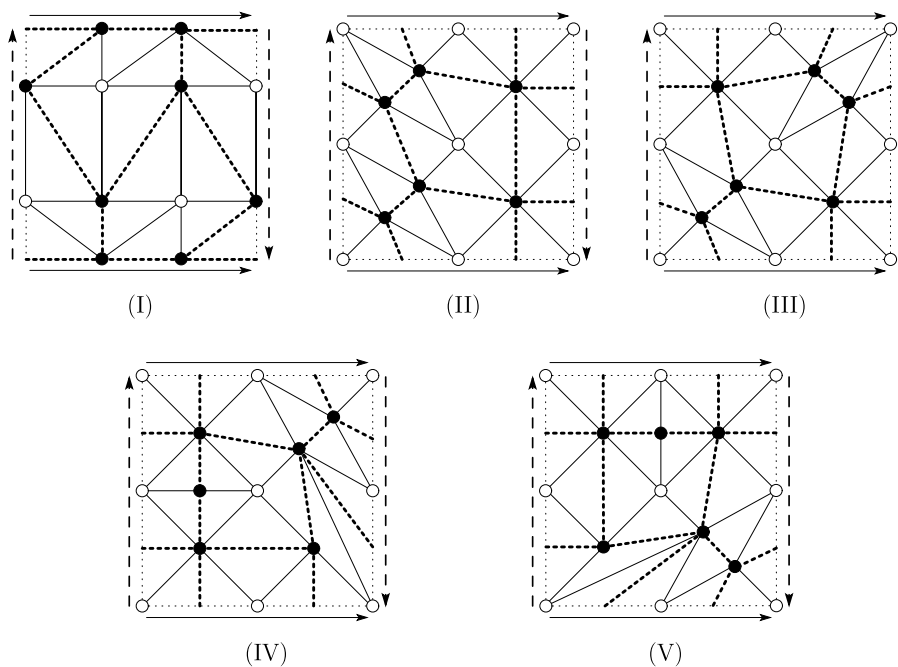
Fig. 1. Forbidden structures for the 2-extendability of 5-connected graphs on K^2.

and characterized the forbidden structures for graphs having such a property. (These topics, which state the extendability of graphs on surfaces with low genera, were also treated in the research of restricted matching extension by Aldred and Plummer [2].) Actually in the proofs in [6], they needed the fact that those graphs in the theorem are 1-extendable; in fact, Thomas and Yu's results in [10,11] guarantee the property.

Although a 5-connected graph on the Klein bottle is not 1-extendable in general, we could prove the following theorem by carrying out topological arguments on the Klein bottle, which is our main result in the paper.

Theorem 2. *A* 5-*connected graph of even order embedded on the Klein bottle is* 2-*extendable if and only if it has none of the structures depicted in Fig.* 1.

In the figure, to obtain the Klein bottle, we have to identify the top and the bottom of the dotted rectangle in parallel, and the right-hand side and left-hand side in anti-parallel. To get an actual graph which are not 2-extendable, replace each of the white vertices with a connected planar graph of odd order and choose additional edges from edges drawn by thick dotted lines so that they include at least one independent pair of edges. The edges between a white vertex and a black vertex may split into several edges with a common black end. Furthermore, the resulting graph should be simple and 5-connected. For example, all dotted edges in each of (I), (II) and (IV) cannot be included simultaneously since if we do so, there would be multiple edges between two black vertices. Therefore, if one wants a triangulation on the Klein bottle, then only (III) and (V) may be used.

By observing those figures, the following corollary related to representativity is an easy consequence:

Corollary 3. *A* 5-*connected and* 4-*representative graph on the Klein bottle with even order is* 2-*extendable.*

1. $\{e_1, e_2\}$-blocker

First we shall prepare two key lemmas to prove Theorem 2. They are basically the same ones as given by Plummer [9]. The point is the existence of a set S of vertices satisfying the two conditions in the following lemma. Plummer called such a set a $\{e_1, e_2\}$-*blocker*:

Lemma 4. *Suppose that there is an independent pair of edges* $e_1 = u_1v_1$ *and* $e_2 = u_2v_2$ *in a graph G of even order which does not extend to a perfect matching. Then G contains a set S of vertices such that:*

(i) $S \supset \{u_1, v_1, u_2, v_2\}$, *and*
(ii) $|S| \le o(G - S) + 2$,

where $o(H)$ *stands for the number of odd components, that is, components of H each of which consists of an odd number of vertices.*

Proof. By the assumption in the lemma, put $G' = G - \{u_1, v_1, u_2, v_2\}$. Since $\{e_1, e_2\}$ does not extend to a perfect matching, G' has no perfect matching. By Tutte's 1-Factor theorem, there is a subset $S' \subset V(G')$ with $|S'| < o(G' - S')$. Put $S = S' \cup \{u_1, v_1, u_2, v_2\}$. Then we have:

$$|S| - 4 = |S'| < o(G' - S') = o(G - S).$$

Since G has even order, we have that $|V(G)| \equiv o(G - S) + |S| \equiv 0 \pmod 2$. Thus, $o(G - S)$ and $|S|$ have the same parity. This implies (ii) in the lemma. $\square$

Let G be a graph and S a subset of $V(G)$. We construct a bipartite graph from G as follows. Remove all even components of $G - S$ and shrink each of odd components of $G - S$ to one vertex, say x_i. Delete the edges joining vertices in S and some edges between S and $X = \{x_1, \ldots, x_{o(G-S)}\}$ so as to eliminate multiple edges. We denote the resulting graph by $B(G, S)$. It is clear that $B(G, S)$ is a bipartite graph with partite sets S and X. This has been called "the BG graph" in [9]. We shall use these notations given above hereafter.

Lemma 5. *Let G be a k-connected graph embedded on a closed surface F^2 with $k \geq 5$ and $\{e_1, e_2\}$ an independent pair of edges in G. If G has an $\{e_1, e_2\}$-blocker S, then:*

$$|S| \leq 2 + \frac{4 - 2\chi(F^2)}{k - 4}.$$

Proof. Perform the deformation to construct $B(G, S)$ from G on the surface F^2. If a self-loop appears in the process of shrinking an odd component, then we should delete it, not contracting it; otherwise, the surface would be pinched. Then $B(G, S)$ is naturally embedded on F^2 but it might have some faces not homeomorphic to a 2-cell. Replace each of such faces with a 2-cell to obtain a closed surface F_0^2 where $B(G, S)$ is 2-cell embedded. Then we have $\chi(F_0^2) \geq \chi(F^2)$. Applying Euler's formula to the bipartite graph $B(G, S)$ on F_0^2 with $|V(B(G, S))| = |S| + o(G - S)$, we have the following inequality:

$$|E(B(G, S))| \leq 2|V(B(G, S))| - 2\chi(F_0^2) \leq 2|S| + 2o(G - S) - 2\chi(F^2).$$

Note that the equality holds in the first inequality only when $B(G, S)$ is embedded on F_0^2 as a *quadrangulation*, that is, one such that each face is quadrilateral.

On the other hand, each vertex x_i has degree at least k since G is k-connected; otherwise, a subset of S consisting of less than k vertices would form a cut in G which separates the odd component corresponding to x_i. This implies that $|E(B(G, S))| \geq k \cdot o(G - S)$. Combining these inequalities on $|E(B(G, S))|$, we have:

$$k \cdot o(G - S) \leq 2|S| + 2o(G - S) - 2\chi(F^2)$$
$$(k - 2)o(G - S) \leq 2|S| - 2\chi(F^2).$$

By the above lemma, we have $|S| - 2 \leq o(G - S)$ and hence,

$$(k - 2)(|S| - 2) \leq 2|S| - 2\chi(F^2)$$
$$(k - 4)|S| \leq 2(k - 2) - 2\chi(F^2).$$

Then, the inequality in the lemma follows. $\square$

Using the above lemma, we prove the following theorem very easily.

Theorem 6. *Every 6-connected graph of even order embedded on the Klein bottle is 2-extendable.*

Proof. Let G be a 6-connected graph embedded on the Klein bottle K^2. For a contradiction, we assume that G is not 2-extendable. Then G has an independent pair of edges $\{e_1, e_2\}$ which cannot extend to a perfect matching and there is an $\{e_1, e_2\}$-blocker S by Lemma 4. Further by Lemma 5, we have $|S| \leq 4$ since $\chi(K^2) = 0$. However, this implies that S would form a 4-cut in G, contrary to G being 6-connected. (Observe that $G - S$ has at least 2 odd components by Lemma 4.) Thus G is 2-extendable. $\square$

2. Embeddings of $K_{3,5}$ into the Klein bottle

An embedding of a graph G on a closed surface F^2 is regarded as an injective continuous map from the one-dimensional topological space G to F^2 when formulating delicate properties of embeddings. Two embeddings $f_1, f_2 : G \to F^2$ are said to be *equivalent* to each other if there is a homeomorphism $h : F^2 \to F^2$ such that $hf_1 = f_2$, and they are *inequivalent* otherwise. Two embeddings f_1 and f_2 are said to be *congruent* to each other if there exists a homeomorphism $h : F^2 \to F^2$ and an automorphism $\sigma : G \to G$ such that $hf_1 = h_2\sigma$.

It is well known that the Klein bottle K^2 admits four types of simple closed curves (e.g., see [5]). Let ℓ be a simple closed curve on K^2. If ℓ bounds a 2-cell, then ℓ is said to be *trivial*, and *essential* otherwise. If ℓ is essential and it separates K^2 into two Möbius bands, then ℓ is called an *equator*. If cutting K^2 along ℓ yields one Möbius band, then ℓ is called a *longitude*. In
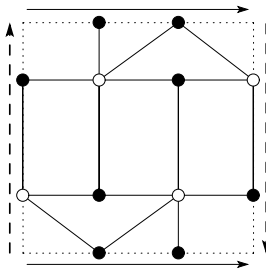
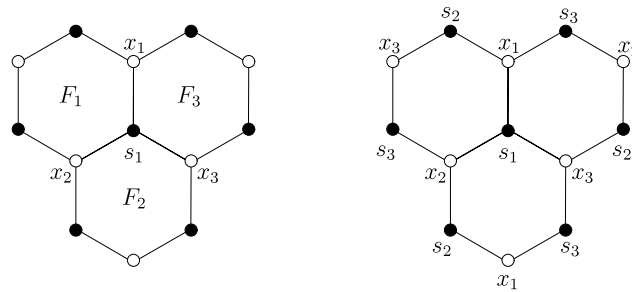**Fig. 2.** The unique embedding of $K_{3,5}$ on the Klein bottle.

Fig. 3. Three 6-gonal faces incident to a vertex s_1.

this case, a tubular neighborhood of ℓ is homeomorphic to a Möbius band. An essential simple closed curve ℓ is said to be a *meridian*, if cutting K^2 along ℓ yields an annulus. Note that a trivial curve and an equator on K^2 are *surface separating*, that is, $K^2 - \ell$ is disconnected.

To prove our main theorem, we consider the re-embeddability of $K_{3,5}$ into the Klein bottle in this section. (It suffices to classify the embeddings of $K_{3,5}$ on the Klein bottle up to congruence for our purpose.) However, we could obtain the following lemma. Throughout the section, we suppose that $K_{3,5}$ has two partite sets $X = \{x_1, x_2, x_3\}$ and $S = \{s_1, s_2, \ldots, s_5\}$.

Lemma 7. *The complete bipartite graph $K_{3,5}$ admits the unique embedding into the Klein bottle, up to congruence, as shown in Fig. 2.*

Proof. For our purpose, we first embed a complete bipartite graph $K_{3,3}$ into the Klein bottle K^2, assuming that two partite sets of the $K_{3,3}$ are $S' = \{s_1, s_2, s_3\}$ and $X = \{x_1, x_2, x_3\}$. (Because any embedding of $K_{3,5}$ on the Klein bottle can be realized by adding two extra vertices s_4 and s_5 to the $K_{3,3}$ on K^2 so that each of them is adjacent to all vertices of X.) If it forms a 2-cell embedding, that is each face corresponds to an open 2-cell, then the number of its faces is exactly 3 by Euler's formula. Furthermore, a combination of sizes of those faces in an embedding is presented by one of (a) (6, 6, 6), (b) (8, 6, 4) and (c) (10, 4, 4).

At first, we suppose (a). Let F_1, F_2 and F_3 be three faces of the embedding. Note that the length of the boundary walk of F_i is 6, for $i = 1, 2, 3$. In this case, any vertex of $K_{3,3}$ appears on the boundary walk of each hexagonal face, that is, it forms a boundary cycle; for otherwise, it would yield multiple edges or an odd cycle. Focus on a vertex of S', say s_1, and assume that $x_1 s_1 x_2$, $x_2 s_1 x_3$ and $x_3 s_1 x_1$ are corners of F_1, F_2 and F_3, respectively, as shown in the left-hand side of Fig. 3. The way to assign labels to other vertices is unique, up to automorphisms that exchange s_2 and s_3 (see the right-hand side of the figure). However, it presents an embedding of $K_{3,3}$ on the torus. Thus, (a) is not the case.

Secondly, suppose (b) and that $|F_1| = 8$, $|F_2| = 6$ and $|F_3| = 4$. First consider the unique 8-gonal face F_1. On the boundary walk of the face, a vertex of X, say x_1, appears twice and also s_1 does so. They have to place in antipodal points of the face, otherwise it immediately yields multiple edges or an odd cycle. Note that the edge $x_1 s_1$ appears twice on the boundary walk. Since we have to identify the boundary walk with edges $x_1 s_1$ in anti-parallel on K^2, a simple closed curve ℓ running through F_1 and passing the middle point of $x_1 s_1$ should be a longitude (see the left-hand of Fig. 4). Moreover, the other four vertices of the 8-gonal face should be distinct and we obtain (A) in the figure; otherwise, it also yields multiple edges, an odd cycle or a vertex of degree 2.

Then we assume the last case (c) in which $|F_1| = 10$, $|F_2| = 4$ and $|F_3| = 4$. Let W denote the boundary walk of F_1. Similarly to (b), there is at least one edge lying twice on W. However in this case, we have to discuss two possibilities on the location of the edge: (i) $W = s_1 x_1 s_a x_b s_c x_1 s_1 x_d s_e x_f$ and (ii) $W = s_1 x_1 s_a x_b s_1 x_1 s_c x_d s_e x_f$ for $a, b, c, d, e, f \in \{2, 3\}$. In (i), a simple closed curve ℓ running through F_1 and $x_1 s_1$ (as well as the case (b)) should be either an equator or a meridian.

If the former occurs, s_1 would become a cut vertex since ℓ is surface separating. Hence we suppose the latter (see the left-hand side of Fig. 5). In the figure, unlabeled vertices have to become s_2, s_3, x_2 and x_3 by the simplicity of the graph. Then, there is an edge appearing on W twice other than $x_1 s_1$, which should be pasted in anti-parallel, say $x_2 s_2$. Eventually, after joining x_3 and s_3, we obtain (B) in the figure.
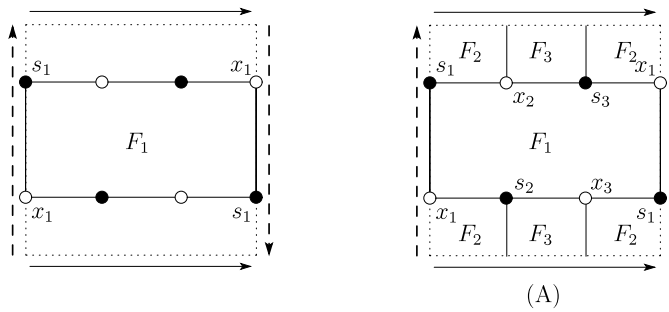
Fig. 4. The 8-gonal face of $K_{3,3}$ on K^2.
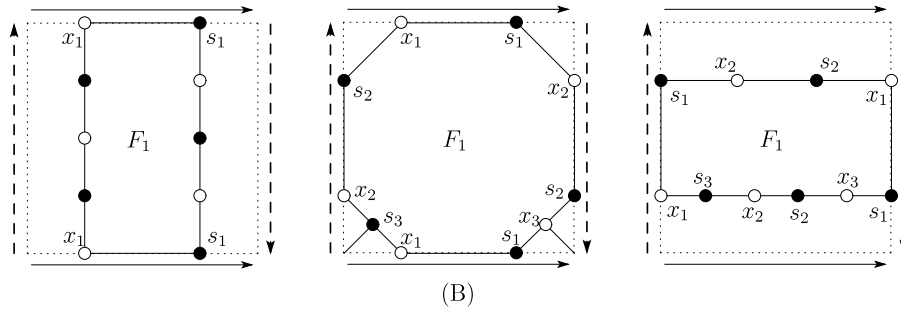
Fig. 5. The 10-gonal face of $K_{3,3}$ on K^2.
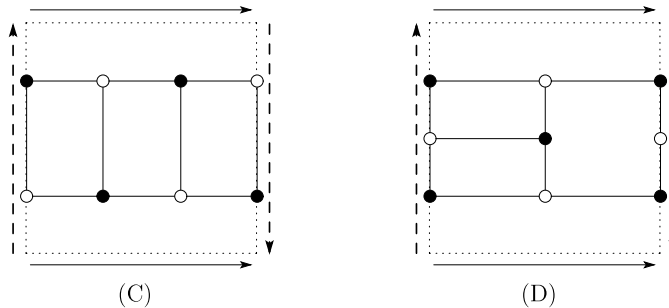
Fig. 6. $K_{3,3}$ in the Möbius band $M_1 \subset K^2$.

Next, we assume (ii) and hence a simple closed curve ℓ passing through F_1 and $x_1 s_1$ is 1-sided. We take ℓ so as to run along longitude and obtain the right-hand side of Fig. 5. Considering the simplicity, we assign labels uniquely, up to automorphism. Therefore, also in this case, we have got (B) in the figure.

Finally, we consider the case when $K_{3,3}$ on the Klein bottle K^2 does not form a 2-cell embedding. Since $K_{3,3}$ is not planar, the embedding on K^2 has a unique region which is homeomorphic to a Möbius band. The boundary cycle C of the region corresponds to an equator of the K^2 and cutting the K^2 along C yields two Möbius bands M_1 and M_2; assume that $M_1 \cup C$ contains all vertices of the $K_{3,3}$. It is also known that the way to embed $K_{3,3}$ into the projective plane is unique, up to congruence; it has a single 6-gonal and three quadrangular faces. Therefore, the length of C is either 4 or 6 and hence we obtain (C) and (D) in Fig. 6. Now we have listed all the embeddings of $K_{3,3}$ on the Klein bottle, which are (A), (B), (C) and (D).

As the final step in the proof, we add two vertices s_4 and s_5 into one or two faces of $K_{3,3}$ on K^2 and join vertices by edges so that the resulting graph becomes $K_{3,5}$. It is easy to see that we cannot add s_i to a quadrangular face for $i = 4, 5$, since s_i could not have degree 3. Further, a hexagonal face includes at most one such vertex. For example, we try to add them in (D). By the above observation, we can add at most one of s_4 and s_5 to the unique hexagonal face in M_1 and hence the other Möbius band M_2 have to include at least one vertex of s_4 and s_5; assume that s_5 is in M_2. However, it is impossible since s_5 cannot be adjacent to the third vertex of X in M_1. So, we conclude that (D) does not extend to the embedding of $K_{3,5}$.

Similarly, by considering the positions of s_4 and s_5, we have two ways to add them in each of (A) and (B) and only one way in (C); as a result, we obtain (A-1), (A-2), (B-1), (B-2) and (C-1) in Fig. 7 under unlabeled sense. However, after assigning the numbers to vertices as in the figure, we can find that each of the five embeddings of $K_{3,5}$ has one 6-gonal face 123456 and six quadrilateral faces 1284, 1436, 2367, 2547, 2568 and 4768. This means that those five embeddings are mutually congruent, that is, $K_{3,5}$ admits a unique embedding on K^2 as shown in Fig. 2, up to congruence. Hence, the lemma follows. □
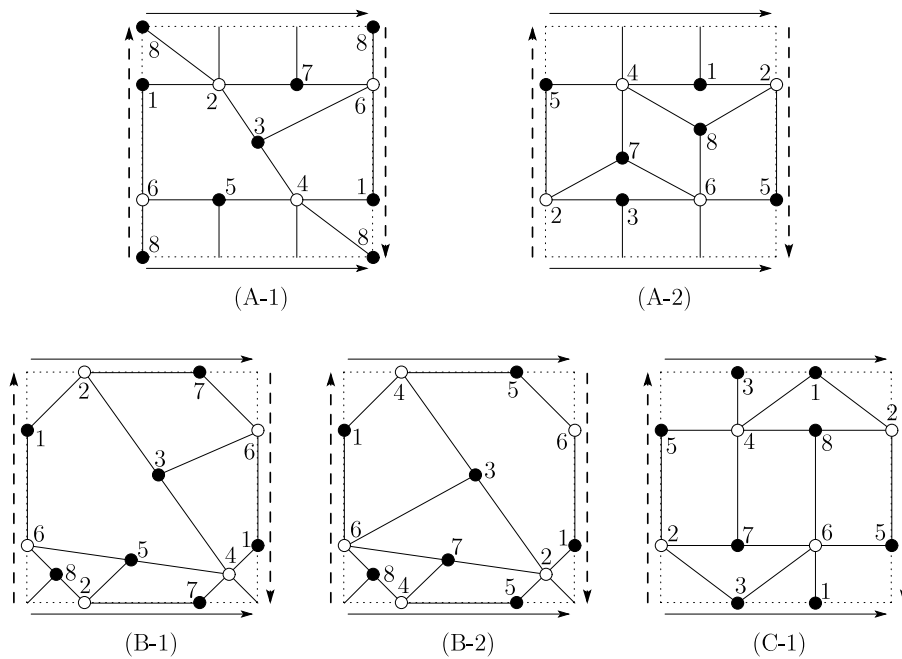
Fig. 7. $K_{3,5}$ on K^2 obtained from (A), (B), (C).
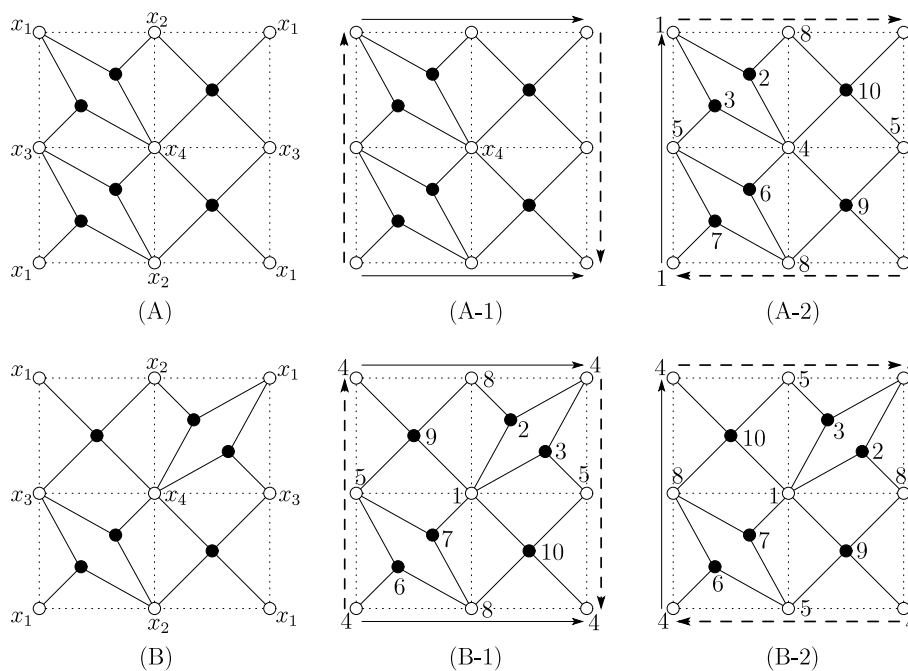
Fig. 8. The 2×2 grid on the Klein bottle.

3. Embeddings of a specified bipartite graph

Similarly to the argument in the previous section, we consider the re-embeddability of a specified bipartite graph, to prove our main theorem.

Lemma 8. *Let H be a bipartite graph with partite sets S and X which can be embedded into the Klein bottle. If H satisfies* (1) $|S| = 6$ *and* $|X| = 4$ *and* (2) *H admits no 4-cut* $S' \subset S$, *then an embedding of H in the Klein bottle is one of* (A − 1), (B − 1), (C − 1) *and* (C − 2) *in* Figs. 8 *and* 9 *in unlabeled sense.*
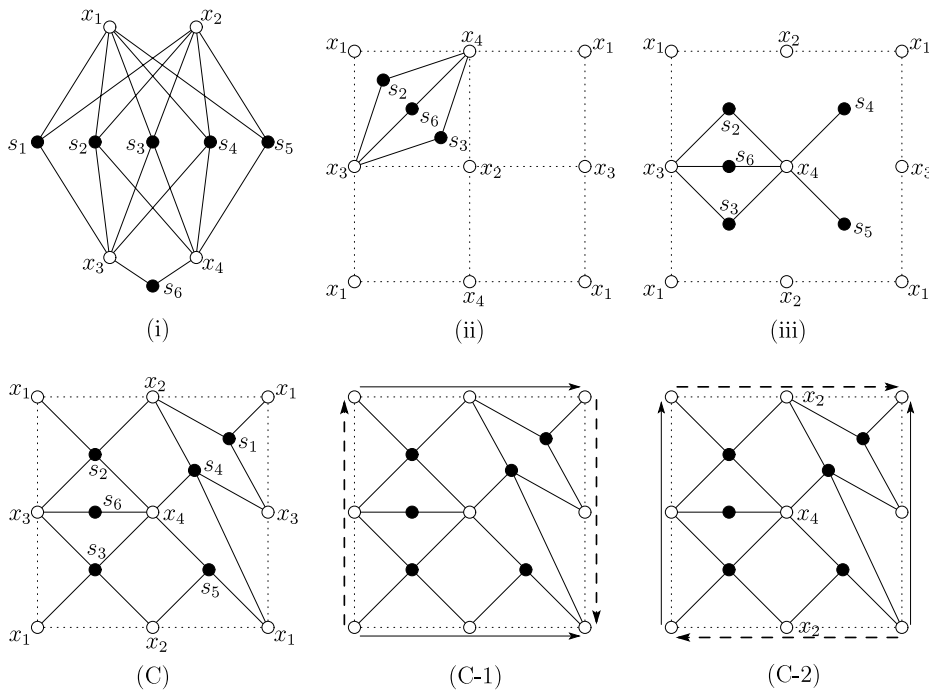
Fig. 9. Re-embeddings of a bipartite graph H into the Klein bottle.

Proof. Suppose that H has two partite sets $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ and $X = \{x_1, x_2, x_3, x_4\}$. Since H is bipartite, H embedded on the Klein bottle should be an even embedding (i.e., each face is bounded by a cycle of even length). Then by Euler's formula, we have $|E(H)| \leq 2|V(H)| = 20$. Furthermore by (2) in the lemma, each vertex of X has degree at least 5 and hence H has at least 20 edges. This implies that H has exactly 20 edges, that is, each x_i has degree 5. Further, it should be a quadrangulation on the Klein bottle; since it satisfies $|E(H)| = 2|V(H)|$. Note that every vertex of a quadrangulation of a closed surface has degree at least 2.

First, we assume that each of $s_1, s_2, \ldots, s_6$ has degree at least 3. Suppose that H is embedded on the Klein bottle, and we focus on a vertex in X, say x_1. We may assume that x_1 is adjacent to $s_1, \ldots, s_5$ and they lie around x_1 in this cyclic order. Let $x_1 s_j w_j s_{j+1}$ $(j = 1, \ldots, 4)$ and $x_1 s_5 w_5 s_1$ be the boundary cycles of quadrilateral faces incident to x_1. Since H is a bipartite graph with $|X| = 4$, we have $\{w_1, \ldots, w_5\} = \{x_2, x_3, x_4\}$. This implies that $w_1 = w_3 = x_2$, $w_2 = w_4 = x_3$ and $w_5 = x_4$ for example; otherwise, H would not be simple. Thus, we find a pair of simple closed curves ℓ_2 and ℓ_3 which cross each other at x_1 transversely and such that $\ell_i \cap H = \{x_1, x_i\}$ for $i = 2, 3$.

If one of ℓ_2 and ℓ_3, say ℓ_2, is an equator then $\{x_1, x_2\}$ clearly forms a 2-cut since an equator of the Klein bottle is surface separating. Therefore, we have the following two cases, up to symmetry: (a) Each of ℓ_2 and ℓ_3 is a longitude. (b) ℓ_2 is a meridian and ℓ_3 is a longitude. First, we suppose (a). Now, $K^2 - (\ell_2 \cup \ell_3)$ has two connected components, one of which is homeomorphic to an open 2-cell D while the other is to a Möbius band without its boundary denoted by M. Under the conditions, one of D and M includes two of $s_1, \ldots, s_5$ and the other contains x_4 and the remaining three vertices of $s_1, \ldots, s_5$. Since x_4 has degree 5, it must be adjacent to five vertices of S. However, two vertices of S are in the other region and hence it is impossible.

Thus we may assume (b). Cut open the Klein bottle along ℓ_2 and ℓ_3. Then we obtain a rectangle such that its four corners are labeled with x_1 and the middle points on each parallel pair of sides have the same label, x_2 or x_3. Further, it contains x_4 inside. The same situation as for x_1 holds for x_4 and there is a similar pair of simple closed curves crossing each other at x_4. Since x_4 does not appear twice around x_1, each of such simple closed curves passes through x_2 or x_3, missing x_1. Thus, we obtain the grid depicted by dotted lines in Fig. 8 (now, it suffices to see (A) in the figure).

The vertices $s_1, \ldots, s_6$ must lie within these four rectangular regions so as to make a quadrangulation with edges missing the grid lines. Clearly, each rectangular region contains either one or two of them since all s_i have degree at least 3; note that if there is a region which contains no such vertex, the whole embedding could not become a quadrangulation. Since each x_i has degree 5, we find two solutions depicted by (A) and (B) of Fig. 8. (Note that each x_i has exactly three neighbors of degree 3 and two neighbors of degree 4. This condition restricts the possibilities of the way to put vertices.) In fact, for each configuration, there are two ways to take ℓ_2 on K^2, that is, it is either a meridian or a longitude. Thus, we obtain four bipartite quadrangulations (A-1), (A-2), (B-1) and (B-2) from (A) and (B) in the figure, respectively. However, (A-2), (B-1) and (B-2) are mutually equivalent, up to congruence. (When we assign the labels to those graphs as in the figure, it is easy to confirm that their face sets become same.) Furthermore, (A-1) and (B-1) are regarded as different embeddings of H by the

following reason: For each white vertex x_i of (A-1), its three black neighbors with degree 3 lie continuously around x_i, while not so in (B-1). Hence we take only (A-1) and (B-1) from this case.

Secondly, we consider the case when H has a vertex of degree 2. Since $\deg(x_i) = 5$ for each $i = 1, \ldots, 4$, we may assume that s_6 has degree 2 and that its neighbors are x_3 and x_4. Note that $\{x_1, x_2\} \cup \{s_1, \ldots, s_5\}$ induces a complete bipartite graph $K_{2,5}$. Now suppose that there is another vertex of degree 2, say s_5. However in this case, each of x_3 and x_4 is adjacent to $s_1, \ldots, s_4$ and s_6 and hence $\{s_1, \ldots, s_4\}$ would form a 4-cut which separates s_5 and s_6, contrary to (2) of the lemma. Therefore, we may assume that each s_i has degree at least 3 for $i = 1, \ldots, 5$. Considering the symmetry of the graph, we can depict it as (i) in Fig. 9.

Since all neighbors of x_1 have degree at least 3, we can cut open the Klein bottle, similarly to the previous case. As a result, we obtain the two rectangles (ii) and (iii) in Fig. 9; we have to consider the two ways to cut the Klein bottle since there exists no automorphism that exchanges x_2 and x_3 (see (i) again). Note that (iii) has no grid line in the interior since x_1 and x_4 do not have the symmetrical structure. Furthermore, note that $H - \{s_6\}$ should also be a quadrangulation and it has a face bounded by $x_3 s_2 x_4 s_3$, up to symmetry of the graph; two common neighbors of x_3 and x_4 should be taken from $\{s_2, s_3, s_4\}$.

However, we can easily exclude (ii) in the figure since s_2 cannot be adjacent to x_2 without crossing over the dotted lines. Thus, we assume (iii). First, we connect x_1 and s_5 by an edge. Although there are four x_1's appearing at the corners of the rectangle, s_5 must be incident to the (nearest) lower right one. (For example, if it is incident to the lower left x_1, then s_3 cannot be adjacent to x_2. Further, if it is incident to the upper right one, s_4 must be incident to the left x_3. Under the conditions, s_2 cannot be adjacent to x_1 and x_2. The upper left one is clearly not the case.) By the repetition of similar arguments, s_5 (also s_3) must be incident to the lower x_1 and x_2, and s_2 must be incident to the upper x_1 and x_2. Moreover, by considering the adjacency of s_1, we eventually obtain (C) in Fig. 9.

Similarly to the previous case, we have two ways to assign a meridian and a longitude to (C). Thus, we obtain (C-1) and (C-2) from this case; they are not congruent to each other since (C-2) admits a simple closed curve passing through only x_4 and x_2 along a longitude, but (C-1) does not admit such a curve. □

4. Proof of our main theorem

Proof of Theorem 2. At first, we prove the sufficiency. Substituting $k = 5$ in the inequality of Lemma 5, we have $|S| \leq 6$ for an $\{e_1, e_2\}$-blocker S and hence $|S| = 5$ or 6 since S forms a cut in the 5-connected graph G.

Case 1. *Suppose that* $|S| = 5$. By Lemma 4, we have $o(G - S) \geq 3$. Construct the bipartite graph $B(G, S)$ with two partite sets $S = \{s_1, \ldots, s_5\}$ and $X = \{x_1, \ldots, x_m\}$ ($m \geq 3$). Since each x_i has degree at least 5, it must be adjacent to all of $s_1, \ldots, s_5$. By Euler's formula, it easily follows that $4|V(B(G, S))| - 2|E(B(G, S))| \geq 0$ for a bipartite graph on K^2. Now we substitute $|V(B(G, S))| = m + 5$ and $|E(B(G, S))| = 5m$ in the above inequality, and obtain $20 - 6m \geq 0$. Thus, we have $m \leq 3$ and hence m is exactly 3 by combining $m \geq 3$.

Therefore, we only have to consider the case that $B(G, S)$ is isomorphic to $K_{3,5}$. Since $B(G, S)$ is neither planar nor projective planar, the surface F_0^2 appearing in the proof of Lemma 5 must be the Klein bottle with $F_0^2 = F^2$ and $B(G, S)$ is 2-cell embedded there. By Lemma 7, we had already got the unique embedding of $K_{3,5}$ as shown in Fig. 2.

Now we try to recover G itself from the embedding of $K_{3,5}$ on the Klein bottle, which is $B(G, S)$. Recall that we have removed all even components of $G - S$ to construct $B(G, S)$. Each even component, if any, must lie in a face of $B(G, S)$ and is joined to some of $s_1, \ldots, s_5$, which are black in the figure. However, it can be adjacent to at most three black vertices, which form a cut separating the even component. This is contrary to G being 5-connected. Therefore, there is no even component of $G - S$.

Since S is an $\{e_1, e_2\}$-blocker, each of e_1 and e_2 must lie in a face of $B(G, S)$ so as to join two of $s_1, \ldots, s_5$. There may be other edges joining black vertices, as suggested by thick dotted lines in (I) of Fig. 1. This is actually the first forbidden structure for 2-extendability of 5-connected graphs on K^2.

Case 2. *Suppose that* $|S| = 6$. In this case, $B(G, S)$ has two partite sets $S = \{s_1, \ldots, s_6\}$ and $X = \{x_1, \ldots, x_m\}$ for $m \geq 4$, by Lemma 4 again. By the same argument in Case 1, we have $|V(B(G, S))| = m + 6$ and $|E(B(G, S))| \geq 5m$ and obtain an inequality $24 - 6m \geq 0$. This implies that $m \leq 4$, that is, $m = 4$.

In this case, $B(G, S)$ completely satisfies the conditions of Lemma 8. Therefore, we obtain (II), (III), (IV) and (V) of Fig. 1, by adding disjoint e_1 and e_2 and other diagonals joining black vertices from (A-1), (B-1), (C-1) and (C-2) of Lemma 8, respectively. (Also in this case, G admits no even component by the same reason as that of the previous case.) These are the remaining four forbidden structures.

Now we shall show the necessity. That is, we prove that if G has one of the forbidden structures depicted in Fig. 1, then G is not 2-extendable. Let S be the black vertices in the figure. Then the number of odd components is equal to $|S| - 2$ in each structure. As is mentioned in the description of those structures, it must contain an independent pair of edges e_1 and e_2 joining two of the black vertices, which should be drawn by thick dotted lines in the figure.

Suppose that $\{e_1, e_2\}$ extends to a perfect matching M. Then at least one vertex in each odd component corresponding to x_i is joined to one of the black vertices by an edge belonging to M. Such black vertices should be all distinct and are $|S| - 2$ in total. However, we have already spent four black vertices to cover both e_1 and e_2 and hence there remain only $|S| - 4$ black vertices as candidates for those. Thus, it is impossible to complete a perfect matching so that it contains e_1 and e_2, a contradiction. Therefore, G is not 2-extendable. □

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