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Cycles through a given arc and certain partite sets in almost regular multipartite tournaments

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Abstract

If x is a vertex of a digraph D, then we denote by $d^+(x)$ and $d^-(x)$ the outdegree and the indegree of x, respectively. The global irregularity of a digraph D is defined by $i_g(D) = \max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\}$ over all vertices x and y of D (including $x = y$). If $i_g(D) = 0$, then D is regular and if $i_g(D) \leqslant 1$, then D is almost regular.

A c-partite tournament is an orientation of a complete c-partite graph. In 1998, Guo and Kwak showed that, if D is a regular c-partite tournament with $c \geqslant 4$, then every arc of D is in a directed cycle, which contains vertices from exactly m partite sets for all $m \in \{4, 5, \ldots, c\}$. In this paper we shall extend this theorem to almost regular c-partite tournaments, which have at least two vertices in each partite set. An example will show that there are almost regular c-partite tournaments with arbitrary large c such that not all arcs are in directed cycles through exactly 3 partite sets. Another example will show that the result is not valid for the case that $c = 4$ and there is only one vertex in a partite set.

Keywords: Multipartite tournaments; Almost regular multipartite tournaments; Cycles

1. Terminology and introduction

In this paper, all digraphs are finite without loops and multiple arcs. The vertex set and arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D, then we write $x \to y$ and say x *dominates* y, and if X and Y are two disjoint vertex sets or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. Furthermore, $X \leadsto Y$ denotes the fact that there is no arc leading from Y to X. We also say that the set Y is *weakly dominated* by X. For the number of arcs from X to Y we write $d(X, Y)$. If D is a digraph, then the *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. Therefore, if there is the arc $xy \in E(D)$, then y is an *outer neighbor* of x and x is an *inner neighbor* of y. The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are called the *outdegree* and *indegree* of x, respectively. For a vertex set X of D, we define $D[X]$ as the subdigraph induced by X. If we speak of a *cycle*, then we mean a directed cycle, and a cycle of length n is called an *n-cycle*. If we replace in a digraph D every arc xy by yx, then we call the resulting digraph the *converse* of D, denoted by D^{-1}.

There are several measures of how much a digraph differs from being regular. Yeo [11] defines the *global irregularity* of a digraph D by

$$i_g(D) = \max_{x \in V(D)} \{d^+(x), d^-(x)\} - \min_{y \in V(D)} \{d^+(y), d^-(y)\}.$$

If $i_g(D) = 0$, then D is *regular* and if $i_g(D) \leqslant 1$, then D is called *almost regular*.

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A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices. If $V_1, V_2, \dots, V_c$ are the partite sets of a c-partite tournament D and the vertex x of D belongs to the partite set V_i, then we define $V(x) = V_i$. If D is a c-partite tournament with the partite sets $V_1, V_2, \dots, V_c$ such that $|V_1| \leqslant |V_2| \leqslant \cdots \leqslant |V_c|$, then $|V_c| = \alpha(D)$ is the independence number of D, and we define $\gamma(D) = |V_1|$. If $|V_i| = n_i$ for $i = 1, 2, \dots, c$, then we speak of the *partition-sequence* $(n_i) = n_1, n_2, \dots, n_c$.

This article deals with cycles containing a given arc. One result about this theme was worked out by the authors in [7–10].

Theorem 1.1. *If D is an almost regular c-partite tournament and $e \in E(D)$ is an arbitrary arc of D, then the following holds*:

(a) *If $c \geqslant 8$, then e is contained in an n-cycle for each $n \in \{4, 5, \dots, c\}$.*
(b) *If $c = 7$ and there are at least two vertices in every partite set, then e is contained in an n-cycle for each $n \in \{4, 5, \dots, c\}$.*

In this article, the length of the cycles are not important, but the number of partite sets, which are contained in the cycle. In 1991, Goddard and Oellermann [2] proved the following generalization of Moon's [5] theorem that every strong tournament is vertex pancyclic.

Theorem 1.2 (Goddard and Oellermann [2]). *Every vertex of a strongly connected c-partite tournament D belongs to a cycle that contains vertices from exactly m partite sets for each $m \in \{3, 4, \dots, c\}$.*

Inspired by this theorem, in 1998 Guo and Kwak [4] (see also Guo [3]) studied cycles containing a given arc and vertices from exactly $m \leqslant c$ partite sets in regular c-partite tournaments. In a first step they proved the following theorem:

Theorem 1.3 (Guo and Kwak [4]). *Let D be a regular c-partite tournament with $c \geqslant 3$. Then the following holds*:

(i) *Every arc of D is in a cycle, which contains vertices from exactly 3 or exactly 4 partite sets.*
(ii) *If $c \leqslant 5$ or the cardinality common to the partite sets of D is odd, then every arc of D is in a cycle, which contains vertices from exactly 3 partite sets.*

Using this theorem as basis of induction, they showed that the following three theorems are valid.

Theorem 1.4 (Guo and Kwak [4]). *Let D be a regular c-partite tournament with $3 \leqslant c \leqslant 5$. Then every arc of D is in a cycle that contains vertices from exactly m partite sets for all m with $3 \leqslant m \leqslant c$.*

Theorem 1.5 (Guo and Kwak [4]). *Let D be a regular c-partite tournament with $c \geqslant 4$. Then every arc of D is in a cycle that contains vertices from exactly m partite sets for all m with $4 \leqslant m \leqslant c$.*

Theorem 1.6 (Guo and Kwak [4]). *Let D be a regular c-partite tournament with $c \geqslant 3$. If the cardinality common to all partite sets of D is odd, then every arc of D is in a cycle that contains vertices from exactly m partite sets for all m with $3 \leqslant m \leqslant c$.*

Note that Theorem 1.6 implies Alspach's [1] theorem that every regular tournament is arc pancyclic, since every partite set of a tournament has the cardinality exactly 1.

The aim is now to carry these results of Guo and Kwak over to almost regular multipartite tournaments. In a first step, we will extend Theorem 1.3 by showing that every arc of an almost regular c-partite tournament is in a cycle containing vertices from exactly 3 or exactly 4 partite sets, if $c \geqslant 4$ or if $c \geqslant 3$ and there are at least two vertices in each partite set. Examples will show that there are multipartite tournaments with an arbitrary large number of partite sets that have arcs which are not in cycles through exactly 3 partite sets. A further example will demonstrate that the condition $c \geqslant 4$ is important, if there is only one vertex in at least one partite set. Using these results as basis of induction, we will derive the main result of this paper.

Theorem 1.7. *Let D be an almost regular c-partite tournament with $c \geqslant 4$. If there are at least two vertices in each partite set, then every arc of D is in a cycle that contains vertices from exactly m partite sets for all m with $4 \leqslant m \leqslant c$.*

An example will show that the condition that there are at least two vertices in each partite set is necessary, at least for $c = 4$.

2. Preliminary results

The following results play an important role in our investigations.

Lemma 2.1 (Tewes, Volkmann and Yeo [6]). *If $V_1, V_2, \ldots, V_c$ are the partite sets of an almost regular c-partite tournament D such that $|V_1| \leqslant |V_2| \leqslant \cdots \leqslant |V_c|$, then $|V_c| \leqslant |V_1| + 2$.*

Lemma 2.2 (Tewes, Volkmann and Yeo [6]). *If D is an almost regular multipartite tournament, then for every vertex x of D we have*

$$\frac{|V(D)| - \alpha(D) - 1}{2} \leqslant d^+(x), d^-(x) \leqslant \frac{|V(D)| - \gamma(D) + 1}{2}.$$

The following observations can be found in [9] by Volkmann and Winzen.

Lemma 2.3. *Let D be an almost regular multipartite tournament and x a vertex of D with $|V(x)| = p$. Then we observe that*

$$\frac{|V(D)| - p - 1}{2} \leqslant d^+(x), d^-(x) \leqslant \frac{|V(D)| - p + 1}{2}.$$

In this article, we treat the case of an almost regular multipartite tournament D with $\alpha(D) = r$, $\alpha(D) = r+1$ or $\alpha(D) = r+2$ and $\gamma(D) = r$. This leads to the following two remarks:

Remark 2.4. Let $\alpha(D) = r$. In this case, Lemma 2.2 yields for all $x \in V(D)$ that

$$\frac{(c-1)r - 1}{2} \leqslant d^+(x), d^-(x) \leqslant \frac{(c-1)r + 1}{2}.$$

Hence, if r is even or if c is odd, then we see that $d^+(x) = d^-(x) = ((c-1)r)/2$ and that D is regular.

Remark 2.5. If $\alpha(D) = r + 2$, $\gamma(D) = r$ and $i_g(D) \leqslant 1$, then $|V(D)| - r$ is even. So the bounds in Lemma 2.3 can be improved by

$$d^+(x), d^-(x) = \frac{|V(D)| - r - 2}{2} \quad \text{if } |V(x)| = r + 2$$

or

$$d^+(x), d^-(x) = \frac{|V(D)| - r}{2} \quad \text{if } |V(x)| = r.$$

Consequently, for the case that $\alpha(D) = r + 2$, instead of Lemma 2.2, we can use the following result:

$$\frac{|V(D)| - r - 2}{2} \leqslant d^+(x), d^-(x) \leqslant \frac{|V(D)| - r}{2}.$$

Now let us summarize some results of Lemma 2.3 and Remark 2.5.

Corollary 2.6. *If D is an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $r = |V_1| \leqslant |V_2| \leqslant \cdots \leqslant |V_c| \leqslant r + 2$, then for every vertex x of D we have*

$$\frac{|V(D)| - r - 2}{2} \leqslant d^+(x), d^-(x).$$

3. Main results

Let D be an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $r = |V_1| \leqslant |V_2| \leqslant \cdots \leqslant |V_c|$. If ab is an arbitrary arc of D such that $a \in V_i$ and $b \in V_j$ with $1 \leqslant i, j \leqslant c$, then the following partition of $V(D)$ is useful in the proofs of the next theorems:

$$A_1 = N^-(b) \cap V_i, \quad A_2 = N^+(b) \cap V_i,$$

$$B_1 = N^+(a) \cap V_j, \quad B_2 = N^-(a) \cap V_j,$$

$$X = N^-(a) \cap \left(\bigcup_{l=1}^{c} V_l - (V_i \cup V_j) \right),$$

$$Y = N^+(a) \cap N^-(b) \cap \left(\bigcup_{l=1}^{c} V_l - (V_i \cup V_j) \right),$$

$$Z = N^+(a) \cap N^+(b) \cap \left(\bigcup_{l=1}^{c} V_l - (V_i \cup V_j) \right).$$

Note that some of the defined sets (clearly except A_1 and B_1) might be empty.

Suppose that $X = \emptyset$. Then it follows that $N^-(a) = B_2$ and hence $N^+(a) = V(D) - (B_2 \cup V_i)$. If we set $d^+(a) = d^-(a) + \Delta_a$ with $\Delta_a \in \{-1, 0, 1\}$ and $\sum_{k \neq i,j} |V_k| = (c-2)r + h$ with $0 \leqslant h \leqslant 2(c-2)$, then we observe that $\Delta_a = |V(D) - (B_2 \cup V_i)| - |B_2| = |V_j| + (c-2)r + h - 2|B_2|$. As $|B_2| = |V_j| - |B_1|$ we obtain

$$|V_j| + \Delta_a = 2|B_1| + (c-2)r + h. \tag{1}$$

Theorem 3.1. *Let D be an almost regular multipartite tournament with the partite sets $V_1, V_2, \ldots, V_c$. If $c \geqslant 4$, then every arc of D is in a cycle containing vertices from exactly 3 or exactly 4 partite sets. If $|V_1| = |V_2| = \cdots = |V_c| = r$, then this result also holds for $c = 3$.*

Proof. According to Lemma 2.1 we can distinguish the three cases that $1 \leqslant r = |V_1| \leqslant |V_2| \leqslant \cdots \leqslant |V_c| = r + m$ with $m = 0, 1, 2$. Thus, we see that $|V(D)| = cr + k$ with $k = 0$, if $m = 0$, $1 \leqslant k \leqslant c - 1$, if $m = 1$, and $2 \leqslant k \leqslant 2c - 2$, if $m = 2$. If $m = 0$ and $c = 3$, then, according to Remark 2.4, D is regular, and Theorem 1.3 of Guo and Kwak yields the desired result. So, if $m = 0$, we can investigate the case that $c \geqslant 4$.

Let ab be an arbitrary arc of D such that $a \in V_i$ and $b \in V_j$ with $1 \leqslant i, j \leqslant c$, and let $A_1, A_2, B_1, B_2, X, Y, Z, \Delta_a$ and h be defined as in the beginning of this section.

Suppose that ab is not in a cycle, which contains vertices from exactly 3 partite sets. In particular, ab is not in a 3-cycle. Under this assumption, we firstly study the domination relationships among the partition sets of $V(D)$ listed above.

Firstly, we observe that

$$X \to b, \text{ i.e.,} \quad N^-(a) \cap N^+(b) \cap \left(\bigcup_{l=1}^{c} V_l - (V_i \cup V_j) \right) = \emptyset, \tag{2}$$

since otherwise, if there is a vertex $x \in X$ such that $b \to x$, then $abxa$ is a 3-cycle, a contradiction.

Now, we suppose that $X = \emptyset$. Since $c \geqslant 4$, (1) yields that $r + 3 \geqslant |V_j| + \Delta_a \geqslant 2 + 2r$, from which we obtain $r = 1$, $|B_1| = 1$, $h = 0$, $\Delta_a = 1$ and $|V_j| = 3$. By Remark 2.5, the fact that $\Delta_a \neq 0$ implies that $|V(a)| = |V_i| = r + 1 = 2$. Furthermore, we observe that $d^-(a) = |B_2| = |V_j| - |B_1| = 2$. Since $h = 0$, it remains to consider the partition-sequence $1, 1, 2, 3$. If $Z = \emptyset$, then we conclude that $|Y| = |V(D) - (V_i \cup V_j)| = 2$, and thus, it follows that $d^-(b) \geqslant 3$, because of Remark 2.5 and $|V_j| = 3 = r + 2$ a contradiction. Hence, we observe that there is a vertex $z \in Z$ and $|V(z)| = 1$. Remark 2.5 yields that $d^+(z) = d^-(z) = 3$. Since $\{a, b\} \to z$, there is a vertex $b_2 \in B_2$ such that $z \to b_2$ and $abzb_2a$ is a cycle with vertices from exactly 3 partite sets, a contradiction.

These considerations lead to $X \neq \emptyset$. Analogously, we see that the case $Z = \emptyset$ is impossible.

If there is an arc $a_2 \to x$ (respectively, $z \to b_2$) from A_2 to X (respectively, Z to B_2), then aba_2xa (respectively, $abzb_2a$) is a cycle containing vertices from exactly 3 partite sets, a contradiction. Hence,

$$X \to A_2 \quad \text{and} \quad B_2 \to Z. \tag{3}$$

If there is an arc $z \to a_2$ (respectively, $b_2 \to x$) from Z to A_2 (respectively, B_2 to X), then we also have $B_2 \to a_2$ (respectively, $b_2 \to A_2$), because otherwise, if there is a vertex $b_2 \in B_2$ (respectively, $a_2 \in A_2$) such that $a_2 \to b_2$, then $abza_2b_2a$ (respectively, aba_2b_2xa) is a cycle through exactly 3 partite sets, a contradiction. But this yields

$$d^-(a_2) \geqslant |X| + |B_2| + |\{b, z\}| = d^-(a) + 2$$

$$(\text{respectively, } d^+(b_2) \geqslant |Z| + |A_2| + |\{a, x\}| = d^+(b) + 2),$$

a contradiction to $i_g(D) \leqslant 1$. Hence,

$$A_2 \to Z \quad \text{and} \quad X \to B_2. \tag{4}$$

Suppose now that the arc ab also does not belong to any cycle with vertices of exactly 4 partite sets. A first consequence, we observe, is that $X \rightsquigarrow Z$, since otherwise, if there are vertices $z \in Z$ and $x \in X$ such that $z \to x$, then $abzxa$ is a cycle with vertices from exactly 4 partite sets, a contradiction.

Assume that there exist vertices $b_1 \in B_1 - \{b\}$ and $x \in X$ such that $b_1 \to x$. If there is a vertex $a_2 \in A_2$ such that $a_2 \to b_1$, then aba_2b_1xa is a cycle through exactly 3 partite sets, a contradiction. If there is a vertex $z \in Z$ such that $z \to b_1$, then $abzb_1xa$ is a cycle containing vertices from exactly 3 or 4 partite sets, a contradiction. Altogether, we see that $b_1 \to Z \cup A_2 \cup \{x\}$ which implies $d^+(b_1) \geqslant d^+(b) + 1$. Because of $i_g(D) \leqslant 1$, we conclude that $d^+(b_1) = d^+(b) + 1$ and $A_1 - \{a\} \to b_1$. If there are vertices $z \in Z$ and $a_1 \in A_1 - \{a\}$ such that $z \to a_1$, then $abza_1b_1xa$ is a cycle with vertices from exactly 3 or exactly 4 partite sets, a contradiction. Together with (3) and (4), for every vertex $z \in Z$, this yields

$$\begin{aligned} d^-(z) &\geqslant |X| + |V_i| + |B_2| + |\{b_1, b\}| - |V(z) - \{z\}| \\ &\geqslant \begin{cases} d^-(a) + 2 & \text{if } |V(z)| \leqslant r + 1, \\ d^-(a) + 1 & \text{if } |V(z)| = r + 2, \end{cases} \end{aligned}$$

in both cases a contradiction either to $i_g(D) \leqslant 1$ or to Remark 2.5. Hence, we see that $X \to B_1$.

Now, assume that there are vertices $a_1 \in A_1 - \{a\}$ and $z \in Z$ such that $z \to a_1$. If there is a vertex $x \in X$ such that $a_1 \to x$, then $abza_1xa$ is a cycle containing vertices from exactly 3 or exactly 4 partite sets, a contradiction. Together with (3) and (4), for every vertex $x \in X$, this yields

$$\begin{aligned} d^+(x) &\geqslant |V_j| + |A_2| + |Z| + |\{a, a_1\}| - |V(x) - \{x\}| \\ &\geqslant \begin{cases} d^+(b) + 2 & \text{if } |V(x)| \leqslant r + 1, \\ d^+(b) + 1 & \text{if } |V(x)| = r + 2, \end{cases} \end{aligned}$$

in both cases a contradiction either to $i_g(D) \leqslant 1$ or to Remark 2.5. Summarizing our results, we see that

$$X \rightsquigarrow Z \cup V_j \cup A_2 \cup \{a\} \quad \text{and} \quad V_i \cup X \cup B_2 \cup \{b\} \rightsquigarrow Z. \tag{5}$$

This leads to the following lower bounds for all $x \in X$ (respectively, all $z \in Z$):

$$\begin{aligned} d^+(x) &\geqslant |V_j| + |Z| + |A_2| + |\{a\}| - |V(x) - \{x\}| \\ &\geqslant \begin{cases} d^+(b) + 2 & \text{if } |V(x)| = r, \\ d^+(b) + 1 & \text{if } |V(x)| = r + 1, \\ d^+(b) & \text{if } |V(x)| = r + 2, \end{cases} \\ d^-(z) &\geqslant |V_i| + |B_2| + |X| + |\{b\}| - |V(z) - \{z\}| \\ &\geqslant \begin{cases} d^-(a) + 2 & \text{if } |V(z)| = r, \\ d^-(a) + 1 & \text{if } |V(z)| = r + 1, \\ d^-(a) & \text{if } |V(z)| = r + 2. \end{cases} \end{aligned}$$

To get no contradiction, it has to be $|V(x)|, |V(z)| \geqslant r + 1$ for all $x \in X$ and $z \in Z$. Furthermore, we conclude that the lower bounds of $d^+(x)$ and $d^-(z)$ must not increase by one, that means $|V_i| = |V_j| = r$, $V(x) - \{x\} \subseteq Z$ for all $x \in X$ and $V(z) - \{z\} \subseteq X$ for all $z \in Z$. If $r \geqslant 2$, then, because of $|V(x)| \geqslant r + 1$ and $V(x) - \{x\} \subseteq Z$ for all $x \in X$, there are at least two vertices $z_1, z_2 \in Z$ with $V(z_1) = V(z_2)$, a contradiction to $V(z) - \{z\} \subseteq X$ for all $z \in Z$. Hence, we examine the case that $r = 1$. This implies $V_i = \{a\}$, $V_j = \{b\}$ and $B_2 = A_2 = B_1 - \{b\} = A_1 - \{a\} = \emptyset$. Furthermore, we conclude that $d^+(b) = |Z|$ and $d^+(a) = |Z| + |Y| + |\{b\}|$ which yields $|Y| = 0$, $d^+(a) = d^+(b) + 1$ and, since $|V_j| = |V_i| = r$, Remark 2.5 yields $|V_c| = r + 1$. Because of $V(D) - (V_i \cup V_j) \subseteq X \cup Z$ and $c \geqslant 4$, there are at least two partite sets V_{x_1} and V_{x_2} in $V(D) - (V_i \cup V_j)$ such that $V_{x_1} = \{x_1, z_1\}$ and $V_{x_2} = \{x_2, z_2\}$. Furthermore, the fact that $V(x) - \{x\} \subseteq Z$ for all $x \in X$ and $V(z) - \{z\} \subseteq X$ for all $z \in Z$ implies that one vertex of V_{x_1} (respectively, V_{x_2}) is in X and the other one in Z. So, without loss of generality, let $x_1, x_2 \in X$ and $z_1, z_2 \in Z$ and $x_1 \to x_2$. But now we observe that

$$d^+(x_1) \geqslant |V_j| + |Z| - |V(x_1) - \{x_1\}| + |A_2| + |\{a, x_2\}| = d^+(b) + 2,$$

a contradiction to $i_g(D) \leqslant 1$. This completes the proof of the theorem. $\square$

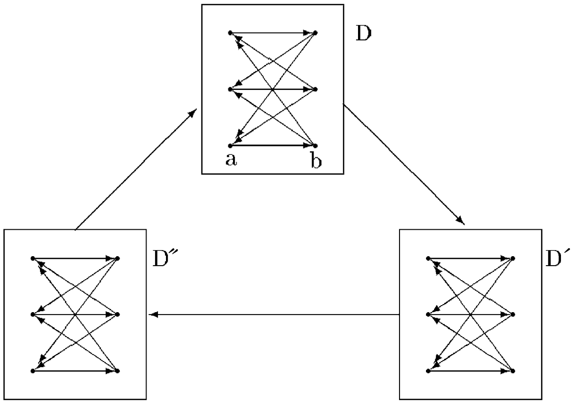

Fig. 1. An almost regular 6-partite tournament with the property that the arc ab is in no cycle through exactly 3 partite sets.

The following example shows that the supplement that every arc is in a cycle which consists of vertices of exactly 3 or 4 partite sets is essential, since not every arc of an almost regular multipartite tournament is in a cycle containing vertices from exactly 3 partite sets.

Example 3.2. Let $V_1=\{a,x_2,x_3\}$ and $V_2=\{b,y_2,y_3\}$ be the two partite sets of a digraph D such that $a\to b\to x_2\to y_2\to x_3\to y_3\to a$, $b\to x_3$, $y_2\to a$ and $y_3\to x_2$. Furthermore, let D' and D'' be copies of D such that $D\to D'\to D''\to D$. The resulting 6-partite tournament H (see also Fig. 1) is almost regular, but the arc ab is not in any cycle containing vertices from exactly three partite sets.

Let G, G', G'' be three copies of H such that $G\to G'\to G''\to G$. The resulting 18-partite tournament is almost regular, but no copy of the arc ab is in a cycle containing vertices from exactly three partite sets.

If we continue this process, we arrive at almost regular c-partite tournaments with arbitrary large c which contain arcs that do not belong to any cycle through exactly three partite sets.

In the case that the maximal difference of the cardinality of the partite sets is exactly 2, Theorem 3.1 also holds, if the multipartite tournament consists of only three partite sets.

Theorem 3.3. *Let D be an almost regular 3-partite tournament with the partite sets V_1,V_2,V_3 such that $1\leqslant r=|V_1|\leqslant|V_2|\leqslant|V_3|=r+2$. Then every arc of D is in a cycle containing vertices of all partite sets.*

Proof. Let ab be an arbitrary arc of D. Suppose that ab is not in any cycle, containing vertices of all partite sets. Obviously, we have $|V(D)|=3r+k$ with $2\leqslant k\leqslant 4$. Let $a\in V_i$ and $b\in V_j$ with $1\leqslant i,j\leqslant 3$. If we define A_1,A_2,B_1,B_2,X,Y,Z,h and Δ_a as in the beginning of this section, then, following the same lines as in Theorem 3.1, we observe that

$$X\to A_2\cup B_2\cup\{a,b\}\to Z. \tag{6}$$

Suppose that $X=\emptyset$. Let $V_l=V(D)-(V_i\cup V_j)$. Since, $c=3$, from (1) we get $|V_j|+\Delta_a=2|B_1|+r+h$. This equality implies $B_1=\{b\}$, $B_2=V_j-\{b\}$ and $0\leqslant h\leqslant 1$. If $h=1$, then it follows that $\Delta_a=1$, $|V_j|=r+2$ and $|V_l|=r+1$. By Remark 2.5 we have $|V(a)|=|V_i|=r+1$. This is a contradiction since there is no partite set with r vertices. Hence, let $h=0$ and thus $|V_l|=r$ and $0\leqslant\Delta_a\leqslant 1$. First, we assume that $\Delta_a=0$ and thus, according to (1), $|V_j|=r+2$. If there is a vertex $z\in Z$, then (6) implies that $d^-(z)\geqslant|V_j|+1=r+3$ and $d^+(z)\leqslant|V_i|-1\leqslant r+1$, a contradiction. Consequently, we can consider the case that $Y=V_l$. If $|V_i|=r$, then we arrive at the contradiction $r+1=|Y|+1\leqslant d^-(b)\leqslant d^+(b)+1\leqslant|V_i|=r$. Since the partition-sequence $r,r+1,r+2$ is impossible, it remains to treat the case that $|V_i|=r+2$. To get no contradiction to $i_g(D)\leqslant 1$, it follows that $A_2=V_i-\{a\}$. If there are vertices $a_2\in A_2$ and $y\in Y$ such that $a_2\to y$, then we conclude that $B_2\to y$, since otherwise, if there is a vertex $b_2\in B_2$ such that $y\to b_2$, then aba_2yb_2a is a cycle through all 3 partite sets. But now we arrive at the contradiction $d^-(y)\geqslant r+3$ and $d^+(y)\leqslant r+1$. Hence, let $Y\to A_2$, which implies that $A_2\to B_2\to Y$. If $a_2,a_2'\in A_2$, $b_2,b_2'\in B_2$ and $y\in Y$, then $aba_2b_2ya_2'b_2'a$ is a cycle through all partite sets, a contradiction. Second, let $\Delta_a=1$. Since $|V_c|=r+2$, Remark 2.5 yields $|V_i|=r+1$, and thus $|V_j|=r+2$, a contradiction to $i_g(D)\leqslant 1$.

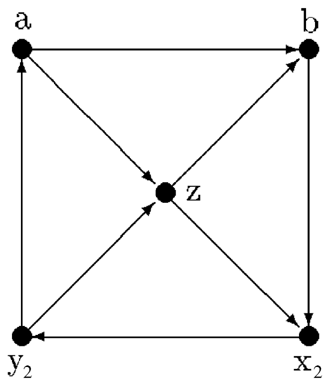

Fig. 2. An almost regular 3-partite tournament with the property that the arc ab is in no cycle through exactly 3 partite sets.

Analogously, we see that the case $Z = \emptyset$ is impossible. Consequently, it remains to consider the case that $X, Z \neq \emptyset$. Now, analogously to Theorem 3.1, we get relationships (5) and the conditions $|V_i| = |V_j| = r = 1$ and $|V_c| = r + 1$, a contradiction. □

Nevertheless Theorem 3.1 cannot be improved in the sense that in all almost regular c-partite tournaments with $c \geqslant 3$, every arc is in a cycle containing vertices from exactly 3 or 4 partite sets. This can be seen in the following simple example, which shows a 3-partite tournament with an arc ab that is not contained in any cycle through all partite sets.

Example 3.4. Let $V_1 = \{a, x_2\}$, $V_2 = \{b, y_2\}$ and $V_3 = \{z\}$ be the three partite sets of the multipartite tournament D such that $a \to b \to x_2 \to y_2 \to z \to x_2$ and $y_2 \to a \to z \to b$ (see Fig. 2). Then the arc ab is not contained in any cycle with vertices of exactly 3 (and clearly also not four) partite sets.

In the last example, there is one partite set containing only one vertex. If we add the condition that there are at least two vertices in every partite set, then we can improve Theorem 3.1.

Theorem 3.5. *Let D be an almost regular multipartite tournament with the partite sets $V_1, V_2, \ldots, V_c$. If $c \geqslant 3$ and there are at least two vertices in each partite set, then every arc of D is in a cycle containing vertices from exactly 3 or exactly 4 partite sets.*

Proof. If $c \geqslant 4$ or $|V_1| = |V_2| = |V_3|$, then the assertion holds with Theorem 3.1. If $r = |V_1| \leqslant |V_2| \leqslant |V_3| = r + 2$, then the assertion follows from Theorem 3.3. Therefore, it remains to consider the case that $c = 3$ and $2 \leqslant r = |V_1| \leqslant |V_2| \leqslant |V_3| = r + 1$.

Let ab be an arbitrary arc of D. Suppose that ab is not in any cycle, containing vertices of all partite sets. Obviously, we have $|V(D)| = 3r + k$ with $1 \leqslant k \leqslant 2$. Let $a \in V_i$ and $b \in V_j$ with $1 \leqslant i, j \leqslant 3$. If we define $A_1, A_2, B_1, B_2, X, Y, Z, h$ and Δ_a as in the beginning of this section, then, following the same lines as in Theorem 3.1, we observe that

$$X \to A_2 \cup B_2 \cup \{a, b\} \to Z. \tag{7}$$

Suppose that $X = \emptyset$. Let $V_l = V(D) - (V_i \cup V_j)$. With $c = 3$ and the fact that $|V_j| \leqslant r + 1$, (1) implies $B_1 = \{b\}$, $h = 0$, $\Delta_a = 1$, $|V_l| = r$, $|V_j| = r + 1$ and $|B_2| = r$. If there is a vertex $z \in Z$, then (7) yields that $d^-(z) \geqslant |V_j| + 1 = r + 2$ and $d^+(z) \leqslant |V_i| - 1 \leqslant r$, a contradiction. Hence, let $Y = V_l$. If $|V_i| = r$, then we arrive at the contradiction $r + 1 = |V_l| + 1 \leqslant d^-(b) \leqslant d^+(b) + 1 \leqslant |A_2| + 1 \leqslant r$. Hence, let us suppose that $|V_i| = r + 1$. To get no contradiction to $i_g(D) \leqslant 1$, it follows that $|A_2| = r$. If there are vertices $a_2 \in A_2$ and $y \in Y$ such that $a_2 \to y$, then we deduce that $B_2 \to y$, since otherwise, if there is a vertex $b_2 \in B_2$ such that $y \to b_2$, then aba_2yb_2a is a cycle with vertices from all partite sets, a contradiction. But this yields the contradiction $d^-(y) \geqslant r + 2$ and $d^+(y) \leqslant r$. Consequently, it follows that $Y \to A_2$, and thus $A_2 \to B_2 \to Y$. If $a_2, a_2' \in A_2$, $b_2, b_2' \in B_2$ and $y \in Y$, then $aba_2b_2ya_2'b_2'a$ is a cycle through all 3 partite sets, a contradiction.

Analogously, we observe that the case $Z = \emptyset$ is impossible. Consequently, it remains to treat the case that $X, Z \neq \emptyset$. Now, analogously to Theorem 3.1, we get relationships (5) and the condition $|V_i| = |V_j| = r = 1$, a contradiction to $r \geqslant 2$. This completes the proof of the theorem. □

We take Theorem 3.5 as basis of induction to show Theorem 1.7. Next, we will present the induction step.

Theorem 3.6. *Let D be an almost regular c-partite tournament with $c \geqslant 4$ and at least two vertices in each partite set. If an arc of D is in a cycle that contains vertices from exactly m partite sets for some m with $3 \leqslant m < c$, then it is also in a cycle that contains vertices from exactly $m+1$ partite sets.*

Proof. Let $V_1, V_2, \ldots, V_c$ be the partite sets of D such that $2 \leqslant r = |V_1| \leqslant |V_2| \leqslant \cdots \leqslant |V_c| = r + o$ with $o = 0$, $o = 1$ or $o = 2$. Obviously, we have $|V(D)| = cr + k$ with $k = 0$, if $o = 0$, $1 \leqslant k \leqslant c-1$, if $o = 1$, and $2 \leqslant k \leqslant 2c-2$, if $o = 2$. Let v_1v_2 be an arc that is in a cycle, say $C = v_1v_2 \ldots v_tv_1$, which contains vertices from exactly m partite sets for some $3 \leqslant m < c$. Suppose that v_1v_2 is not part of a cycle containing vertices from exactly $m+1$ partite sets. Assume without loss of generality that $v_1 \in V_i$ and $v_2 \in V_j$ for some $1 \leqslant i, j \leqslant c$. If $I = \{i_{m+1}, \ldots, i_c\}$ is the maximal set of indices such that $V(C) \cap V_l = \emptyset$ for all $l \in I$, then we define the sets X and Y by

$$X = N^-(v_1) \cap \left(\bigcup_{l \in I} V_l \right), \qquad Y = N^+(v_1) \cap \left(\bigcup_{l \in I} V_l \right).$$

It is clear that $X \cup Y = \bigcup_{l \in I} V_l$ and every vertex of $X \cup Y$ is adjacent with all vertices in C.

Firstly, let us suppose that $X \neq \emptyset$. If there is a vertex $x \in X$ such that $v_t \to x$, then $v_1v_2 \ldots v_txv_1$ is a cycle through exactly $m+1$ partite sets, a contradiction. If such a vertex does not exist, then $X \to v_t$. Since $X \to \{v_1, v_t\}$, we observe that, if some $v_i \in V(C)$ dominates a vertex $x \in X$, then let $n = \max\{l \mid v_l \to x\}$ and $v_1v_2 \ldots v_nxv_{n+1} \ldots v_tv_1$ is a cycle through exactly $m+1$ partite sets. Now, we assume that $X \to V(C)$.

Now, let $H = N^+(v_2) - V(C)$. If there is an arc $h \to x$ with $h \in H$ and $x \in X$, then let firstly be $h \in V_l$ with $l \notin I$. In this case $v_1v_2hxv_3 \ldots v_tv_1$ is a cycle through exactly $m+1$ partite sets, a contradiction. Consequently, let $h \in V_l$ with $l \in I$. If $m = 3$, then $v_1v_2hxv_1$ is a cycle through exactly 4 partite sets, a contradiction. Otherwise, if $m \geqslant 4$, then let p be the index such that $\{v_p, v_{p+1}, \ldots, v_t, v_1\} - V(v_2)$ consists of vertices from exactly $m-2$ partite sets. In this case, $v_1v_2hxv_p \ldots v_tv_1$ is a cycle containing vertices of exactly $m+1$ partite sets, a contradiction. For all $x \in X$, this leads to

$$d^+(x) \geqslant |H - (V(x) - \{x\})| + |V(C)|,$$

whereas

$$d^+(v_2) \leqslant |H| + |V(C)| - 2.$$

If $H \cap V(x) = \emptyset$, then we arrive at a contradiction to $i_g(D) \leqslant 1$. Hence, let $y \in H \cap V(x)$. Since $H \cap X = \emptyset$, we conclude that $y \in Y$. Now let $z \in N^-(x)$ and assume that $y \to z$. If $z \in V_l$ with $l \notin I$, then $v_1v_2yzxv_3 \ldots v_tv_1$ is a cycle through exactly $m+1$ partite sets, a contradiction. Thus, let $z \in V_l$ with $l \in I$. If $m = 3$, then $v_1v_2yzxv_1$ is a cycle through exactly 4 partite sets, and if $m \geqslant 4$, then we choose the index p as above and $v_1v_2yzxv_p \ldots v_tv_1$ is a cycle through exactly $m+1$ partite sets, in both cases a contradiction. Hence, let $N^-(x) \to y$. If $y \to v_i$ for some $3 \leqslant i \leqslant t$, then let $n = \min\{q \mid 2 \leqslant q \leqslant i-1, v_q \to y\}$. Now, $v_1v_2 \ldots v_nyv_{n+1} \ldots v_tv_1$ is a cycle through exactly $m+1$ partite sets, a contradiction. Altogether, we see that $\{v_1, v_2, \ldots, v_t\} \cup N^-(x) \to y$, and thus it follows that

$$d^-(y) \geqslant d^-(x) + t \geqslant d^-(x) + 3,$$

a contradiction to $i_g(D) \leqslant 1$.

Consequently, there remains to consider the case that $X = \emptyset$. This implies that $v_1 \to Y$ and $Y = \bigcup_{l \in I} V_l$. Now, we distinguish different cases.

Case 1: Let there be a vertex $y \in Y$ such that $v_2 \to y$. Then we have $V(C) \to y$, since otherwise let $n = \min\{z \mid y \to v_z\}$. Then $v_1v_2 \ldots v_{n-1}yv_n \ldots v_tv_1$ is a cycle through exactly $m+1$ partite sets, a contradiction. If $v_1 \rightsquigarrow N^+(y)$, then it follows that $d^-(y) = |V(C)| + |N^-(y) - V(C)|$ and $d^-(v_1) \leqslant |V(C)| - 2 + |N^-(y) - V(C)|$, a contradiction to $i_g(D) \leqslant 1$. Therefore, there is a 3-cycle v_1yzv_1. Obviously, the case $z \in Y \cup V(C)$ is impossible, and thus $v_1v_2 \ldots v_tyzv_1$ is a cycle through exactly $m+1$ partite sets, a contradiction.

Altogether we see that there remains the case $Y \to v_2$.

Case 2: Suppose that there exists a vertex $y \in Y$ such that $v_3 \to y$. As in Case 1 we observe that in this case $V(C) - \{v_2\} \to y$. In the following, we will denote the sets F and H by $F = N^-(y) - V(C)$ and $H = N^+(y) - V(C)$, respectively. If there is a 3-cycle v_1yzv_1, then, analogously as in Case 1, we arrive at a contradiction. Hence, let $v_1 \rightsquigarrow N^+(y)$. It follows that $d^-(y) = |V(C)| - 1 + |F|$ and $d^-(v_1) \leqslant |V(C)| - 2 + |F|$. Because of $i_g(D) \leqslant 1$, this leads to $N^-(v_1) = (V(C) - \{v_1, v_2\}) \cup F$, $d^-(y) = d^-(v_1) + 1$, $V(v_1) - \{v_1\} \subseteq N^+(y)$ and $Y - V(y) \subseteq N^+(y)$. Since $r \geqslant 2$, we conclude that $V(v_1) - \{v_1\} \neq \emptyset$. Let $H' = H - Y$. Then we have $\{v_4, v_5, \ldots, v_t\} \rightsquigarrow H'$, because otherwise, if there are vertices $h' \in H'$ and v_l such that $h' \to v_l$ for some $4 \leqslant l \leqslant t$, then $v_1v_2 \ldots v_{l-1}yh'v_l \ldots v_tv_1$ is a cycle containing vertices from exactly $m+1$ partite sets, a contradiction. Furthermore, if there are vertices $f \in F$ and $h' \in H'$ such that $h' \to f$, then $v_1v_2\ldots v_tyh'fv_1$ is a cycle through exactly $m+1$ partite sets, a contradiction. Summarizing our results, we see that $(F\cup\{y,v_1,v_4,v_5,\ldots,v_t\})\rightsquigarrow H'$.

Subcase 2.1: Assume that there are vertices $h'\in H'$ and $y'\in V(y)-\{y\}$ such that $h'\to y'$. It follows that $F\to y'$, since otherwise, if there is a vertex $f\in F$ such that $y'\to f$, then $v_1v_2\ldots v_tyh'y'fv_1$ is a cycle through exactly $m+1$ partite sets, a contradiction. If there exists a vertex $v_l\in V(C)$ with $4\leqslant l\leqslant t$ such that $y'\to v_l$, then $v_1v_2\ldots v_{l-1}yh'y'v_l\ldots v_1$ is a cycle containing vertices from exactly $m+1$ partite sets, a contradiction. Hence, let $(\{v_1,v_4,\ldots,v_t,h'\}\cup F)\to y'$. We arrive at

$$d^-(y')\geqslant|F|+|V(C)|-1=d^-(y)=d^-(v_1)+1.$$

To get no contradiction to $i_g(D)\leqslant 1$, it follows that $y'\to(H-\{h'\})\cup\{v_3\}$. If there is a vertex v_l $(4\leqslant l\leqslant t)$ such that $v_2\to v_l$, then $v_1v_2v_l\ldots v_tyh'y'v_3\ldots v_{l-1}v_1$ is a cycle through exactly $m+1$ partite sets, a contradiction. If there is a vertex $f\in F$ such that $v_2\to f$, then $v_1v_2fyh'y'v_3\ldots v_tv_1$ is a cycle containing vertices from exactly $m+1$ partite sets, a contradiction. If $v_2\to h'$, then $v_1v_2h'y'v_3\ldots v_tv_1$ is a cycle through exactly $m+1$ sets, also a contradiction. Hence, we have $(F\cup\{h',v_1,v_4,\ldots,v_t\}\cup Y)\rightsquigarrow v_2$, and thus

$$d^+(v_2)\leqslant|H|-1-|Y-V(y)|-|V(v_2)\cap H|+|\{v_3\}|\leqslant|H|,$$

whereas $d^+(y)=|H|+1$. This implies that $v_2\to H-\{h'\}$ and $H'':=H'-\{h'\}=H-\{h'\}$. If there exist vertices $h''\in H''$ and $y''\in Y-\{y\}$ such that $h''\to y''$, then analogously as above, we observe that $h''\to v_2$, a contradiction. Hence, let $Y=V(y)\to H''$. According to Corollary 2.6, we have $d^+(y)\geqslant 3$, and thus $|H|\geqslant 2$, which means that $H''\neq\emptyset$. Consequently, there is a vertex $h''\in H''$ such that $d^+_{D[H'']}(h'')\leqslant(|H|-2)/2$. Summarizing our results, we arrive at

$$|H|\leqslant d^+(h'')\leqslant\frac{|H|-2}{2}+2.$$

Since $|H|\geqslant 2$, this yields $|H|=2$ and $h''\to h'$. Now, $v_1v_2h''h'y'v_3\ldots v_tv_1$ is a cycle through all $m+1$ partite sets, a contradiction.

Subcase 2.2: Suppose that $V(y)\to H'$. Since $V(v_1)-\{v_1\}\subseteq H'$, the observations above yield that $(\{v_4,v_5,\ldots,v_t\}\cup F)\to(V(v_1)-\{v_1\})(\subseteq H')$. This implies that

$$\begin{aligned}d^-(v_1')&\geqslant|F|+|V(C)|-3+|V(y)|\geqslant|F|+|V(C)|-1\\&=d^-(v_1)+1\end{aligned}$$

for all vertices $v_1'\in V(v_1)-\{v_1\}$. To get no contradiction to $i_g(D)\leqslant 1$, it follows that $|V(y)|=2$ and $(V(v_1)-\{v_1\})\to\{v_2,v_3\}$. Analogously as in Subcase 2.1, replacing the path $yh'y'v_3$ by $yv_1'v_3$, we see that $(F\cup\{v_4,v_5,\ldots,v_t\})\rightsquigarrow v_2$. Hence, we arrive at

$$d^+(v_2)\leqslant|H|-|Y-V(y)|-|V(v_2)\cap H|-|V(v_1)\cap H|+1\leqslant|H|-r+2\leqslant|H|,$$

whereas $d^+(y)=|H|+1$. This implies that $v_2\to H-V(v_1)=:H''$, $|H\cap V(v_1)|=1$ and $Y-V(y)=\emptyset$, which means $H'=H$. Following the same lines as in Subcase 2.1, replacing there h' by v_1', we arrive at a contradiction.

Summarizing the investigations of Case 2, we see that $Y\to v_3$. Observing the converse D^{-1} of D, we conclude that $v_t\to Y$ and therefore $t\geqslant 4$.

Case 3: Finally, let $\{v_t,v_1\}\to Y\to\{v_2,v_3\}$. Let us define the sets U and W by $W=N^+(v_2)-V(C)$ and $U=N^-(v_1)-V(C)$, respectively. It is not difficult to show that, if there is an arc leading from W to Y (respectively, from Y to U), or if $Y\to W$ (respectively, $U\to Y$) and there is an arc from W to v_1 (respectively, from v_2 to U), then the multipartite tournament contains a cycle through v_1v_2 and exactly $m+1$ partite sets, a contradiction. Hence, we may assume that $Y\cup\{v_1,v_2\}\rightsquigarrow W$ and $U\rightsquigarrow Y\cup\{v_1,v_2\}$ and $U\cap W=\emptyset$.

If there exists a vertex $v_l\in V(C)$ such that $v_2\to v_l$ and $v_{l-1}\to v_1$, then obviously $l\geqslant 4$ and $v_1v_2v_l\ldots v_tyv_3\ldots v_{l-1}v_1$ is a cycle through exactly $m+1$ partite sets for some $y\in Y$, a contradiction. Therefore, from now on, we investigate the case that $v_1\to v_{l-1}$ or $V(v_1)=V(v_{l-1})$, if $v_2\to v_l$.

If there are vertices $u\in U$ and $v_l\in V(C)$ with $l\geqslant 4$ such that $v_2\to v_l$ and $v_{l-1}\to u$, then $v_1v_2v_l\ldots v_tyv_3\ldots v_{l-1}uv_1$ is a cycle through exactly $m+1$ partite sets, a contradiction. Hence, we may assume that $u\to v_{l-1}$ or $V(u)=V(v_{l-1})$, if $v_2\to v_l$. Analogously, we see that $v_{l+1}\to w$ or $V(w)=V(v_{l+1})$, if $w\in W$ and $v_l\to v_1$ with $l<t$.

If there is an arc $w\to u$ from W to U, then $v_1v_2wuyv_3\ldots v_tv_1$ is a cycle containing vertices from exactly $m+1$ partite sets, a contradiction. Therefore, we have $U\rightsquigarrow W$.

If $y \in Y$ is an arbitrary vertex, then these results yield the following three lower bounds:

$$\begin{aligned} |N^+(v_1)| &\geqslant |Y| + |W| + |N^+(v_2) \cap V(C)| - |V(v_1) - \{v_1\}| \\ &\geqslant |V(y)| + |N^+(v_2)| - |V(v_1) - \{v_1\}| \\ &\geqslant \begin{cases} |N^+(v_2)| & \text{if } |V(v_1)| \leqslant r+1, \\ |N^+(v_2)| - 1 & \text{if } |V(v_1)| = r+2, \end{cases} \end{aligned} \tag{8}$$

$$\begin{aligned} |N^+(u)| &\geqslant |Y| + |W| + |N^+(v_2) \cap V(C)| - 1 + |\{v_1, v_2\}| - |V(u) - \{u\}| \\ &\geqslant |V(y)| + |N^+(v_2)| + 1 - |V(u) - \{u\}| \\ &\geqslant \begin{cases} |N^+(v_2)| + 1 & \text{if } |V(u)| \leqslant r+1, \\ |N^+(v_2)| & \text{if } |V(u)| = r+2, \end{cases} \end{aligned} \tag{9}$$

for every $u \in U$ and

$$\begin{aligned} |N^-(w)| &\geqslant |Y| + |U| + |N^-(v_1) \cap V(C)| - 1 + |\{v_1, v_2\}| - |V(w) - \{w\}| \\ &\geqslant |V(y)| + |N^-(v_1)| + 1 - |V(w) - \{w\}| \\ &\geqslant \begin{cases} |N^-(v_1)| + 1 & \text{if } |V(w)| \leqslant r+1, \\ |N^-(v_1)| & \text{if } |V(w)| = r+2, \end{cases} \end{aligned} \tag{10}$$

for every $w \in W$. If the right-hand side of (8) increases by at least two or the right-hand side of (9) or (10) increases by at least one, then we arrive at a contradiction either to $i_g(D) \leqslant 1$ or to Remark 2.5. This leads to $|V(u)|,\ |V(w)| \geqslant r+1$ for $u \in U$ and $w \in W$. Another consequence is that $|Y| = r$, if $U \cup W \neq \emptyset$, and $|Y| \leqslant r+1$, if $U \cup W = \emptyset$. Anyway, Y consists of exactly one partite set. Furthermore, bounds (8)–(10) yield $|U|, |W| \leqslant 1$, since otherwise, the right-hand side of (9) or (10) increases by one, a contradiction. Let $U \neq \emptyset$ and $u \in U$. Because of $v_1 \to v_2$, we conclude that $v_t \to u$, since otherwise the right-hand side of (9) increases by one, a contradiction. If we observe the cycle $C' = b_1 b_2 \ldots b_{t+1} b_1 := v_1 v_2 \ldots v_t u v_1$ such that $b_1 = v_1$, then we see that C' fulfills $\{b_{t+1}, b_1\} \to Y \to \{b_2, b_3\}$. Hence, we can replace C by C', which means that, without of generality, we may suppose that $U = \emptyset$. Analogously, it remains to treat the case that $W = \emptyset$.

Let $y \in Y$. If we define $U' = N^-(y) - V(C)$ and $W' = N^+(y) - V(C)$, then we conclude that $V(D) = V(y) \cup V(C) \cup U' \cup W'$. Let $w' \in W'$. If $w' \to v_1$, then it follows that $w' \in U$, and thus we have $w' \in N^-(y) - V(C)$, a contradiction to the definition of W'. Since $W = \emptyset$, this yields $v_1 \rightsquigarrow w' \rightsquigarrow v_2$ and the right-hand side of (8) increases by one. Analogously, we observe that $v_1 \rightsquigarrow u' \rightsquigarrow v_2$ for each $u' \in U'$. To get no contradiction in (8), it has to be $|U' \cup W'| \leqslant 1$.

Subcase 3.1: Suppose that $m = 3$, and thus $c = 4$. Let $V_b = V(D) - (Y \cup V(v_1) \cup V(v_2))$. We observe that $N^-(v_1) \cap V_b \neq \emptyset$, since otherwise, we arrive at

$$\frac{3r+k-2}{2} \leqslant d^-(v_1) \leqslant |V(v_2) - \{v_2\}| \leqslant r \quad \text{if } |V(v_2)| \leqslant r+1,$$

$$\frac{3r+k-2}{2} \leqslant d^-(v_1) \leqslant |V(v_2) - \{v_2\}| = r+1 \quad \text{if } |V(v_2)| = r+2,\ |V(v_1)| \geqslant r+1 \text{ and thus } k \geqslant 3$$

and

$$\frac{3r+k}{2} = d^-(v_1) \leqslant |V(v_2) - \{v_2\}| = r+1 \quad \text{if } |V(v_2)| = r+2 \text{ and } |V(v_1)| = r,$$

in all cases a contradiction. If $N^+(v_2) \cap (V(C) - \{v_3\}) = \emptyset$, then Corollary 2.6 yields $(3r+k-2)/2 \leqslant d^+(v_2) \leqslant 2$, a contradiction.

Suppose that there exists an index $q \geqslant 4$ as small as possible such that $v_2 \to v_q$ and that there is an index $l < q$ with $v_l \to v_1$. This index l let be chosen as large as possible. Now, let us observe the cycle $C' = v_1 v_2 v_q \ldots v_t y v_3 \ldots v_l v_1$. If C' does not contain vertices from all the 4 partite sets, then we conclude that $V_b \subseteq V(D) - V(C') \subseteq [\{v_{l+1}, \ldots, v_{q-1}\} \cup U' \cup W']$. Since $v_1 \rightsquigarrow U' \cup W' \cup \{v_{l+1}, \ldots, v_{q-1}\}$, we arrive at $N^-(v_1) \cap V_b = \emptyset$, a contradiction.

Altogether, we see that an index q chosen as above does not exist. Let y_1 be the largest index such that $v_2 \to v_{y_1}$. This implies that $v_1 \rightsquigarrow \{v_2, v_3, \ldots, v_{y_1-1}\}$. If $v_{y_1} \to v_1$, then we have the 3-cycle $v_1 v_2 v_{y_1} v_1$, a contradiction to $t \geqslant 4$. Hence, we deduce that $N^-(v_1) \subseteq \{v_{y_1+1}, v_{y_1+2}, \ldots, v_t\}$. If there is no arc leading from v_3 to $\{v_{y_1+1}, v_{y_1+2}, \ldots, v_t\}$, then we arrive at

$$d^-(v_3) \geqslant d^-(v_1) + |Y| + |\{v_1, v_2\}| - |V(v_3) - \{v_3\}|$$
$$\geqslant \begin{cases} d^+(v_1) + 2 & \text{if } |V(v_3)| \leqslant r+1, \\ d^+(v_1) + 1 & \text{if } |V(v_3)| = r+2, \end{cases}$$

in both cases a contradiction. Therefore, let $y_2 > y_1$ be the largest index such that $v_3 \to v_{y_2}$. Firstly, let $v_l \to y$ for some $y \in Y$ and $4 \leqslant l \leqslant y_2 - 1$ (notice that, because of $y_1 \geqslant 4$, it has to be $y_2 \geqslant 5$). This yields $v_a \to y$ for all $l \leqslant a \leqslant t$, since otherwise, we can find a cycle through all 4 partite sets, a contradiction. Let x_1 be the smallest index in $\{4, 5, \ldots, y_1\}$ such that $v_2 \to v_{x_1}$. Now, let us observe the cycle $C' := v_1 v_2 v_{x_1} \ldots v_{y_2-1} y v_3 v_{y_2} \ldots v_t v_1$. If C' does not contain vertices from all 4 partite sets, then we conclude that $V_b \subseteq \{v_4, v_5, \ldots, v_{x_1-1}\} \cup U' \cup W'$, and thus $N^-(v_1) \cap V_b = \emptyset$, a contradiction. Hence, we arrive at $Y \to \{v_2, v_3, v_4, v_5, \ldots, v_{y_2-1}\}$, and thus $d^+(y) \geqslant d^+(v_2) + 1$ for all $y \in Y$, $y_2 = y_1 + 1$ and $v_2 \to \{v_3, \ldots, v_{y_1}\}$, which means $\{v_3, \ldots, v_{y_1}\} \cap V(v_2) = \emptyset$. Let x_2 be the first index such that $v_{x_2} \to v_1$ ($x_2 \geqslant y_2$). If $\{v_{x_2+1}, \ldots, v_t\} \rightsquigarrow v_4$, then we conclude that

$$d^-(v_4) \geqslant d^-(v_1) - 1 + |Y| + |\{v_1, v_2, v_3\}| - |V(v_4) - \{v_4\}|$$
$$\geqslant \begin{cases} d^-(v_1) + 2 & \text{if } |V(v_4)| \leqslant r+1, \\ d^-(v_1) + 1 & \text{if } |V(v_4)| = r+2, \end{cases}$$

in both cases a contradiction. Therefore, let $v_4 \to v_{y_3}$ with $y_3 > y_2$. If we notice that either $v_3 \in V_b$ or $v_4 \in V_b$, then we observe that $v_1 v_2 v_3 v_{y_2} y v_4 v_{y_3} \ldots v_t v_1$ is a cycle through all 4 partite sets, a contradiction.

Subcase 3.2: Let $m \geqslant 4$ and thus $c \geqslant 5$. Using Corollary 2.6, we arrive at $d^+(v_2) \geqslant ((c-1)r + k - 2)/2 \geqslant \frac{7}{2}$, which means $d^+(v_2) \geqslant 4$ and v_2 has at least four outer neighbors in $V(C)$.

Suppose that there is an index $q \geqslant 4$ as small as possible such that there is an index $l < q$ with $v_l \to v_1$. This index l let be chosen as large as possible. If the cycle $C' = v_1 v_2 v_q \ldots v_t y v_3 \ldots v_l v_1$ does not contain vertices from all $m+1$ partite sets, then the remaining partite sets have to be in $\{v_{l+1}, \ldots, v_{q-1}\} \cup U' \cup W'$. Furthermore, the choice of the indices l and q implies $v_1 \rightsquigarrow \{v_{l+1}, \ldots, v_{q-1}\} \rightsquigarrow v_2$. If the partite sets, which do not appear in C' are only part of $\{v_{l+1}, \ldots, v_{q-1}\}$, then there are at least two vertices v_{x_1} and v_{x_2} such that $v_1 \rightsquigarrow \{v_{x_1}, v_{x_2}\}$ and $\{v_{x_1+1}, v_{x_2+1}\} \rightsquigarrow v_2$ which leads to a contradiction to (8). Let $w' \in W'$ be part of a partite set that does not appear in C'. Hence, we have $U' = \emptyset$, $l + 1 = q - 1$ and $v_{l+1} \in V(w')$, since otherwise, the right-hand side of (8) increases by at least two, a contradiction. Therefore, there are vertices from exactly m partite sets in C'. Now, we see that $r = 2$ and $|V(w')| = r = 2$. This and the fact that $v_1 \to v_2$ yield $q \geqslant 5$. If $w' \to v_3$, then $v_1 v_2 v_q \ldots v_t y w' v_3 \ldots v_l v_1$ is a cycle with vertices from exactly $m+1$ partite sets, a contradiction. If $q \geqslant 6$ and $w' \to v_b$ with $4 \leqslant b \leqslant l$, then we observe inductively that $v_1 v_2 v_q \ldots v_t y v_3 \ldots v_{b-1} w' v_b \ldots v_l v_1$ is a cycle with vertices from m+1 partite sets, a contradiction. Hence, let $\{v_3, \ldots, v_l\} \to w'$. If there is a vertex $y' \in V(y) - \{y\}$ such that $w' \to y'$, then $v_1 v_2 v_q \ldots v_t y w' y' v_3 \ldots v_l v_1$ is a cycle with vertices from exactly $m+1$ partite sets, a contradiction. If there is a vertex v_b in $V(C)$ with $4 \leqslant b \leqslant t$ such that $v_b \to y$ and $w' \to v_{b+1}$ ($t + 1 \equiv 1$), then $v_1 v_2 \ldots v_b y w' v_{b+1} \ldots v_1$ is a cycle containing vertices from exactly $m+1$ partite sets, a contradiction.

Firstly, let $v_l \to y$. This implies $\{v_l, v_{l+1}, \ldots, v_t, v_1\} \to y$, and thus $N^+(y) \subseteq \{w', v_2, \ldots, v_{l-1}\}$, which means $d^+(y) \leqslant l - 1$. Because of Corollary 2.6, on the other hand, we have $d^+(y) \geqslant ((c-1)r + k - 1)/2 \geqslant \frac{7}{2}$, which implies $l \geqslant 5$. Altogether, it follows that

$$d^-(w') \geqslant d^-(y) - 2 + |Y| + l - 2 \geqslant d^-(y) + 3,$$

a contradiction to $i_g(D) \leqslant 1$. Otherwise, if $y \to v_l$, then, it follows that

$$d^-(w') \geqslant d^-(y) - 1 + |Y| + 1 \geqslant d^-(y) + 2,$$

again a contradiction to $i_g(D) \leqslant 1$.

Altogether, we see that an index q chosen as above does not exist. Let z' be the largest index such that $v_2 \to v_{z'}$ (notice that $z' \geqslant 6$). This implies that $v_1 \rightsquigarrow \{v_2, v_3, \ldots, v_{z'-1}\}$, and thus $N^-(v_1) \subseteq \{v_{z'}, v_{z'+1}, \ldots, v_t\}$. If there is a vertex $y \in Y$ such that $v_{z'-1} \to y$, then it follows that $\{v_{z'-1}, \ldots, v_t, v_1\} \to y$, and thus, we have $d^-(y) \geqslant d^-(v_1) + 2$, a contradiction to $i_g(D) \leqslant 1$. Therefore, we may assume that $Y \to \{v_2, v_3, \ldots, v_{z'-1}\}$. Let z'' be the smallest index such that $v_{z''} \to v_1$.

Firstly, let $v_2 \rightsquigarrow v_{z'-2}$. Then there exists an arc from $v_{z'-2}$ to $\{v_{z''+1}, \ldots, v_t\}$, since otherwise, we observe that

$$d^-(v_{z'-2}) \geqslant d^-(v_1) - 1 + |\{v_{z'-3}, v_1, v_2\}| + |Y| - |V(v_{z'-2}) - \{v_{z'-2}\}|$$
$$\geqslant \begin{cases} d^-(v_1) + 2 & \text{if } |V(v_{z'-2})| \leqslant r+1, \\ d^-(v_1) + 1 & \text{if } |V(v_{z'-2})| = r+2. \end{cases}$$

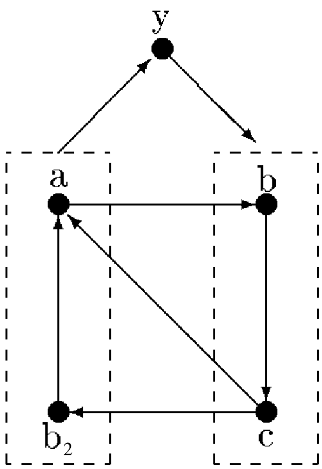

Fig. 3. An almost regular 4-partite tournament with the property that the arc ab is in no cycle through exactly 4 partite sets.

Both cases yield a contradiction, either to $i_g(D) \leqslant 1$ or to Remark 2.5. Consequently, let $v_{z'-2} \to v_{y_1}$ with $y_1 \in \{z''+1,\ldots,t\}$. Let $y \in Y$ and let $y_2 < y_1$ be the largest index such that $v_{y_2} \to v_1$. If $C' := v_1v_2\ldots v_{z'-2}v_{y_1}\ldots v_t y v_{z'-1}\ldots v_{y_2}v_1$ does not contain vertices of exactly $m+1$ partite sets, then there is a partite set V_b such that $V_b \subseteq \{v_{y_2+1}, v_{y_2+2},\ldots,v_{y_1-1}\} \cup U' \cup W'$. Since $v_1 \rightsquigarrow \{v_{y_2+1}, v_{y_2+2},\ldots,v_{y_1-1}\} \cup U' \cup W'$ and $\{v_{y_2+2}, v_{y_2+3},\ldots,v_{y_1}\} \cup U' \cup W' \rightsquigarrow v_2$, (8) implies that $|V_b| \leqslant 1$, a contradiction to $r \geqslant 2$.

Secondly, let $v_{z'-2} \to v_2$. Since $v_1 \rightsquigarrow v_{z'-3}$, this yields that the right-hand side of (8) increases by 1. To get no contradiction, it follows that $v_2 \rightsquigarrow v_{z'-1}$ and $\{v_{z'}, v_{z'+1},\ldots,v_t\} \to v_1$, which means that $z' = z''$. This implies that there is an arc from $v_{z'-1}$ to $\{v_{z'+1}, v_{z'+2},\ldots,v_t\}$, since otherwise, we observe that

$$d^-(v_{z'-1}) \geqslant d^-(v_1) - 1 + |\{v_{z'-2}, v_1, v_2\}| + |Y| - |V(v_{z'-1}) - \{v_{z'-1}\}|$$
$$\geqslant \begin{cases} d^-(v_1)+2 & \text{if } |V(v_{z'-1})| \leqslant r+1, \\ d^-(v_1)+1 & \text{if } |V(v_{z'-1})| = r+2. \end{cases}$$

Both cases yield a contradiction, either to $i_g(D) \leqslant 1$ or to Remark 2.5. Consequently, let $v_{z'-1} \to v_{z_1}$ with $z_1 \in \{z'+1, z'+2,\ldots,t\}$. If there is a vertex $y \in Y$ such that $v_{z'} \to y$, then we conclude that $\{v_{z'}, v_{z'+1},\ldots,v_t,v_1\} \to y$ and $v_1v_2v_{z'}\ldots v_{z_1-1}yv_3\ldots v_{z'-1}v_{z_1}\ldots v_tv_1$ is a cycle with vertices from exactly $m+1$ partite sets, a contradiction. Hence, let $Y \to v_{z'}$. For an arbitrary vertex $y \in Y$, it follows that $v_1v_2\ldots v_{z'-1}v_{z_1}\ldots v_t y v_{z'}\ldots v_{z_1-1}v_1$ is a cycle through $m+1$ partite sets, a contradiction. This completes the proof of the theorem. □

Combining the results of Theorems 3.5 and 3.6, we arrive at Theorem 1.7.

The next example shows that the condition that there are at least two vertices in each partite set is necessary, at least for $c = 4$.

Example 3.7. Let $V_1 = \{a\}$, $V_2 = \{b, b_2\}$, $V_3 = \{c\}$, and $V_4 = \{y\}$ be the partite sets of a 4-partite tournament such that $a \to b \to c \to b_2 \to y \to c \to a \to y \to b$ and $b_2 \to a$ (see Fig. 3). The resulting 4-partite tournament is almost regular, however, the arc ab is on a cycle with vertices from exactly 3 partite sets, but not from all 4 partite sets.

4. Open problems

The results in the last section lead us to the following problems:

Problem 4.1. *Let D be a c-partite tournament with $i_g(D) \leqslant i$ and at least r vertices in each partite set. For all i, find the smallest values $g(i)$ and $f(i, g(i))$ with the property that every arc of D is contained in a cycle through m partite sets for all $m \in \{4, 5,\ldots,c\}$, if $r \geqslant g(i)$ and $c \geqslant f(i, g(i))$.*

According to Theorems 1.5 and 1.7, we have $g(0)=1$, $f(0,1)=4$, $g(1)=2$ and $f(1,2)=4$.

Problem 4.2. *Let D be a c-partite tournament with $i_g(D) \leqslant i$ and r vertices in each partite set. For all i, c and r find optimal values $g_1(i,c,r)$ and $g_2(i,c,r)$ such that every arc of D is contained in a cycle through exactly m partite sets for all $g_1(i,c,r) \leqslant m \leqslant g_2(i,c,r)$.*

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