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# Affine semipartial geometries and projections of quadrics 

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#### Abstract

Debroey and Thas introduced semipartial geometries and determined the full embeddings of semipartial geometries in $\mathrm{AG}(n, q)$ for $n=2$ and 3 . For $n>3$ there is no such classification. A model of a semipartial geometry fully embedded in $\mathrm{AG}(4, q), q$ even, due to Hirschfeld and Thas, is the $\operatorname{spg}\left(q-1, q^{2}, 2,2 q(q-1)\right)$ constructed by projecting the quadric $Q^{-}(5, q)$ from a point of $\operatorname{PG}(5, q) \backslash Q^{-}(5, q)$. In this paper this semipartial geometry is characterized amongst the $\operatorname{spg}\left(q-1, q^{2}, 2,2 q(q-1)\right.$ ) (of which there is an infinite family of non-classical examples due to Brown) by its full embedding in $\mathrm{AG}(4, q)$. (C) 2003 Elsevier Science (USA). All rights reserved.


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## 1. Introduction

A semipartial geometry [9] with parameters $s, t, \alpha, \mu$, also denoted by $\operatorname{spg}(s, t, \alpha, \mu)$, is a partial linear space $\mathscr{S}=(\mathscr{P}, \mathscr{B}, \mathrm{I})$ of order $(s, t)$, such that for each anti-flag $(x, L)$, the incidence number $\alpha(x, L)$, being the number of points on $L$ collinear with $x$, equals 0 or a constant $\alpha(\alpha>0)$ and such that for any two points which are not collinear, there are $\mu(\mu>0)$ points collinear with both ( $\mu$-condition).

A semipartial geometry with $\alpha=1$ is called a partial quadrangle. It was introduced by Cameron [5] as a generalization of a generalized quadrangle. Semipartial geometries generalize at the same time the partial quadrangles and the partial

[^0]geometries, which are partial linear spaces of order $(s, t)$, such that for each anti-flag $(x, L)$, the incidence number $\alpha(x, L)=\alpha$ (the $\mu$-condition is automatically satisfied). Partial geometries with $\alpha=1$ are the well-known generalized quadrangles. See for instance [13] for more information on generalized quadrangles and [6,7] for more information on partial and semipartial geometries. A semipartial which is not a partial geometry, nor a partial quadrangle will be called proper.

The point graph $\Gamma$ of a semipartial geometry is strongly regular. For a point $x$ of $\mathscr{S}$ we will denote by $\Gamma(x)$ the set of points of $\mathscr{S}$ different from $x$ and collinear to $x$.

## 2. Semipartial geometries and generalized quadrangles

In [2] Brown gives the following general construction method for $\operatorname{spg}\left(q-1, q^{2}, 2,2 q(q-1)\right)$. Let $\mathscr{S}$ be a generalized quadrangle of order $\left(q, q^{2}\right)$ containing a subquadrangle $\mathscr{S}^{\prime}$ of order $q$. If $x$ is a point of $\mathscr{S} \backslash \mathscr{S}^{\prime}$, then each line of $\mathscr{S}$ incident with $x$ is incident with a unique point of $\mathscr{S}^{\prime}$ and the set $\mathcal{O}_{x}$ of such points is an ovoid of $\mathscr{S}^{\prime}$. (An ovoid of a generalized quadrangle is a set of points such that each line of the generalized quadrangle is incident with a unique point of the set.) The ovoid $\mathcal{O}_{x}$ is said to be subtended by $x$. A rosette of ovoids of $\mathscr{S}^{\prime}$ is a set of $q$ ovoids meeting pairwise in an exactly one fixed point of $\mathscr{S}^{\prime}$. If $L$ is a line of $\mathscr{S} \backslash \mathscr{S}^{\prime}$, then the ovoids of $\mathscr{S}^{\prime}$ subtended by the points of $\mathscr{S} \backslash \mathscr{S}^{\prime}$ incident with $L$ form a rosette of $\mathscr{S}^{\prime}$.

If for a subtended ovoid $\mathcal{O}_{x}$ there is a point $y$ of $\mathscr{S} \backslash \mathscr{S}^{\prime}, y \neq x$, such that $\mathcal{O}_{y}=\mathcal{O}_{x}$, then $\mathcal{O}_{x}$ is said to be doubly subtended. If each ovoid of $\mathscr{S}^{\prime}$ subtended by a point of $\mathscr{S} \backslash \mathscr{S}^{\prime}$ is doubly subtended, then $\mathscr{S}^{\prime}$ is said to be doubly subtended in $\mathscr{S}$. If $\mathscr{S}^{\prime}$ is doubly subtended in $\mathscr{S}$, then the incidence structure with point set the subtended ovoids of $\mathscr{S}^{\prime}$; line set the rosettes of subtended ovoids of $\mathscr{S}^{\prime}$; and incidence containment is an $\operatorname{spg}\left(q-1, q^{2}, 2,2 q(q-1)\right)$.

The generalized quadrangle $Q(4, q)$ is doubly subtended in $Q^{-}(5, q)$ and hence by Brown's construction yields a semipartial geometry which is better known as the Metz model of $\mathrm{TQ}(4, q)$ (we use the notation as introduced in [6]). For $q$ odd and $\sigma \in \operatorname{Aut}(\operatorname{GF}(q))$ the generalized quadrangle $Q(4, q)$ is also doubly subtended in the Kantor translation generalized quadrangle associated with $\sigma$ [12]. Two such generalized quadrangles associated with field automorphisms $\sigma_{1}$ and $\sigma_{2}$, respectively, are isomorphic if and only if $\sigma_{1}=\sigma_{2}$ or $\sigma_{1}=\sigma_{2}^{-1}$, and similarly for the $\operatorname{spg}\left(q-1, q^{2}, 2,2 q(q-1)\right)$. In the case where $\sigma$ is the identity the Kantor construction yields $Q^{-}(5, q)$ and the associated $\operatorname{spg}\left(q-1, q^{2}, 2,2 q(q-1)\right)$ is the Metz model of TQ $(4, q)$.

An embedding of a partial linear space in $\mathrm{AG}(n, q)$ is a representation of the geometry with point set a subset of the point set of $\operatorname{AG}(n, q)$; line set a subset of the line set of $\mathrm{AG}(n, q)$; and incidence inherited from $\mathrm{AG}(n, q)$. The geometry is fully embedded if the embedding has the additional property that for every line $L$ of $\mathrm{AG}(n, q)$ that is also a line of the geometry, each point of $\mathrm{AG}(n, q)$ that is incident with $L$ is a point of the geometry. It is also required that $\mathrm{AG}(n, q)$ is generated by the
point set of the geometry. In the same way one can define a full embedding of a partial linear space in $\operatorname{PG}(n, q)$.

Let $\mathscr{S}$ be a generalized quadrangle fully embedded in a projective space $\operatorname{PG}(n, q)$, hence $\mathscr{S}$ is classical and $n=3,4$ or 5 [4]. Let $p$ be a point of $\operatorname{PG}(n, q)$ and let $\Pi$ be a hyperplane of $\operatorname{PG}(n, q)$ not containing $p$. Let $\mathscr{P}_{1}$ be the projection of the point set of $\mathscr{S}$ from $p$ onto $\Pi$ and let $\mathscr{P}_{2}$ be the set of points of $\Pi$ on a tangent through $p$ at $\mathscr{S}$. Consider the incidence structure $\mathscr{S}_{p}=\left(\mathscr{P}_{p}, \mathscr{L}_{p}, \mathrm{I}_{p}\right)$ with $\mathscr{P}_{p}=\mathscr{P}_{1} \backslash \mathscr{P}_{2}, \mathscr{L}_{p}$ the set of lines of $\Pi$ with $q$ points in $\mathscr{P}_{p}$ and incidence $\mathrm{I}_{p}$ inherited from the projective space. If $\mathscr{S}=Q^{-}(5, q)$ (fully embedded in $\mathrm{PG}(5, q)$ ) or $\mathscr{S}=H\left(4, q^{2}\right)$ (fully embedded in $\left.\mathrm{PG}\left(4, q^{2}\right)\right)$ the incidence structure $\mathscr{S}_{p}$ is a semipartial geometry.

Assume $\mathscr{S}=Q^{-}(5, q)$ is fully embedded in $\operatorname{PG}(5, q)$ and $p$ is not on the quadric $Q^{-}(5, q)$, then Hirschfeld and Thas [11] proved that projection yields an $\operatorname{spg}(q-$ $\left.1, q^{2}, 2,2 q(q-1)\right)$ that is isomorphic to the semipartial geometry $\mathrm{TQ}(4, q)$. For the other examples we refer to [7]. If $q$ is even, the Hirschfeld-Thas model of TQ $(4, q)$ yields a semipartial geometry which is fully embedded in $\operatorname{AG}(4, q)$.

In [8] Debroey and Thas classified the proper semipartial geometries that may be fully embedded in $\operatorname{AG}(n, q)$ for $n=2$ and 3 , as well as the possible models for the embeddings in these cases. For $n>3$ there is no such classification. There are two examples known, one being the Hirschfeld-Thas model of TQ $(4, q), q$ even.

We will prove the following main theorem.
Main Theorem. Let $\mathscr{S}$ be a semipartial geometry $\operatorname{spg}\left(q-1, q^{2}, 2,2 q(q-1)\right)$ fully embedded in $\mathrm{AG}(4, q)$. Then $q=2^{h}$ and $\mathscr{S}$ is the Hirschfeld-Thas model of $\mathrm{TQ}(4, q)$.

## 3. The $\boldsymbol{\operatorname { s p g }}\left(q-1, q^{2}, 2,2 q(q-1)\right)$ embedded in $\mathbf{A G}(4, q)$

In this section, let $\mathscr{S}$ be an $\operatorname{spg}\left(q-1, q^{2}, 2,2 q(q-1)\right)$ fully embedded in $\mathrm{AG}(4, q), q \neq 2$.

Let $\Pi_{\infty}$ denote the hyperplane at infinity of $\operatorname{AG}(4, q)$. The line set of $\mathscr{S}$ is a subset of the line set of $\operatorname{AG}(4, q)$, which in turn is a subset of the line set of $\operatorname{PG}(4, q)$, the projective completion of $\operatorname{AG}(4, q)$. Thus a line of $\mathscr{S}$ will be said to intersect $\Pi_{\infty}$ in the point of $\Pi_{\infty}$ incident with the line in $\operatorname{PG}(4, q)$. The same symbol will be used to refer to such a line in the three different contexts.

For a point $x$ of $\mathscr{S}$, let $\theta_{x}$ denote the set of $q^{2}+1$ points in $\Pi_{\infty}$ determined by the intersection of $\Pi_{\infty}$ with the lines of $\mathscr{S}$ through $x$. Since $\alpha=2$ any line $N$ of $\Pi_{\infty}$ intersects $\theta_{x}$ in at most three points. A line of $\Pi_{\infty}$ intersecting $\theta_{x}$ in $0,1,2$ or 3 points will be referred to as an external line, tangent, secant or 3 -secant, respectively.

Let $(x, L)$ be an antiflag of $\mathscr{S}, M=\langle x, L\rangle \cap \Pi_{\infty}$ and $p=L \cap \Pi_{\infty}$. If $\alpha(x, L)=$ 0 , then $M$ is either a tangent of $\theta_{x}$ at $p$ or an external line of $\theta_{x}$, while for $\alpha(x, L)=2$, we obtain that either $p \notin \theta_{x}$ and $M$ intersects $\theta_{x}$ in two points, or $p \in \theta_{x}$ and $M$ intersects $\theta_{x}$ in three points.

Lemma 1. Let $x$ be a point of the semipartial geometry $\mathscr{S}$ and let $M$ be a projective line of $\Pi_{\infty}$ intersecting $\theta_{x}$ in three points $p_{1}, p_{2}, p_{3}$. Then all of the points of $\langle M, x\rangle \backslash M$ are points of $\mathscr{S}$ and the $3 q$ affine lines in $\langle M, x\rangle$ through $p_{1}, p_{2}$ or $p_{3}$ are exactly the lines of $\mathscr{S}$ contained in the plane $\langle M, x\rangle$. Furthermore, $q$ is a power of 3 .

Proof. Let $y$ be a point of $\left\langle x, p_{1}\right\rangle \backslash\left\{x, p_{1}\right\}$. Since $\alpha\left(y,\left\langle x, p_{2}\right\rangle\right)=2$ we obtain a line $\langle y, z\rangle$ of $\mathscr{S}$ with $z \in\left\langle x, p_{2}\right\rangle \backslash\left\{x, p_{2}\right\}$ which also intersects $\left\langle x, p_{3}\right\rangle$ in the point $u$. If $u \neq p_{3}$, then $\alpha(x,\langle y, z\rangle)>2$, a contradiction, and so $u=p_{3}$. Similarly since $\alpha\left(y,\left\langle x, p_{3}\right\rangle\right)=2$ it follows that the line $\left\langle y, p_{2}\right\rangle$ is a line of $\mathscr{S}$. Since this is true for any $y \in\left\langle x, p_{1}\right\rangle \backslash\left\{x, p_{1}\right\}$ we have that each affine line through $p_{2}$ or $p_{3}$ is a line of $\mathscr{S}$. Clearly by similar arguments we also have that each affine line through $p_{1}$ is a line of $\mathscr{S}$. If $N$ is any line of $\langle M, x\rangle$ not incident with $p_{1}, p_{2}$ or $p_{3}$, then $N$ cannot be a line of $\mathscr{S}$ since for any point $y$ of $\langle M, x\rangle \backslash M$ not on $N$ we would have $\alpha(y, N)>2$.

Now let the affine lines of $\langle M, x\rangle$ through $p_{1}$ be labelled $L_{1}, \ldots, L_{q}$. For any $L_{i}$ there are $q^{3}(q-1) / 2$ antiflags $\left(y, L_{i}\right)$ of $\mathscr{S}$ with incidence number 2 , and hence $q^{2}\left(q^{2}-1\right) / 2-q^{3}(q-1) / 2-q=\left(q^{3}-q^{2}\right) / 2-q$ antiflags $\left(z, L_{i}\right)$ with incidence number 0 . Counting the number of points $z$ of $\mathscr{S}$ such that $z \in\langle M, x\rangle$ or $\alpha\left(z, L_{i}\right)=$ 0 for some $L_{i}$ we have at most $q^{3}(q-1) / 2$ points, fewer than the total number of points of $\mathscr{S}$. Consequently there exists a point $x^{\prime}$ of $\mathscr{S}$ such that $x^{\prime} \notin\langle M, x\rangle$ and $\alpha\left(x^{\prime}, L_{i}\right)=2$ for all $L_{i}$. Hence there are $2 q$ points of $\langle M, x\rangle$ collinear with $x^{\prime}$ in $\mathscr{S}$. Let this set of points be $\Omega$. Since $|\Omega|=2 q$ it follows that each affine line through $p_{2}$ or $p_{3}$ is incident with 2 points of $\Omega$. If $N$ is any line of $\langle M, x\rangle$ not incident with $p_{1}, p_{2}$ or $p_{3}$, then $\left\langle N, x^{\prime}\right\rangle$ contains at most 3 lines of $\mathscr{S}$ on $x^{\prime}$ and so $N$ contains at most 3 points of $\Omega$. So now consider any $y \in \Omega$. Then lines $\left\langle y, p_{1}\right\rangle,\left\langle y, p_{2}\right\rangle$ and $\left\langle y, p_{3}\right\rangle$ cover 4 points of $\Omega$ while the remaining $q-2$ lines of $\langle M, x\rangle$ on $y$ must cover the remaining $2 q-4$ points of $\Omega$ with at most 2 points of $\Omega \backslash\{y\}$ on a line. Consequently, each such line contains exactly 3 points of $\Omega$. It follows that each line of $\langle M, x\rangle$ not incident with $p_{1}, p_{2}$ or $p_{3}$ is incident with 0 or 3 points of $\Omega$. Now let $p$ be any point of $M \backslash\left\{p_{1}, p_{2}, p_{3}\right\}$. By considering the lines of $\langle M, x\rangle$ on $p$ we see that $\Omega$ may be partitioned into sets of size 3 and so $3 \mid q$.

Lemma 2. Let $x$ and $y$ be two collinear points of $\mathscr{S}$, then a line $M$ of $\Pi_{\infty}$ incident with $p=\langle x, y\rangle \cap \Pi_{\infty}$ is either a tangent of both $\theta_{x}$ and $\theta_{y}$, a secant of both $\theta_{x}$ and $\theta_{y}$ with $M \cap \theta_{x} \cap \theta_{y}=\{p\}$, or a 3-secant of both $\theta_{x}$ and $\theta_{y}$ with $\left|M \cap \theta_{x} \cap \theta_{y}\right|=3$.

Proof. Let $M$ be a line of $\Pi_{\infty}$ incident with $p=\langle x, y\rangle \cap \Pi_{\infty}$. Since $\alpha=2$, both $\left|M \cap \theta_{x}\right|$ and $\left|M \cap \theta_{y}\right|$ are at most 3. If $M \cap \theta_{y}=\{p\}$ and $\left|M \cap \theta_{x}\right|>1$, then this contradicts $\alpha=2$. Hence if $M \cap \theta_{y}=\{p\}$, then it is also the case that $M \cap \theta_{x}=\{p\}$, that is, $M$ is a tangent of both $\theta_{x}$ and $\theta_{y}$.

Assume that $\left|M \cap \theta_{x}\right|=3$, then by Lemma 1 every point of the affine plane $\langle x, M\rangle$ is incident with three lines of $\mathscr{S}$ and belonging to that plane; more particular this holds for the point $y$ and so $\left|M \cap \theta_{y}\right|=3$.

Hence the only possibility which is left is $\left|M \cap \theta_{x}\right|=\left|M \cap \theta_{y}\right|=2$ with $M \cap \theta_{x} \cap \theta_{y}=\{p\}$.

If $\left|M \cap \theta_{x} \backslash \theta_{y}\right|=\left|M \cap \theta_{y} \backslash \theta_{x}\right|=1$, then $M$ is said to be of type $(A)$ with respect to $x$ and $y$. If $\left|M \cap \theta_{x} \cap \theta_{y}\right|=3$, and hence $\left|M \cap \theta_{x} \backslash \theta_{y}\right|=\left|M \cap \theta_{y} \backslash \theta_{x}\right|=0$, then $M$ is said to be of type $(B)$ with respect to $x$ and $y$.

Lemma 3. Let $x$ be a point of $\mathscr{S}$, then $\theta_{x}$ is an ovoid of $\Pi_{\infty}$ and $q$ is even.
Proof. Let $y$ be a point of $\mathscr{S}$ collinear with $x$ and let $p=\langle x, y\rangle \cap \Pi_{\infty}$. By the proof of Lemma 2 a line of $\Pi_{\infty}$ containing $p$ is either tangent to both $\theta_{x}$ and $\theta_{y}$, or is either of type $(A)$ or of type $(B)$ with respect to $x$ and $y$. To prove that $\theta_{x}$ is an ovoid, we have to show that there are no lines of type $(B)$ with respect to $x$ and $y$.

Let $L$ be the line $\langle x, y\rangle$ of $\mathscr{S}$. For any line $N$ of $\mathscr{S}$ intersecting $L$ (in a point of $\mathscr{S}$ ) $\langle L, N\rangle \cap \Pi_{\infty}$ is a line of type $(A)$ or $(B)$ (with respect to $x$ and $y$ ). Let this line be $M$ and suppose that $M$ is of type $(A)$ such that $\theta_{x} \cap M=\left\{p, p_{1}\right\}$ and $\theta_{y} \cap M=\left\{p, p_{2}\right\}$. Let $z=\left\langle p_{2}, y\right\rangle \cap\left\langle p_{1}, x\right\rangle$ and suppose that $z$ is incident with a third line of $\mathscr{S}$ in $\langle M, L\rangle$. Since $z$ is collinear with $x$ and $y$ and $\alpha=2$ this third line must be $\langle z, p\rangle$. As $\alpha(y,\langle z, p\rangle)=2$ there is a third line of $\mathscr{S}$ on $y$ in $\langle M, L\rangle$, contradicting the fact that $M$ is a 2 -secant of $\theta_{y}$. Consequently $z$ is incident with exactly two lines of $\mathscr{S}$ in $\langle M, L\rangle$. Similar arguments show that each point of $\left\langle x, p_{1}\right\rangle \backslash\left\{p_{1}\right\}$ is incident with exactly two lines of $\mathscr{S}$ in $\langle M, L\rangle$. It follows that there are exactly $q+1$ lines of $\mathscr{S}$ in $\langle M, L\rangle$. Also $\alpha=2$ implies that no two of these lines meet on $M$. Hence the lines of $\mathscr{S}$ in $\langle M, L\rangle$ form a dual oval with nucleus $M$, from which it follows that $q$ is even.

By Lemma 1 if $M$ is a line of type $(B)$, then $q$ is a power of 3 . Thus we have two distinct cases for the lines of $\Pi_{\infty}$ through $p$ that are not tangent to both $x$ and $y$ : either they are all of type $(A)$ or all of type $(B)$. In the latter case $3 \mid q$ and the lines through $p$ partition $\theta_{x} \backslash\{p\}$ into sets of size 2 which implies that $2 \mid q$, a contradiction. So we must be in the former case and $q$ is even.

Now suppose that $x$ is an arbitrary point of $\mathscr{S}$ and $p$ a point of $\theta_{x}$. If $y \in\langle x, p\rangle \backslash\{x, p\}$, then by applying the above argument it follows that every line of $\Pi_{\infty}$ on $p$ is either a tangent or a secant of $\theta_{x}$. Hence there are no 3 -secants of $\theta_{x}$ and $\theta_{x}$ is an ovoid.

Corollary 4. Let $x$ and $y$ be two collinear points of $\mathscr{S}$, then $\left|\theta_{x} \cap \theta_{y}\right|=1$.

Proof. Every line of $\Pi_{\infty}$ incident with $p=\langle x, y\rangle \cap \Pi_{\infty}$ is either a tangent to both $\theta_{x}$ and $\theta_{y}$ or is of type $(A)$ with respect to $x$ and $y$.

Lemma 5. Let $x$ and $y$ be two non-collinear points of $\mathscr{S}$ and $p=\langle x, y\rangle \cap \Pi_{\infty}$. Let $M$ be any line of $\Pi_{\infty}$ incident with $p$. Then one of the following is the case:
(i) $M$ is secant to both $\theta_{x}$ and $\theta_{y}$ and $M \cap \theta_{x} \cap \theta_{y}=\emptyset$;
(ii) $M$ is tangent to both $\theta_{x}$ and $\theta_{y}$ at a point of $\theta_{x} \cap \theta_{y}$; or
(iii) $M$ is external to both $\theta_{x}$ and $\theta_{y}$.

Furthermore $\theta_{x} \cap \theta_{y}$ is an oval with nucleus $p$.

Proof. Suppose that $r \in \theta_{x} \cap \theta_{y}$. We show that $\langle r, p\rangle$ is tangent to both $\theta_{x}$ and $\theta_{y}$. Suppose that $\langle r, p\rangle$ is a secant line of at least one of the ovoids, say $\theta_{x}$. Hence $\left(\theta_{x} \cap\langle r, p\rangle\right) \backslash\{r\}=\{u\}$ for some point $u$. Let $z=\langle u, x\rangle \cap\langle r, y\rangle$. Then $x$ and $z$ are collinear in $\mathscr{S}$ while $\left|\theta_{x} \cap \theta_{z}\right| \geqslant 2$, contradicting Corollary 4. Hence $\langle p, r\rangle$ is a tangent line of both ovoids.

Now we show that $M$ is secant to $\theta_{x}$ if and only if it is secant to $\theta_{y}$. Therefore we first assume that $M$ intersects $\theta_{x}$ in the point $v$ and $\theta_{y}$ in the point $w$, with $v \neq w$. Then $\langle v, x\rangle$ intersects $\langle w, y\rangle$, and so $\alpha(y,\langle x, v\rangle)=2$. This implies that $M$ intersects $\theta_{y}$ in the distinct points $w$ and $w^{\prime}$, and moreover $w, w^{\prime} \notin \theta_{x} \cap \theta_{y}$. Similarly, since $\alpha(x,\langle y, w\rangle)=2$, it follows that $M$ intersects $\theta_{x}$ in the distinct points $v$ and $v^{\prime}$, with $v, v^{\prime} \notin \theta_{x} \cap \theta_{y}$. In other words, $M$ intersects both ovoids in two points outside their intersection. Since $|\Gamma(x) \cap \Gamma(y) \cap\langle x, y, M\rangle|=4$ and $|\Gamma(x) \cap \Gamma(y)|=\mu=2 q(q-1)$ it follows that there are exactly $q(q-1) / 2$ lines incident with $p$ that are secant to both $\theta_{x}$ and $\theta_{y}$. Since this is the number of secants of an ovoid incident with a point not on the ovoid this means that the set of lines of $\Pi_{\infty}$ incident with $p$ and secant to $\theta_{x}$ is also the set of lines incident with $p$ and secant to $\theta_{y}$.

By Lemma $3, q$ is even and consequently the $q+1$ tangents of $\theta_{x}$ incident with $p$ are contained in a plane $\pi_{x}$ on $p$ and similarly the $q+1$ tangents of $\theta_{y}$ incident with $p$ are contained in a plane $\pi_{y}$. There are two cases to consider: $\pi_{x}=\pi_{y}$ and $\pi_{x} \cap \pi_{y}$ is a line incident with $p$. First suppose that $\pi_{x}=\pi_{y}$. It follows that the tangents of $\theta_{x}$ incident with $p$ are precisely the tangents of $\theta_{y}$ incident with $p$ with a common point of tangency. Consequently $\theta_{x} \cap \theta_{y}$ is an oval of $\pi_{x}$ with nucleus $p$. So in this case $\left|\theta_{x} \cap \theta_{y}\right|=q+1$. Now suppose that $\pi_{x} \cap \pi_{y}$ is a line $L$ incident with $p$. The line $L$ is a tangent of both $\theta_{x}$ and $\theta_{y}$ at a point $o \in \theta_{x} \cap \theta_{y}$. If $M \neq L$ then by arguments above $M$ must be external to $\theta_{y}$. From this it follows that $\pi_{x}$ is the tangent plane of $\theta_{y}$ at $o$ and similarly $\pi_{y}$ is the tangent plane of $\theta_{x}$ at $o$. Since $\langle p, o\rangle$ is the only line of $\Pi_{\infty}$ incident with $p$ that is tangent to both $\theta_{x}$ and $\theta_{y}$ it follows that $\theta_{x} \cap \theta_{y}=\{o\}$, and so $\left|\theta_{x} \cap \theta_{y}\right|=1$.

It is now shown that the case $\left|\theta_{x} \cap \theta_{y}\right|=1$ cannot occur. Suppose that $\left|\theta_{x} \cap \theta_{y}\right|=$ 1. Let $M$ be secant of both $\theta_{x}$ and $\theta_{y}$. It follows by arguments above that if $\theta_{x} \cap M=$ $\left\{v, v^{\prime}\right\}$ and $\theta_{y} \cap M=\left\{w, w^{\prime}\right\}$, then $\left\{v, v^{\prime}, w, w^{\prime}\right\}$ are four distinct points. Let $\left\{x=x_{1}, x_{2}, \ldots, x_{q}\right\}$ be the set of $q$ points of $\mathscr{S}$ incident with the line $L=\langle x, v\rangle$. By Corollary 4, $\theta_{x_{i}} \cap \theta_{x_{j}}=\{v\}$ for $i, j \in\{1, \ldots, q\}, i \neq j$, and by a consequence of Lemma 2, the ovoids $\theta_{x_{1}}, \ldots, \theta_{x_{q}}$ have a common tangent plane at $v, \pi_{v}$ say. It follows that the ovoids $\theta_{x_{1}}, \ldots, \theta_{x_{q}}$ partition the points of $\Pi_{\infty} \backslash \pi_{v}$ into $q$ sets of size $q^{2}$. Without loss of generality assume that $y$ is collinear with the points $x_{2}$ and $x_{3}$ of $L$, so by Corollary $4\left|\theta_{y} \cap \theta_{x_{2}}\right|=\left|\theta_{y} \cap \theta_{x_{3}}\right|=1$. By above arguments it follows that for $i=4, \ldots, q,\left|\theta_{y} \cap \theta_{x_{i}}\right|=1$ or $q+1$.
Suppose that $\left|\pi_{v} \cap \theta_{y}\right|=1$, then since $v \notin \theta_{y}$ the ovoids $\theta_{x_{1}}, \ldots, \theta_{x_{q}}$ partition the $q^{2}$ points of $\theta_{y} \backslash\left(\pi_{v} \cap \theta_{y}\right)$ into $q$ sets with size either 1 or $q+1$. This requires $q-1$ sets of size $q+1$ and 1 set of size 1 . However $\left|\theta_{x_{i}} \cap \theta_{y}\right|=1$ for $i=1,2$ and 3 , a contradiction. Now suppose that $\left|\pi_{v} \cap \theta_{y}\right|=q+1$, then since $v \notin \theta_{y}$ the ovoids $\theta_{x_{1}}, \ldots, \theta_{x_{q}}$ partition the $q^{2}-q$ points of $\theta_{y} \backslash\left(\pi_{v} \cap \theta_{y}\right)$ into $q$ sets with size either 1 or $q+1$. This requires $q-2$ sets of size $q+1$ and 2 sets of size 1 , again a contradiction.

It follows that $\left|\theta_{x} \cap \theta_{y}\right|$ cannot be 1 and so $\pi_{x}=\pi_{y}$ and $\theta_{x} \cap \theta_{y}$ is an oval of $\pi_{x}$ with nucleus $p$.

Theorem 6. Let $\mathscr{S}$ be a semipartial geometry $\operatorname{spg}\left(q-1, q^{2}, 2,2 q(q-1)\right)$ fully embedded in $\mathrm{AG}(4, q)$. Then $q=2^{h}, \mathscr{S}$ is isomorphic to $\mathrm{TQ}(4, q)$ and is fully embedded as the Hirschfeld-Thas model.

Proof. Let $\mathscr{S}$ be a semipartial geometry $\operatorname{spg}\left(q-1, q^{2}, 2,2 q(q-1)\right)$ fully embedded in $\mathrm{AG}(4, q)$. If $q=2$, then $\mathscr{S}$ coincides with its point graph which is the unique complete graph on six vertices and the result follows. Hence we may assume that $q>2$.
Let $\mathscr{K}=\Pi_{\infty} \cup \mathscr{P}$, where $\mathscr{P}$ is the point set of $\mathscr{S}$. The intersections of $\mathscr{K}$ with a plane of $\operatorname{PG}(4, q)$ are now considered which will allow the use of a result of Hirschfeld and Thas in [10] in order to prove the theorem. So let $\pi$ be a plane of $\operatorname{PG}(4, q)$. If $\pi \subset \Pi_{\infty}$, then $\pi \subset \mathscr{K}$; so suppose that $\pi \not \subset \Pi_{\infty}$ and that $\pi \cap \Pi_{\infty}$ is the line $M$.

Suppose that $\pi$ contains a point $x$ of $\mathscr{S}$. Then $M$ may either be a secant, tangent or external line of $\theta_{x}$.

Suppose that $M$ is a secant line of $\theta_{x}$. This is the case if and only if there exists an antiflag $(x, L)$ of $\mathscr{S}$ contained in $\pi$ such that $\alpha(x, L)=2$. By the proof of Lemma 3 the lines of $\mathscr{S}$ in $\pi$ form a dual oval $\mathscr{D}$ with nucleus $M$ and these are all the lines of $\mathscr{S}$ in $\pi$. Let $z$ be any point of $\mathscr{S} \cap \pi$ and not collinear in $\mathscr{S}$ with $x$. Then by Lemma 5, M is a secant of $\theta_{z}$; hence $z$ is incident with exactly two lines of the dual oval $\mathscr{D}$. It follows that $\pi \cap \mathscr{K}$ is a dual hyperoval; or equivalently the complement of a maximal arc of type $(0, q / 2)$.

Next suppose that $M \cap \theta_{x}=\{p\}$. Hence $M$ is a tangent of $\theta_{x}$ at $p$ and all points of $\pi \cap \mathscr{S}$ are not collinear in $\mathscr{S}$ with $x$. If $y$ is such a point of $\mathscr{S}$ on $\pi$, then by Lemma 5 $M$ is a tangent of $\theta_{y}$ at $p$ and so $\langle p, y\rangle$ is a line of $\mathscr{S}$. It follows that lines of $\mathscr{S}$ in $\pi$ are incident with $p$ and that all points of $\mathscr{S}$ on $\pi$ are incident with such a line. Let $z$ be a point of $M \backslash\{p\}$, and let $N$ be a secant of $\theta_{x}$ incident with $z$. By the above the plane $\langle N, x\rangle$ meets $\mathscr{K}$ in a dual hyperoval and since $z \notin \theta_{x}$ it follows that the line $\langle z, x\rangle$ is not a line of $\mathscr{S}$. Hence $\langle z, x\rangle$ is incident with exactly $q / 2$ points of $\mathscr{S}$ and so $\pi$ meets the line set of $\mathscr{S}$ in exactly $q / 2$ lines each intersecting $M$ in $p$. So $M$ is a tangent of $\theta_{x}$ if and only if $\pi$ meets $\mathscr{K}$ in the point set of $q / 2+1$ concurrent lines.

Finally suppose that $M$ is an external line of $\theta_{x}$. Let $y$ be any point of $M$ and let $L$ be a secant of $\theta_{x}$ incident with $y$. The line $\langle x, y\rangle$ is incident with $q / 2$ points of $\mathscr{S}$. Hence each line of $\pi$ incident with $x$ is incident with $q / 2$ points of $\mathscr{S}$. If $z$ is any other point of $\mathscr{S}$ in $\pi$, then since $x$ and $z$ are not collinear and $M$ is an external line of $\theta_{x}$ it follows by Lemma 5 that $M$ is also an external line of $\theta_{z}$. Hence $\pi$ meets $\mathscr{K}$ in a maximal arc of type $(0, q / 2)$, and $M$ is an external line to this maximal arc.

By the above discussion a plane section of $\mathscr{K}$ is one of the following sets: (i) a single line; (ii) the entire plane; (iii) a maximal arc of type ( $0, q / 2$ ), plus an external line; (iv) a dual hyperoval, or equivalently, the complement of a maximal arc of type $(0, q / 2)$; or (v) $q / 2+1$ concurrent lines.

From this list it follows that with respect to the intersection with lines $\mathscr{K}$ is a set of points of type ( $1, q / 2+1, q+1$ ).

Actually, it is possible to show that no planes of type (i) occur, but we do not need this. The set $\mathscr{K}$ does contain plane sections of type (iv), and for $q=4, \mathscr{K}$ has no plane section that is either a unital or a subplane. Hence by [10, Theorem 6] the set $\mathscr{K}$ is the projection of a non-singular quadric of $\operatorname{PG}(5, q)$ onto $\operatorname{PG}(4, q)$. Any plane contained in $\mathscr{K}$ is also contained in $\Pi_{\infty}$ which can only be the case if $\mathscr{K}$ is the projection of an elliptic quadric $Q^{-}(5, q)$ onto $\operatorname{PG}(4, q)$.

We can rephrase as follows our result for an $\operatorname{spg}\left(q-1, q^{2}, 2,2 q(q-1)\right)$ constructed from a doubly subtended subquadrangle of order $q$ of a generalized quadrangle of order $\left(q, q^{2}\right)$.

Corollary 7. Let $\mathscr{G}$ be a generalized quadrangle of $\operatorname{order}\left(q, q^{2}\right), \mathscr{G}^{\prime}$ a doubly subtended subquadrangle of $\mathscr{G}$ of order $q$, and $\mathscr{S}$ the $\operatorname{spg}\left(q-1, q^{2}, 2,2 q(q-1)\right)$ constructed from $\mathscr{G}$ and $\mathscr{G}^{\prime}$. If $\mathscr{S}$ may be fully embedded in $\mathrm{AG}(4, q)$, then $\mathscr{S}=\mathrm{TQ}(4, q), \mathscr{G}=Q^{-}(5, q)$, $\mathscr{G}^{\prime}=Q(4, q)$ and $q=2^{h}$.

Proof. By Theorem $6 \mathscr{S} \cong \mathrm{TQ}(4, q)$ and $q=2^{h}$. Since $\mathscr{S}$ (in the model of Metz) may be constructed from the doubly subtended subquadrangle $Q(4, q)$ of $Q^{-}(5, q)$, it follows from [2, Theorem 3.3] that $\mathscr{G}^{\prime}=Q(4, q)$ and $\mathscr{S}$ is the model of Metz in $Q(4, q)$. Since $Q(4, q)$ is doubly subtended in $\mathscr{G}$ with all subtended ovoids being elliptic quadrics on $Q(4, q)$, it follows that $\mathscr{G}=Q^{-}(5, q)[1,3]$.

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