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Trajectories Joining Critical Points

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1. INTRODUCTION

In this paper we shall be concerned with the existence of trajectories joining a pair of critical points, or more generally, a pair of compact invariant sets. Our framework will be abstract dynamical systems. We shall present applications and examples among both finite and infinite dimensional differential equations.

The existence of connecting orbits has attracted the attention of many researchers. (In Gelfand [1963], a footnote on page 299 explicitly indicates that this is an interesting problem.) A variety of techniques have been employed to cope with the problem. The tools which are employed include consideration of degree and index theories, isolating blocks, dimensionality of stable and unstable manifolds, bifurcation type arguments, fixed point theory, etc. (Consult Conley [1976], Conley and Smoller [1975, 1978], Conlon [1980], Foy [1964], Gordon [1974], and Howard and Koppel [1975]. Some sort of differentiability and smoothness is required in any of the cited works and therefore the applicability to infinite dimensional problems is limited. In some of these works the global connecting orbit was found only in a local neighborhood of some critical point.

In this paper we pursue a rather naive approach. It would not be applicable unless some a priori demands on the global behavior of solutions are met. However, when the method works it is capable of handling infinite

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dimensional problems and it leads to natural algorithms for a numerical detection of the connecting orbit.

The idea is simple enough to be displayed here. We treat the existence of connecting orbits in two stages. First we look for ε-connecting orbits, namely, orbits which connect in finite time the ε-neighborhoods of the critical sets. If for every $\varepsilon > 0$ such an orbit exists, we turn to the next step and study the limiting behavior of these orbits as $\varepsilon \to 0$, hoping that the limit produces the desired connecting orbit.

The second stage of this program can be studied in a general setting. We devote Section 3 to the abstract investigation, and the examples in the later sections demonstrate the validity of the conditions.

The first stage of the program, namely, establishing the existence of ε-connecting orbits, depends heavily on the geometrical structure of the equation and the location of the two critical sets. Our contribution is to analyze some particular examples.

Here is an outline of the paper. In Section 2 we present our terminology and notation. In Section 3 we study the limit of ε-connecting orbits as $\varepsilon \to 0$. We present conditions that guarantee that the limit exists and produces an orbit which connects the critical sets in a weak sense (Theorem 3.3). We provide counterexamples which show that these conditions cannot be dropped. We give a further condition guaranteeing existence of a strong connecting orbit. The section is concluded with remarks on weak connecting orbits and on the numerical detection of the connecting orbit.

In Section 4 we analyze in detail a nonautonomous ordinary differential equation previously analyzed by Conley [1976]. We apply our theory to get a connecting orbit and present some numerical results which show the connecting orbit.

In Section 5 we deal with Lagrange p-stable flows. We present an abstract result concerning existence of connecting orbits in the case where the union of ω-limit sets is precompact. In Section 6 we apply these results to various examples of nonlinear evolution equations, of parabolic and hyperbolic types, in infinite dimensional Banach space.

2. Terminology and Notations

The considerations of this paper can be applied under quite general circumstances, but at the price of complicated notations. In particular many of the results hold for equations without uniqueness, for equations without global existence, and for nonautonomous equations. We shall work in a more restrictive framework in order to make the ideas more transparent. Most of the results hold also for systems with discrete time. We occasionally note what modifications should be made to include this case.

A *semiflow* on a metric space X is a continuous mapping

$$\pi(t, x): [0, \infty) \times X \to X$$

satisfying $\pi(0, x) = x$ and $\pi(t+s, x) = \pi(t, \pi(s, x))$ for $x \in X$ and s, t in $[0, \infty)$.

Typically X is the state space for a differential equation and $\pi(t, x_0)$ is the solution $x(t)$ in positive time with the initial condition $x(0) = x_0$. The properties of π reflect the uniqueness and global existence of solutions for positive time, the well-posedness and the autonomacy of the equation.

Let I be an interval in the real line. A function $U(t): I \to X$ is an *orbit* on I if $U(t+s) = \pi(t, U(s))$ whenever $t \geqslant 0$ and s and $s+t$ belong to I. If $I = (-\infty, \infty)$ then U is called a *full orbit* or a *full solution* of π.

A subset $B \subset X$ is *positively invariant* with respect to π if $x_0 \in B$ implies $\pi(t, x_0) \in B$ for all $t \geqslant 0$. It is *invariant* if for every $x_0 \in B$ a full solution U exists such that $U(0) = x_0$ and $U(t) \in B$ for $t \in (-\infty, \infty)$. A point x_0 is a *rest point* if $\pi(t, x_0) = x_0$ for all $t \geqslant 0$. Let U be an orbit on an interval (r, ∞). The ω-limit set of U, denoted $\omega(U)$ is the set of all limits $z = \lim U(t_k)$ for sequences $t_k \to \infty$. If U is a full orbit then its α-limit set is the set $\alpha(U)$ of all limits $z = \lim U(t_k)$ for sequences $t_k \to -\infty$. It follows from the continuity of π that the α-limit sets and ω-limit sets are positively invariant.

Let B_1 and B_2 be two subsets of X. The full solution U of π is a *connecting orbit* between B_1 and B_2 if both $\omega(U)$ and $\alpha(U)$ are not empty, $\operatorname{dist}(U(t), B_1) \to 0$ as $t \to -\infty$ and $\operatorname{dist}(U(t), B_2) \to 0$ as $t \to +\infty$. Here $\operatorname{dist}(a, B) = \inf\{d(a, b): b \in B\}$ and $d(\cdot, \cdot)$ is the metric on X. The orbit U on an interval $[r, s]$ is an *ε-connecting orbit* between B_1 and B_2 if $\operatorname{dist}(U(r), B_1) \leqslant \varepsilon$ and $\operatorname{dist}(U(s), B_2) \leqslant \varepsilon$.

3. Limits of ε-Connecting Orbits

Let B_1 and B_2 be two disjoint, compact and positively invariant sets, with respect to the semiflow π. Suppose also that for every $\varepsilon > 0$ there is an ε-connecting orbit between B_1 and B_2. We intend to derive a connecting orbit between B_1 and B_2 as an appropriate limit of the ε-connecting orbits, as $\varepsilon \to 0$. This scheme would not work out unless some restrictions are imposed on the flow, as indicated by the following examples.

Example 3.1. Consider the flow on R^2 generated by the ordinary differential equations $\dot{\eta} = 0$ and $\dot{\xi} = \eta^2 + \xi(\xi - 1)^2 (2 - \xi)$. Clearly, for every $\varepsilon > 0$ there is an ε-connecting orbit between the rest point $(0, 0)$ and the rest point $(2, 0)$. For instance, the solution with initial condition $(0, \varepsilon)$ will generate such an orbit. However, a connecting orbit does not exist. The natural limit consists of orbits connecting $(0,0)$ to $(1,0)$ and the latter to $(2,0)$. The reason is that the ε-connecting orbits are not isolated from the rest point $(1,0)$. In general they should be isolated from ω-limit points not included in $B_1 \cup B_2$.

EXAMPLE 3.2. Consider the flow on R^2 generated by the ordinary differential equations $\dot{\xi} = -\eta\,|\xi|$ and $\dot{\eta} = 1-\eta^2$. For every $\varepsilon > 0$ there is an ε-connecting orbit between the two rest points $(0,-1)$ and $(0,1)$. For instance, the solution with initial condition $(0,-1+\varepsilon)$ will generate such an orbit. However, the lack of compactness in the family of these approximations prevents using the desired limit. Indeed, the natural limit of the ε-connecting orbits consists of the two unbounded full orbits $(e^t,-1)$ and $(e^{-t},1)$ for $t \in (-\infty,\infty)$.

If we rule out similar counterexamples by adding the appropriate conditions, we get the following result:

THEOREM 3.3. *Let B_1 and B_2 be two disjoint, compact and positively invariant sets. Suppose that for every $\varepsilon > 0$ there is an ε-connecting orbit $U_\varepsilon\colon I_\varepsilon \to X$ between B_1 and B_2. Denote by C the closure of all values $U_\varepsilon(t)$ for $t \in I_\varepsilon$ and $\varepsilon > 0$. Suppose that C is compact and that any ω-limit set which is contained in C is contained already in $B_1 \cup B_2$. Then there is a full orbit U of π such that $\varnothing \neq \omega(U) \subset B_2$ and $\alpha(U) \cap B_1 \neq \varnothing$.*

Proof. Let δ_k, $k = 0,1,2,\ldots$ be a sequence of positive real numbers strictly decreasing to zero and such that $2\delta_0 \leqslant \min\{d(b_1,b_2)\colon b_1 \in B_1, b_2 \in B_2\}$. The continuity on the time variable t implies that if $\operatorname{dist}(y,B_1) \leqslant \delta_k$ and $\operatorname{dist}(\pi(t,y),B_1) \geqslant \delta_{k-1}$ then for a certain $s \in [0,t]$ the inequalities $\delta_k \leqslant \operatorname{dist}(\pi(s,y),B_1) \leqslant \delta_{k-1}$ hold.

Consider the sequence U_{δ_j}, which for convenience will be denoted U_j, of δ_j-connecting orbits between B_1 and B_2. Suppose U_j is defined on an interval I_j. Let $[\sigma_{j,1},\tau_{j,1}]$ be a subinterval of I_j such that $\operatorname{dist}(U_j(\sigma_{j,1}),B_1)$ and $\operatorname{dist}(U_j(\tau_{j,1}),B_2)$ are both less than or equal to δ_1 and $\operatorname{dist}(U_j(t),B_i) > \delta_1$ whenever $\sigma_{j,1} < t < \tau_{j,1}$ and $i = 1,2$. Such an interval $[\sigma_{j,1},\tau_{j,1}]$ exists since each U_j is in particular a δ_1-connecting orbit between B_1 and B_2. Once the intervals $[\sigma_{j,1},\tau_{j,1}]$ are chosen the intervals $[\sigma_{j,k},\tau_{j,k}]$ are uniquely determined for $j \geqslant k$ as follows: Let $[\sigma_{j,k},\tau_{j,k}]$ be the smallest interval containing $[\sigma_{j,1},\tau_{j,1}]$ and such that both $\operatorname{dist}(U(\sigma_{j,k}),B_1)$ and $\operatorname{dist}(U(\tau_{j,k}),B_2)$ are smaller than or equal to δ_k. Such intervals exist since U_j is a δ_k-connecting orbit if $j \geqslant k$.

We claim: For every fixed k the lengths $\tau_{j,k} - \sigma_{j,k}$ are uniformly bounded for $j = k, k+1,\ldots$.

Proof of the claim: Suppose that for a subsequence $\{i\}$ of $\{j\}$ the values

$\tau_{i,k}-\sigma_{i,k}$ tend to infinity as $i\to\infty$. The sequence $U(\sigma_{i,k})$ has a converging subsequence, say to y. This follows from the compactness of C. For each $t\geqslant 0$ the corresponding subsequence of $U(\sigma_{i,k}+t)=\pi(t, U(\sigma_{i,k}))$ converges to $\pi(t,y)$. Since for i large enough $t\leqslant\tau_{i,k}-\sigma_{i,k}$ it follows that both $\operatorname{dist}(\pi(t,y),B_i)$ for $i=1,2$, are greater than or equal to δ_k. Since the entire orbit $\pi(t,y)$ on $[0,\infty)$ is contained in the compact set C, its ω-limit set is not empty, and it cannot be contained in $B_1\cup B_2$. This contradicts a condition in the statement of the theorem.

We also claim: The length $\tau_{j,k}-\sigma_{j,k}$ tend to ∞ as $k\to\infty$, this uniformly for $j\geqslant k$.

Proof of the claim: As $k\to\infty$ the distances $\operatorname{dist}(U(\sigma_{j,k}),B_1)$ and $\operatorname{dist}(U(\tau_{j,k}),B_2)$ tend to 0 uniformly for $j\geqslant k$. The convergence $\tau_{j,k}-\sigma_{j,k}\to\infty$ follows now from the equality $U(\tau_{j,k})=\pi(\tau_{j,k}-\sigma_{j,k}, U(\sigma_{j,k}))$ and from the compactness and positive invariance of B_1.

We start now with the construction of the weakly connecting orbit. Let $\{j_1\}$ be a subsequence of $\{j\}$ such that $U_{j_1}(\sigma_{j_1,1})$ converges, say to y_1. Such a subsequence exists by the compactness of C, and also $\operatorname{dist}(y_1,B_1)\geqslant\delta_2$ by the construction of $\sigma_{j,1}$. We shall now show that the ω-limit set of $\pi(t,y_1)$ (for $t\geqslant 0$) is nonempty and contained in B_2. Since $\pi(t,y_1)=\lim U(\sigma_{j_1,1}+t)$ for $t\geqslant 0$, it follows that $\operatorname{dist}(\pi(t,y_1),B_1)\geqslant\delta_2$ for all $t\geqslant 0$. Therefore the ω-limit set has an empty intersection with B_1. Since the entire trajectory $\pi(t,y_1)$ for $t\geqslant 0$ is included in the compact set C it follows that its ω-limit set is not empty and is also included in C. From our assumptions it follows now that the ω-limit set is included in B_2.

We shall now continue the orbit $\pi(t,y_1)$ backwards and embed it in a full solution which has a nonempty α-limit set which intersects B_1. To this end we proceed in a diagonal fashion. Let $\{j_2\}$ be a subsequence of $\{j_1\}$ such that both $U(\sigma_{j_2,2})$ and $\sigma_{j_2,1}-\sigma_{j_2,2}$ converge, say to y_2 and s_1. The existence of such a subsequence follows from the compactness of C and the first of the preceding claims. Then $\pi(s_1,y_2)=y_1$. Inductively, suppose $\{j_{k-1}\}$ is defined, and let $\{j_k\}$ be a subsequence such that $U(\sigma_{j_k,k})$ converges, say to y_k, and $\sigma_{j_k,k-1}-\sigma_{j_k,k}$ converges, say to s_{k-1}. The same argument as before shows the existence of such a subsequence. Then $\pi(s_{k-1},y_k)=y_{k-1}$. This is done for all natural k. Define a full solution of π as follows: $U(t)=\pi(t-(s_1+\cdots+s_k),\ y_{k+1})$ if $t>-(s_1+\cdots+s_k)$. The second of the above claims implies that the summation of all s_i diverges, hence $U(t)$ is defined for all real t. The definition does not depend on the choice of k since $\pi(s_k,y_{k+1})=y_k$. Finally, the α-limit set of U is not empty (by the compactness of C) and in particular contains a limit point of the sequence y_k. Since $\operatorname{dist}(y_k,B_1)\leqslant\delta_k$ it follows that $\alpha(U)\cap B_1\neq\varnothing$. This completes the proof.

A remark on the discrete time case. Theorem 3.3 holds also when the time is discrete. The same proof can be used with only one modification. The sequence δ_k cannot be chosen arbitrarily since an arbitrary sequence might not satisfy the property mentioned in the first paragraph of the proof. However, the existence of particular sequences which satisfy this property follows from the compactness and positive invariance of B_1.

The possibility that an orbit will be a weak connecting orbit but not a (strong) connecting orbit will be demonstrated in Remark 3.5. We shall now state a useful condition guaranteeing that this does not happen. Recall that the function V is a *Liapunov function* of π if it is continuous and $V(x) \geqslant V(\pi(t, x))$ for every $t \geqslant 0$ and every x in the domain of V. Here it is demanded that the domain of definition of V is positively invariant with respect to π.

THEOREM 3.4. *Suppose that in addition to the conditions of Theorem* 3.3 *there exists a positively invariant neighborhood N of B_1 and a Liapunov function V on N such that $V(\pi(t, x)) < V(x)$ if $t > 0$ and $x \in N \backslash B_1$. Then the connecting orbit guaranteed by Theorem* 3.3 *satisfies $\alpha(U) \subset B_1$.*

Proof. Let $U(t)$ be the connecting orbit guaranteed by Theorem 3.3. Since its values are in C it follows that $\alpha(U)$ is compact. Since $\alpha(U)$ has a nonempty intersection with B_1 and since B_1 is positively invariant it follows that if $\alpha(U)$ has an element y not in B_1 then $\alpha(U)$ has such an element in any neighborhood of B_1. This would contradict the existence of the Liapunov function V since a Liapunov function is always constant on an α-limit set, and an α-limit set is positively invariant.

Remark 3.5. Existence of weakly joining orbits.

Consider the flow with phase portrait as drawn in Fig. 1. There is a

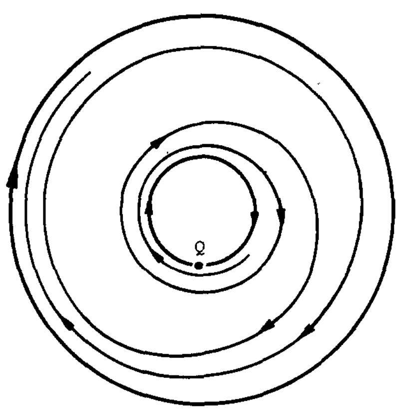


FIGURE 1

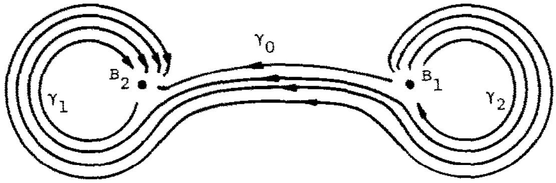


FIGURE 2

weakly joining orbit between the rest point Q and the outer circle, but there is no joining orbit whose α-limit set is equal to Q.

Remark 3.6. The limit process as a numerical scheme.

The limit process, in the proof of Theorem 3.3, suggests a way of calculating approximations to the desired orbit. A numerical example will be displayed in the next section. Here we want to note why one should not be tempted to either prove the result or calculate the desired solution by using the following scheme: Let $U_\varepsilon: I_\varepsilon \to X$ be the ε-connecting orbits as stated in the statement of Theorem 3.3. Let ε_j and $\tau_j \in I_{\varepsilon_j}$ be sequences such that $\varepsilon_j \to 0$ and $U(\tau_j)$ converges to an element x_0 which is not included in $B_1 \cup B_2$. (It is not hard to prove existence of such an x_0.) Then use a full orbit through x_0 which is a pointwise limit of $U_{\varepsilon_j}(\tau)$ as the connecting orbit. What might go wrong is demonstrated in Fig. 2. If $x_j = U_j(\tau_j)$ are chosen arbitrarily, the limit x_0 might be in one of the homoclinic orbits γ_1 or γ_2 and the desired connecting orbit γ_0 would not be detected. The specific choice of the intervals $[\sigma_j, \tau_j]$ in the proof of Theorem 3.3 assures that this would not happen.

4. An Example

The application which we present in this section was previously analyzed by Conley [1976].

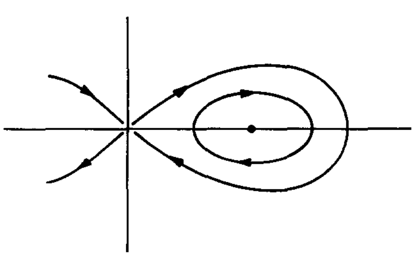
Consider the system of ordinary differential equations in the plane

$$\begin{aligned} \dot{\xi} &= \eta, \\ \dot{\eta} &= -s(t)\,\xi(1-\xi), \end{aligned} \tag{4.1}$$

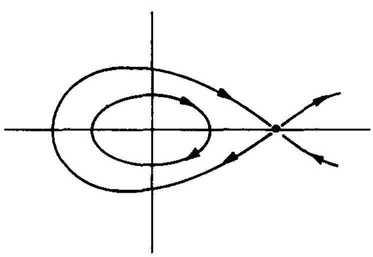
where $s(t)$ is a bounded function on $(-\infty, \infty)$. We shall further assume that $s(t)$ converges to $c_- < 0$ as $t \to -\infty$ and to $c_+ > 0$ as $t \to +\infty$. We want to establish the existence of a solution $(\xi_0(t), \eta_0(t))$ which tends to $(0, 0)$ as $t \to -\infty$ and converges to $(1, 0)$ as $t \to +\infty$. For such a solution, the function $\xi_0(t)$ is a standing wave solution of the nonlinear, autonomous, parabolic, partial differential equation $\partial\xi/\partial\tau = \partial^2\xi/\partial t^2 + s(t)\,\xi(1-\xi)$. The latter equation motivated Conley's work.

The solution $(\xi_0(t), \eta_0(t))$ that we seek is actually a connecting orbit between the rest points $(0, 0)$ and $(1, 0)$. The approximate (or ε-connecting) orbits would be solutions $(\xi_\varepsilon(t), \eta_\varepsilon(t))$ on an interval $[\sigma_\varepsilon, \tau_\varepsilon]$ such that $(\xi_\varepsilon(\sigma_\varepsilon), \eta_\varepsilon(\sigma_\varepsilon))$ and $(\xi_\varepsilon(\tau_\varepsilon), \eta_\varepsilon(\tau_\varepsilon))$ are close, respectively, to $(0, 0)$ and $(1, 0)$ and σ_ε and τ_ε are close, respectively, to $-\infty$ and $+\infty$. Say, $|\xi_\varepsilon(\sigma_\varepsilon)| + |\eta_\varepsilon(\sigma_\varepsilon)| < \varepsilon$, $|\xi_\varepsilon(\tau_\varepsilon) - 1| + |\eta_\varepsilon(\tau_\varepsilon)| < \varepsilon$, $\sigma < -1/\varepsilon$ and $\tau > 1/\varepsilon$. If in addition all the values of $(\xi_\varepsilon(t), \eta_\varepsilon(t))$ are bounded, the limit as $\varepsilon \to 0$ should produce the connecting orbit. To show this we cannot use Theorem 3.3 as it stands. The reason is that the solution funnel of the nonautonomous equation (4.1) does not fit the framework presented in Section 2. But the limiting process does work. One possibility to show this is to follow the proof of Theorem 3.3, and modify it to the nonautonomous case. Another possibility is to follow the construction of Sell [1967], and to embed the nonautonomous flow in a skew product flow as follows. The space X is the product $R^2 \times [-\infty, \infty]$. For $x = (a, b, r) \in R^2 \times [-\infty, \infty]$ the semiflow is defined as $\pi(t, x) = (\xi(r + t), \eta(r + t), r + t)$ where $(\xi(t), \eta(t))$ is the solution of (4.1) satisfying $\xi(r) = a$ and $\eta(r) = b$; this for $|r| < \infty$. For $|r| = \infty$ the semiflow is $\pi(t, x) = (\xi(t), \eta(t), r)$, where $(\xi(t), \eta(t))$ satisfies $\xi(0) = a$, $\eta(0) = b$ and solves the equation $\dot{\xi} = \eta$, $\dot{\eta} = -c\xi(1 - \xi)$ with $c = c_-$ if $r = -\infty$ and $c = c_+$ if $r = +\infty$. It is not hard to check that π is indeed a flow, and that the first two coordinates of a connecting orbit between $(0, 0, -\infty)$ and $(1, 0, +\infty)$ form the desired connecting orbit. (This construction can be generalized, using Sell [1967] and further results, and relax considerably the convergence conditions concerning $s(t)$. We leave out the details.)

We shall now establish the existence of the approximated orbits described above. In Figs. 3a, b the phase portraits of the equations $\dot{\xi} = \eta$ and $\dot{\eta} = -c\xi(1 - \xi)$, with $c = c_-$ and $c = c_+$, respectively, are drawn. These portray, approximately, the behavior of the solution funnel of (4.1) near $t = -\infty$ and $t = +\infty$, respectively. Consider now all the solutions of (4.1) having initial states at the triangle $K = \{(\xi, \eta)\colon \xi \geqslant 0,\ \eta \geqslant 0,\ \xi + \eta \leqslant \varepsilon\}$ and



a



b

FIGURE 3

fixed initial time σ close to $-\infty$. Denote this set by $K(\sigma)$ and by $K(t)$ the image of it at the time t, by the solution funnel. If s is large and σ is close to $-\infty$ it follows from Fig. 3a that $K(\sigma+s)$ contains a point on the half line $\xi=1$, $\eta \geqslant 0$. Since all the images $K(t)$ are connected and since $(0,0)$ is a rest point, namely contained in any $K(t)$, it follows from the two figures (by examining the directions of the local vector fields) that $K(t)$ will contain a point on $\xi=1$, $\eta \geqslant 0$ for all $t \geqslant \sigma+s$. In particular for $t=s_1$ with $s_1 \geqslant 1/\varepsilon$, and s_1 large enough so that Fig. 3b governs, approximately, the evolution of (4.1). The connectedness of K implies then that the image $K(s_1)$ contains a

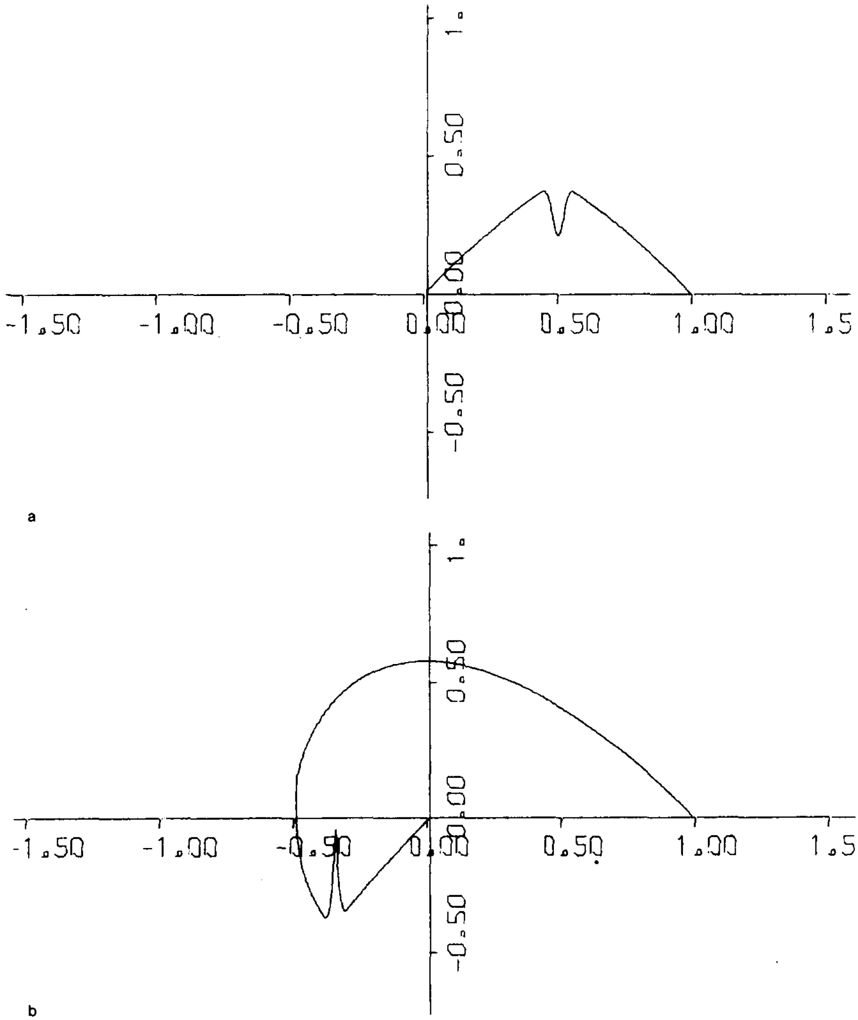


FIGURE 4

point, say z_0, on the trajectory which converges to the saddle point $(1, 0)$. In particular, in finite time, say s_2, the solution through this point will get to an ε-neighborhood of $(1, 0)$. Now the solution through z_0 at time s_1 generates the ε-connecting orbit on the time interval $[\sigma, \sigma + s_1 + s_2]$. Its limit as $\varepsilon \to 0$ will produce a weakly connecting orbit included in the positive orthant of R^2. It is clear however, say by a Liapunov function argument applied to Fig. 3a, that the trajectory is indeed a (strong) connecting orbit.

The same considerations can be applied when $K = \{(a, b): a \leqslant 0, b \leqslant 0, |a| + |b| \leqslant \varepsilon\}$. Then the limit process produces a connecting orbit which starts in the negative orthant of R^2 and then flows around to the positive orthant and to $(1, 0)$.

The ε-connecting orbits can be detected numerically. In Fig. 4a, b we present the outcome of a computer simulation for Eqs. (4.1) with the following data: $s(t) = -1$ for $t \leqslant -0.2$, $s(t) = 1$ for $t \geqslant 0.2$, $s(t)$ is piecewise linear between the values $s(-0.2) = -1$, $s(-0.1) = 1$, $s(0, 1) = -1$ and $s(0.2) = 1$. The method follows the geometrical ideas which were described earlier, using a simple Runge–Kutta method for solving the ordinary differential system.

The initial condition in Fig. 4a is $\xi(-4.5) = \eta(-4.5) = 0.007047$ and the time duration which is drawn is 10.8. (The regularity of $s(t)$ implies that $\sigma = -4.5$ is close enough to $-\infty$.) The initial condition on Fig. 4b is $\xi(-4.5) = \eta(-4.5) = -0.003857$ and the time duration is 12.5. In the two cases the continuation of the solution which is drawn misses slightly the point $(1, 0)$, and continues as a periodic solution, one of the family drawn in Fig. 3b. The accuracy of the method can be seen from the fact that a change of the order 10^{-6} in the initial condition (which produces almost the same approximation) would miss $(1, 0)$ in the other direction, and its continuation will go to infinity along the other direction of the saddle point.

5. Lagrange p-Stable Flows

Many examples share the property that all solutions converge to a prescribed compact set—sometimes to a collection of rest points. Examples are displayed in the next section. If the following property is satisfied, connecting orbits can be guaranteed.

Definition 5.1. The semiflow π is Lagrange p-stable if whenever $x_k \to x_0$ and $t_k \to \infty$ the sequence $\pi(t_k, x_k)$ has a converging subsequence.

The "p" in the previous definition stands for "prolongationally." The term Lagrange stability is commonly used to denote the compactness of positive orbits which yields compactness of ω-limit sets. (See, e.g., Bhatia and Szego [1970, p. 41].) The Lagrange p-stability plays the same role but with respect to the prolongational limit set. (For the latter consult Bhatia and Szego [1970].)

THEOREM 5.2. *Suppose that X is connected and suppose that the semiflow π is Lagrange p-stable. Let $B_1 \cup B_2$ be the closure of all ω-limit sets of orbits of π, and suppose that B_1 and B_2 are compact disjoint and nonempty. Then there is a full orbit U of π with $\alpha(U)$ and $\omega(U)$ nonempty such that either $\alpha(U) \cap B_1 \neq \varnothing$ and $\omega(U) \subset B_2$ or $\alpha(U) \cap B_2 \neq \varnothing$ and $\omega(U) \subset B_1$.*

Proof. The Lagrange p-stability implies in particular that each positive orbit is precompact. Therefore $B_1 \cap B_2 = \varnothing$ implies that an ω-limit set of π is included in either B_1 or in B_2. Denote by X_i the set of points x such that the ω-limit set of the orbit through x is included in B_i—this for $i = 1, 2$. Then $X_1 \cap X_2 = \varnothing$ and $X_1 \cup X_2 = X$. Since X is connected, it follows that one of the X_i is not closed, say X_1. Let $x_0 \in X_1$ and let $x_i \to x_0$ where $x_i \in X_2$. Let y_0 be in the ω-limit set of x_0, i.e., a sequence $t_m \to \infty$ exists with $\pi(t_m, x_0)$ converging to y_0. It is easy to extract then a subsequence $\{x_m\}$ of $\{x_k\}$ such that $\pi(t_m, x_m)$ also converges to y_0. But since $x_m \in X_2$ it follows that for a sequence s_m the values $\pi(s_m, x_m)$ converge to B_2. Clearly the functions $U_m(t) = \pi(t, x_m)$, for $t_m \leqslant t \leqslant s_m$ generate the ε-connecting orbits for any $\varepsilon > 0$. It remains to be shown that the conditions of Theorem 3.3 hold. To this end notice that the compactness of the set C follows from the Lagrange p-stability (since $x_i \to x_0$) and all ω-limit sets are anyway contained in $B_1 \cup B_2$. Theorem 3.3 now yields the desired connecting orbit.

A remark on the discrete time case. Theorem 5.2 holds also for discrete t with only the following modification. It is not enough to require that B_1 and B_2 be disjoint. They ought to be also positively invariant. (The latter is implied by disjointness for continuous time.)

6. APPLICATIONS

We shall apply the method of Section 5 to several infinite dimensional evolution equations. Some of the results are available in the literature and then our interest is not so much in the originality of the results as in the fact that such problems fit into our general framework. We first treat abstract evolutions in the strong topology, analyze examples, and then turn to evolutions in weak topologies. Our main concern is to check that the conditions of Theorem 5.2 hold. For completeness, the analysis will be done in some detail.

Consider the nonlinear evolution equation

$$\frac{du}{dt} = Au(t) + f(u(t)),$$
$$u(t_0) = x, \tag{6.1}$$

where we assume

(H.1) A is the infinitesimal generator of C^0 semigroup e^{At} on the Banach space X, and

(H.2) f is (generally nonlinear) locally Lipschitz map from X into X.

A *weak solution* of (6.1) is a continuous function $u(t)$ defined for $t \in [t_0, t_{\max})$ such that $f(u(\cdot))$ is locally integrable and such that the variation of constants formula

$$u(t) = e^{A(t-t_0)}x + \int_{t_0}^{t} e^{A(t-s)} f(u(s))\, ds$$

is satisfied for all $t \in [t_0, t_{\max})$. (An equivalent formulation is that $u(t_0) = x$, $f(u(\cdot))$ locally integrable and whenever $v \in D(A^*)$ then $\langle u(t), v\rangle$ is absolutely continuous and $d/dt\langle u(t), v\rangle = \langle u(t), A^*v\rangle + \langle f(u(t)), v\rangle$ for almost every t. For the equivalence see Ball [1978, Theorem 5.1] or Balakrishnan [1976, Theorem 4.8.3].) The following information is obtained by a routine adaptation of the contraction mapping argument used in the proof of the Picard theorem for ordinary differential equations. (A good reference is Pazy [1974].)

> Assume (H.1) and (H.2) hold. Then there exists a unique local weak solution $u(t)$ of (6.1) defined on a maximal interval $[t_0, t_{\max})$. If $t_{\max} < \infty$ then $\limsup_{t \to t_{\max}} \|u(t)\| = \infty$. Furthermore, the weak solution depends continuously on the initial data x. (6.2)

From now on we denote the solution of (6.1) by $u(t, x)$. Note that in case global existence is guaranteed, i.e., $t_{\max} \equiv \infty$, then $u(t, x)$ is a semiflow on X.

PROPOSITION 6.1. *Assume* (H.1) *and* (H.2) *and that for every bounded set* $B \subset X$ *a constant* c_B *exists such that* $\|u(t, x)\| \leqslant c_B$ *if* $x \in B$ *and* $t \in [t_0, t_{\max})$. *Then* (6.1) *possesses a unique global weak solution, hence* $u(t, x)$ *is a semiflow. Furthermore, suppose that one of the following holds*:

(H.3) e^{At} *is compact for* $t > 0$ *and* f *maps bounded sets of* X *into bounded sets of* X; *or*

(H.4) $\|e^{At}\| \leqslant Me^{-\mu t}$ *for* $M, \mu > 0$ *constants and* $f\colon X \to X$ *compact.*

Then the semiflow u is Lagrange p-stable.

Proof. Global existence follows from the a priori estimate $\|u(t, x)\| \leqslant c_B$ and the statement (6.2). Lagrange p-stability under (H.3) follows from the obvious modifications of the proof of Lemma 5.3 in Ball [1978] or Theorem 4.1 of Pazy [1975]. Lagrange p-stability under (H.4) follows from Proposition 3.1 of Webb [1979].

EXAMPLE 6.1. Damped sine–Gordon equation. Consider the equation

$$\begin{aligned} w_{tt} + a w_t = w_{xx} + \lambda \sin w, \qquad &\text{for} \quad 0 < x < \pi, \\ w(0, t) = w(\pi, t) = 0, \qquad& \\ w(x, 0) = \varphi(x), \qquad w_t(x, 0) = \psi(x).& \end{aligned} \tag{6.3}$$

(Here the subscripts t and x denote partial differentiation.) For a more general framework consult Webb [1979a]. Equation (6.3) can be written, at least formally, in the form of (6.1) if we set $u = \left(\begin{smallmatrix} w \\ w_t \end{smallmatrix}\right)$, $A = \left(\begin{smallmatrix} 0 & 1 \\ d^2/dx^2 & -a \end{smallmatrix}\right)$ and $f(u) = \left(\begin{smallmatrix} 0 \\ \lambda \sin w \end{smallmatrix}\right)$. It is a standard application of the Hille–Yosida–Phillips theorem to show that A is the infinitesimal generator of a C^0 semigroup e^{At} in the Hilbert space $X = H_0^1(0, \pi) \times L_2(0, \pi)$. Here X is endowed with the energy norm

$$\|(w, w_t)\|^2 = \int_0^\pi (w_x^2 + w_t^2)\, dx.$$

This verifies (H.1). In order to check (H.2) notice that if $u = (w, w_t)$ and $v = (z, z_t)$ then

$$\begin{aligned} \|f(u) - f(v)\|^2 &= \lambda^2 \|\sin w - \sin z\|^2_{L_2(0,\pi)} \\ &\leqslant \lambda^2 \|w - z\|^2_{L_2(0,\pi)} \leqslant \lambda^2 \|u - v\|^2. \end{aligned}$$

The last inequality is a consequence of the Poincare inequality. Thus f is Lipschitz continuous on X.

We claim that (H.4) holds. Let $u^j = (w^j, w_t^j)$ be a bounded sequence in X. Then w^j is a bounded sequence in $H_0^1(0, \pi)$ and hence via the Sobolev imbedding theorem possesses a convergent subsequence in $C(0, \pi)$. Since the sine function is continuous, it follows that $\sin w^j$ also has a convergent subsequence in $C(0, \pi)$, hence $f(u^j) = (0, \lambda \sin w^j)$ possesses a convergent subsequence in X. This proves the compactness of f. An elementary energy argument or a direct calculation via separation of variables shows $\|e^{At}\| \leqslant Me^{-\mu t}$ with $M, \mu > 0$.

We conclude therefore, using (6.2), that Eq. (6.3) has a unique local weak solution. Consider now the Liapunov functional

$$V(u)=\int_0^{\pi}\left(\frac{w_t^2}{2}+\frac{w_x^2}{2}+\lambda\cos w\right)dx.$$

(Compare with Webb [1979a].) A formal computation for sufficiently smooth solutions of (6.3) shows that

$$\dot{V}(u)=\int_0^{\pi} w_t(w_{tt}-w_{xx}-\lambda\sin w)\,dx=-\alpha\int_0^{\pi} w_t^2\,dx.$$

Hence, by standard density arguments, we find that the weak solutions $u(t)$ satisfy

$$V(u(t))\leqslant V(u_0)$$

with $u_0=(\varphi,\psi)$ as the initial condition. The estimates $V(u(t))\geqslant \frac{1}{2}\|u(t)\|^2-\lambda\pi$ and $V(u(t))\leqslant\frac{1}{2}\|u_0\|^2+\lambda\pi$ show that

$$\|u(t)\|^2\leqslant\|u_0\|^2+4\lambda\pi,$$

for all $t\geqslant 0$. Hence, solutions originating at bounded sets stay in bounded sets. In view of Proposition 6.1 we have

LEMMA 6.2. *The weak solutions of* (6.3) *generate a Lagrange p-stable semiflow on* $X=H_0^1(0,\pi)\times L_2(0,\pi)$.

We turn now to the issue of connecting orbits. It is well known (see Callegari and Reiss [1973], Dickey [1976] or Webb [1979a]) that (6.3) possesses equilibrium solutions under the following conditions:

> Let $n=0,1,2,\ldots$. If $n^2<\lambda\leqslant(n+1)^2$ then there are $2n+1$ equilibrium solutions, which we denote u_0, $\pm u_1,\ldots,\pm u_n$, and $u_0=0$. Furthermore, if $1<\lambda$ then u_1 and $-u_1$ are locally asymptotically stable in X, and u_0 and $\pm u_k$ for $k\geqslant 2$ are unstable. (6.4)

THEOREM 6.3. *Suppose* $n^2<\lambda\leqslant(n+1)^2$. *Then there is a full orbit* (w,w_t) *connecting one of the states* u_0, $\pm u_k(k\geqslant 2)$, *to* u_1. *In particular if* $n=1$ *there is a full orbit connecting the equilibrium states* u_0 *and* $-u_1$ *and a full orbit connecting* u_0 *and* u_1.

Proof. The semiflow is Lagrange p-stable and in particular each orbit $u(t,x)$, for $t\geqslant 0$, is precompact. It therefore converges to its ω-limit set and

$\dot{V}$ on the latter should be 0. Hence each individual orbit converges to one of the points u_0, $\pm u_k$; $k = 1,..., n$. Set $B_1 = \{-u_1, u_0, \pm u_k \text{ for } k = 2,..., n\}$, and $B_2 = \{u_1\}$. The Lagrange p-stability and Theorem 5.2 imply that there is a full orbit weakly connecting B_1 and B_2. The existence of the Liapunov function V implies that this orbit is (strongly) connecting one of the points in B_1 to B_2. This point cannot be $-u_1$ since both u_1 and $-u_1$ are asymptotically stable. This completes the proof of the general statement and the case $n = 1$ is indeed a particular case.

EXAMPLE 6.2. Nonlinear heat equation. Consider the equation

$$\begin{aligned} w_t = w_{xx} + \lambda \sin w, \qquad \text{for} \quad 0 < x < \pi,& \\ w(0, t) = w(\pi, t) = 0,& \\ w(x, 0) = \varphi(x).& \end{aligned} \tag{6.5}$$

Consult Chafee and Infante [1974] for a more general setting. If we set $u = w$, $A = d^2/dx^2$ and $f(u) = \lambda \sin u$ then (6.5) is transferred to the form of (6.1). Here we choose X to be $H_0^1(0, \pi)$ ad the domain $D(A) = \{w \in H^3(0, \pi)$: $w = w_{xx} = 0$ at $x = 0$ and $x = \pi\}$. We endow X with the norm

$$\|w\|^2 = \int_0^\pi w_x^2 \, dx.$$

It is a standard matter to check that A generates a C^0 semigroup e^{At} on X, and that $e^{At}: X \to D(A)$ for $t > 0$. Another elementary computation shows that $f: X \to X$ is locally Lipschitz. Hence (H.1), (H.2) and (H.3) are satisfied. By (6.2) Eq. (6.5) has a unique local weak solution. In order to prove global existence we consider the Liapunov functional

$$V(w) = \int_0^\pi \left(\frac{w_x^2}{2} + \lambda \cos w \right) dx.$$

(Compare with Chafee and Infante [1974].) An easy computation shows that for sufficiently smooth solutions

$$\dot{V}(w) = -\int_0^\pi w_t^2 \, dx$$

and a standard density argument then implies

$$V(w(t)) \leqslant V(w_0)$$

for any weak solution $w(t)$ with initial condition w_0. Since $V(w(t)) \geqslant \frac{1}{2}\|w(t)\|^2 - \lambda\pi$ and $V(w_0) \leqslant \frac{1}{2}\|w_0\|^2 + \lambda\pi$ it follows that

$$\|w(t)\|^2 \leqslant \|w_0\|^2 + 4\lambda\pi.$$

Hence bounded sets stay bounded under the semiflow and Proposition 6.1 implies

LEMMA 6.4. *The weak solutions of* (6.5) *generate a Lagrange p-stable semiflow on X.*

We have chosen the form (6.5) specifically such that the equilibrium solutions of (6.5) coincide with those of (6.3). It is also true that the local stability properties of the solutions are the same, i.e., (6.4) is applicable. See Chafee and Infante [1974]. Since the proof of Theorem 6.3 uses only this structure, the Lagrange p-stability and the existence of a Liapunov function we conclude that: Theorem 6.3, (as stated but with (w, w_t) replaced by w and with the same proof) *is valid for the nonlinear heat equation* (6.5).

There are interesting problems arising in applications not covered by Proposition 6.1. The reason is that (H.3) basically restricts us to "parabolic" problems when the semigroup e^{At} is smoothening; on the other hand (H.4) contains a rather stringent requirement on the nonlinear term f. In order to overcome these difficulties we (following Ball and Slemrod [1979]) consider the original semiflow in a weaker topology. The major tool in the imbedding is the following result.

LEMMA 6.5. *Let X be a separable reflexive Banach space. Assume*

(H.5) $f: X \to X$ *is sequentially weakly continuous.*

Let (6.1) *possess a unique weak solution* $u(t, x)$ *on an interval* $[0, T]$ *for every* $x \in X$. *Furthermore suppose that* $\|u(t, x)\| \leqslant$ *constant if* $t \in [0, T]$ *and for x restricted to a bounded set in X. Then* $x_n \to x$ *weakly and* $t \in [0, T]$ *imply* $u(t, x_n) \to u(t, x)$ *weakly.*

A proof of this lemma for the case X in a separable Hilbert space is given in Ball and Slemrod [1979, Theorem 2.3]. The same proof works for a separable reflexive space.

PROPOSITION 6.6. *Let X be a separable reflexive Banach space and assume that* (H.1), (H.2) *and* (H.5) *hold. Suppose that S is a bounded and weakly closed subset of X with the property that* $x \in S$, *implies that the unique local weak solution* $u(t, x)$ *satisfies* $u(t, x) \in S$ *for* $t \in [0, t_{\max})$. *Then for* $x \in S$ *Eq.* (6.1) *possesses a global weak solution and* $u: [0, \infty) \times S \to S$ *is sequentially continuous in both the weak and strong topologies (i.e.,* $x_n \to x$, $t_n \to t$ *imply* $u(t_n, x_n) \to u(t, x)$ *and* $x_n \to x$ *weakly,* $t_n \to t$ *imply* $u(t_n, x_n) \to$

$u(t, x)$ weakly). Furthermore, u: $[0, \infty) \times S \to S$ is a Lagrange p-stable semiflow on S when the latter is endowed with the metrized weak topology.

Proof. From the a priori estimate $u(t, x) \in S$ and (6.2) we see that u defines a semiflow in the strong topology on S. Weak sequential continuity in t and x independently follow from (6.2) and Lemma 6.5. Joint weak continuity follows from Ball [1974, Corollary 3.4], or Chernoff and Marsden [1970]. The boundedness implies that the weak topology is metrizable, and the closedness, hence compactness, in the weak topology trivially implies the Lagrange p-stability.

A crucial assumption in the previous result is the existence of the weakly closed and invariant set S. The following is a device which helps to detect such a set S in the applications.

PROPOSITION 6.7. *Let X be a separable reflexive Banach space and assume that* (H.1), (H.2) *and* (H.5) *hold. Suppose that for every bounded set B a constant c_B exists such that $\|u(t, x)\| \leqslant c_B$ if $x \in B$ and $t \in [t_0, t_{\max})$. Then $t_{\max} = \infty$. Let K be a ball $K = \{x: \|x\| \leqslant a\}$ and let S be the weak closure of $\{u(t, x): x \in K, t \in [0, \infty)\}$. Then S is bounded and $x \in S$ implies $u(t, x) \in S$ for $t \geqslant 0$.*

Proof. $t_{\max} \equiv \infty$ follows from (6.2) and the a priori estimate. The very same estimate implies that S is bounded. Let $x \in S$ and let $t > 0$. Then x is a weak limit of a sequence $u(t_i, x_i)$ with $x_i \in K$. The cntinuity of u is the weak topology (Lemma 6.5) implies that $u(t, x)$ is the weak limit of $u(t_i + t, x_i)$. Hence $u(t, x) \in S$. This completes the proof.

EXAMPLE 6.3. Dynamic buckling of a beam. Consider the nonlinear beam equation

$$w_{tt} + \alpha w_{xxxx} - \left(\beta + k \int_0^l \left[\frac{\partial w}{\partial \xi}(\xi, t)\right]^2 d\xi\right) \frac{\partial^2 w}{\partial x^2} + \delta w_t = 0, \qquad \text{for } 0 < x < l,$$

$$w(0, t) = w_x(0, t) = w(l, t) = w_x(l, t) = 0 \qquad \text{(clamped ends)}, \tag{6.6}$$

$$w(x, 0) = w_0(x), \qquad w_t(x, 0) = w_1(t), \qquad \text{for} \quad 0 < x < l.$$

(Alternatively, we may replace the clamped ends boundary conditions by the hinged end conditions, i.e., $w(0, t) = w_{xx}(0, t) = w(l, t) = w_{xx}(l, t) = 0$.) Here α, k, δ are positive constants (compare with Ball [1973] and [1973a]). If we set

$$u = \begin{pmatrix} w \\ w_t \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -\alpha \dfrac{d^4}{dx^4} & -\delta \end{pmatrix}$$

and

$$f(u) = \begin{pmatrix} 0 \\ \left(\beta + k \int_0^l \left[\frac{\partial w}{\partial \xi}(\xi, t)\right]^2 d\xi\right) \frac{d^2 x}{dx^2} \end{pmatrix},$$

then (6.6) has the form of (6.1). An application of the Hille–Yosida–Phillips theorem will show that A is the infinitesimal generator of a C^0 semigroup e^{At} on the, respectively, Hilbert space $X_1 = H^2(0, l) \cap H_0^1(0, l) \times L^2(0, l)$ for the clamped ends boundary condition, and the Hilbert space $X_2 = H_0^2(0, l) \times L^2(0, l)$ for the hinged ends boundary conditions. Here both X_1 and X_2 are endowed with the "energy" inner product

$$\langle (w, w_t), (v, v_t) \rangle = \int_0^l (\alpha w_{xx} v_{xx} + w_t v_t)\, dx,$$

and then the norm is, clearly, $\|u\|^2 = \langle u, u \rangle$. A long, albeit straightforward, computation shows that f is locally Lipschitz continuous on X_1 and X_2. We now will show that f is actually weakly sequentially continuous. Let $u^n = (w^n, w_t^n)$ be a sequence in X_1 so that $(w^n, w_t^n) \to (w, w_t)$ weakly in X_1. Then $w^n \to w$ weakly in $H^2(0, l)$ and hence (since the injection of $H^2(0, l)$ into $H^1(0, l)$ is compact) $w^n \to w$ in $H_0^1(0, l)$. But then the expression

$$\int_0^l \left(\frac{dw^n}{d\xi}\right)^2 d\xi\, w_{xx}^n - \int_0^l \left(\frac{dw}{d\xi}\right)^2 d\xi\, w_{xx}$$
$$= \int_0^l \left[\left(\frac{dw^n}{d\xi}\right)^2 - \left(\frac{dw}{d\xi}\right)^2\right] d\xi\, w_{xx}^n + \int_0^l \left(\frac{dw}{d\xi}\right)^2 d\xi (w_{xx}^n - w_{xx})$$

converges weakly to 0 in $L^2(0, l)$, this since $w_{xx}^n \to w_{xx}$ weakly in $L^2(0, l)$ and $w_x^n \to w_x$ in $L^2(0, l)$. This then implies that $f(u^n)$ converges weakly to $f(u)$, i.e., f is weakly sequentially continuous. (The last term of the equality converges weakly but not strongly in L_2. Therefore f is not compact.)

Existence of a unique local weak solution is now guaranteed by (6.2) for both types of boundary conditions, on, respectively, X_1 and X_2.

Consider now the Liapunov functional

$$V(u) = \int_0^l \left(\frac{w_2^t}{2} + \alpha \frac{w_{xx}^2}{2} + \beta \frac{w_x^2}{2}\right) dx + \frac{k}{4}\left(\int_0^l w_x^2\, dx\right)^2$$

(compare with Ball [1973a, p. 119]). A formal calculation shows that for sufficiently smooth solutions of (6.6)

$$\dot{V}(u) = -\delta \int_0^l w_t^2\, dx. \tag{6.7}$$

Hence, by standard density arguments we have

$$V(u(t)) \leqslant V(u_0)$$

for any weak solution $u(t)$ with initial value u_0. We now note, first, that $V(u(t)) \geqslant 1/2(\|u(t)\|^2 - \beta^2/2k)$. Secondly, since in either case (of boundary conditions) $w \in H^2(0, l) \cap H_0^1(0, l)$ we have, via the Schwartz inequality,

$$\frac{\pi^2}{l^2} \int_0^l w^2\, dx \leqslant \int_0^l w_x^2\, dx \leqslant \left(\int_0^l w^2\, dx\right)^{1/2} \left(\int_0^l w_{xx}^2\, dx\right)^{1/2}.$$

This, in turn, yields

$$\frac{\pi^2}{l^2} \int_0^l w_x^2\, dx \leqslant \int_0^l w_{xx}^2\, dx.$$

The definition of V then gives the inequality

$$V(u(t)) \leqslant \frac{1}{2}\left(1 + \frac{l^2\beta}{\alpha\pi^2}\right) \|u\|^2 + \frac{k}{4} \frac{l^4}{\pi^4\alpha^2} \|u\|^4.$$

Combining the two inequalities of V implies

$$\|u(t)\|^2 \leqslant \frac{\beta^2}{2k} + \left(1 + \frac{l^2\beta}{\alpha\pi^2}\right) \|u_0\|^2 + \frac{k}{2} \frac{l^4}{\pi^4\alpha^2} \|u_0\|^4,$$

this for t in the maximal interval of existence. We conclude, therefore, that bounded sets stay bounded under the semiflow. Using Propositions 6.6 and 6.7 we have

LEMMA 6.8. *Equation* (6.6) *with either of the boundary conditions generate a semiflow on* X_1, *or, respectively*, X_2. *The corresponding maps* u: $[0, \infty) \times X_1 \to X_1$ and u: $[0, \infty) \times X_2 \to X_2$ *are continuous for both the weak and strong topologies. If* K *is a ball,* (*in* X_1 *or* X_2) *and* S *is the weak closure of* $\{u(t, x)$: $t \geqslant 0$, $x \in K$ *then* S *is positively invariant under the semiflow and the latter is Lagrange p-stable in the metrized weak topology on* S.

We turn now to examine the equilibrium points of the semiflow and their relation to the semiflow. We denote by ω_w and α_w the ω-limit set and the α-limit set with respect to the weak topology.

LEMMA 6.9. *The* ω-*limit set* ω_w *of each orbit generated by* (6.6) *and the* α-*limit set* α_w *of each bounded full orbit of* (6.6) (*in* X_1 *or* X_2 *according to* *the boundary conditions) are contained in the set of equilibrium points. If the latter is a discrete set then each orbit converges weakly to one equilibrium point as $t \to \infty$, and each bounded full orbit converges weakly to one equilibrium point as $t \to -\infty$.*

Proof. We consider the case of clamped ends. The case of hinged ends is analogous. From (6.7) we have

$$V(u(t)) + \delta \int_0^t \int_0^l w_t^2 \, dx \, ds \leqslant V(u_0),$$

whenever u_0 is an initial condition and $u(t)$ for $t \geqslant 0$ is the solution. As $\| u(t) \| \leqslant$ constant, $V(u(t))$ is bounded from below, hence

$$\int_0^t \int_0^l w_t^2 \, dx \, ds \leqslant \text{constant}. \tag{6.8}$$

This along any fixed bounded solution. Let $u(t, u_0)$ be a bounded orbit and fix an element ψ in either $\omega_w(u_0)$ or $\alpha_w(u_0)$. Suppose that $u(t_n, u_0) \to \psi$ weakly. From the weak continuity of the semiflow (Lemma 6.8) we have $u(\tau, u(t_n, u_0)) \to u(\tau, \psi)$ weakly for each $\tau > 0$. In particular $w_t(\tau, u(t_n, u_0)) \to w_t(\tau, \psi)$ weakly in $L^2(0, l)$. As the norm is a weakly lower semicontinuous function we have

$$\liminf_{n \to \infty} \| w_t(\tau, u(t_n, u_0)) \|^2_{L^2(0,l)} \geqslant \| w_t(\tau, \psi) \|^2_{L^2(0,l)},$$

for each $\tau > 0$. Fatou's lemma then implies

$$\liminf_n \int_0^t \| w_t(\tau, u(t_n, u_0)) \|^2_{L^2(0,l)} \, d\tau \geqslant \int_0^t \| w_t(\tau, \psi) \|^2_{L^2(0,\, l)} \, d\tau.$$

The equality $w_t(\tau, u(t_n, u_0)) = w_t(\tau + t_n, u_0)$, implied by the semigroup property of the flow, implies that the last inequality is actually

$$\liminf_n \int_{t_n}^{t_n + t} \| w_t(s, u_0) \|^2_{L^2(0,l)} \, ds \geqslant \int_0^t \| w_t(\tau, \psi) \|^2_{L^2(0,l)} \, d\tau.$$

From (6.8) the left-hand side of the last inequality is zero, so we get

$$\int_0^t \| w_t(\tau, \psi) \|^2_{L^2(0,l)} \, d\tau = 0,$$

this for every $t > 0$. Therefore $w_t(\tau, \psi)$ is identically zero. This clearly implies that ψ is an equilibrium point. The precompactness of the orbits in the weak topology implies that $\omega_w(u_0)$ is indeed nonempty and that each orbit converges weakly to its (weak) ω-limit set. Similarly, α_w of a bounded full orbit is nonempty and the orbit converges weakly to it as $t \to -\infty$. The last statement of the lemma is now clear.

We now turn to the issue of connecting orbits for (6.6) with the hinged ends boundary conditions. It is known that for $\lambda_n = n^2\pi^2/l^2$, $n = 1, 2, \ldots$, if $-\beta \leqslant \lambda_1$ the only equilibrium state is $\varphi_0(x) = 0$, if $\lambda_n < -\beta \leqslant \lambda_{n+1}$ there are $(2n+1)$ equilibrium states $-\varphi_n(x), \ldots, -\varphi_1(x)$, $\varphi_0(x)$, $\varphi_1(x), \ldots, \varphi_n(x)$, with $\varphi_0(x) = 0$. Furthermore, if $\lambda_1 < -\beta$ the equilibrium states $-\varphi_1(x)$ and $\varphi_1(x)$ are weakly asymptotically stable in X_2 and φ_0 and $\pm\varphi_k$, $k = 2, \ldots, n$, are unstable. These results can be found in Ball [1973].

THEOREM 6.10. *Consider Eq.* (6.6) *with the hinged boundary conditions. Suppose $\lambda_n < -\beta \leqslant \lambda_{n+1}$. Then there is a full orbit of the semiflow connecting in the weak topology φ_1 with one of the equilibrium points φ_0, $\pm\varphi_k$, $k = 2, \ldots, n$. If $n = 1$ then there is a full orbit connecting the equilibrium states φ_0 and φ_1 and an orbit connecting the equilibrium states φ_0 and $-\varphi_1$; the connection being in the weak topology.*

Proof. There is a finite number of equilibrium states. Let K be a ball in X_2 containing all of them. Let S be the weakly closed and positively invariant set constructed in Proposition 6.7. By Proposition 6.6 the semiflow u is Lagrange p-stable on S with respect to the weak topology. Set $B_1 = \{\varphi_1\}$ and $B_2 = \{-\varphi_1, \varphi_0, \pm\varphi_k, k = 2, \ldots, n\}$. An application of Theorem 5.2 yields an orbit in S weakly connecting B_1 and B_2. In view of the (weak) asymptotic stability of φ_1 this orbit converges weakly to φ_1, and its (weak) α-limit set intersects B_2. We claim that the α-limit set consists actually of one point in B_2 (which cannot be $-\varphi_1$ in view of the asymptotic stability). Indeed, the connecting orbit is a full orbit in the weakly compact set S, and Lemma 6.9 show that it converges weakly to one equilibrium point as $t \to -\infty$. (This replaces the Liapunov function argument that was used in Theorem 6.3. We cannot use the preceding Liapunov functional for this purpose since it is not continuous in the weak topology.) This completes the proof.

Remark. Results of this form were originally obtained by Ball [1973a]. Similar results may be stated for the clamped ends conditions, except that the expression for λ_n is more complicated. See Timoshenko and Gere [1961, p. 32], for a formula for λ_n.

REFERENCES

BALAKRISHNAN, A. V. [1976], "Applied Functional Analysis," Applications of Mathematics, Vol. 3, Springer-Verlag, New York.

BALL, J. M. [1973]. Stability theory of an extensible beam, *J. Differential Equations* **14**, 399–418.

BALL, J. M. [1973a], Saddle point analysis for an ordinary differential equation in a Banach space, and an application to dynamic buckling of a beam, *in* "Nonlinear Elasticity" (R. W. Dickey, Ed.), pp. 93–160, Academic Press, New York.

BALL, J. M. [1974], Continuity properties of nonlinear semigroups, *J. Funct. Anal.* **17**, 91–103.

BALL, J. M. [1978], On the asymptotic behavior of generalized processes with applications to nonlinear evolution equations, *J. Differential Equations* **27**, 224–265.

BALL, J. M. AND SLEMROD, M. [1979], Feedback stabilization of distributed semilinear control systems, *Appl. Math. Optim.* **5**, 169–179.

BHATIA, N. P. AND SZEGO, G. P. [1970], "Stability Theory of Dynamical Systems," Springer-Verlag, Berlin.

CALLEGARI, A. J. AND REISS, E. L. [1973], Nonlinear stability problems for the sine–Gordon equation, *J. Math. Phys.* **14**, 267–276.

CHAFEE, N. AND INFANTE, E. F. [1974], A bifurcation problem for a nonlinear partial differential equation of parabolic type, *Applicable Anal.* **4**, 17–37.

CHERNOFF, P. AND MARSDEN, J. [1970], On the continuity and smoothness of group actions, *Bull. Amer. Math. Soc.* **76**, 1044–1049.

CONLEY, C. C. [1976], Some aspects of the qualitative theory of differential equations, *in* "Dynamical Systems, an International Symposium," Vol. I (Cesari, Hale and LaSalle, Eds.), pp. 1–12, Academic Press, New York.

CONLEY, C. C. AND SMOLLER, J. A. [1975], The existence of heteroclinic orbits, and applications, *in* "Dynamical Systems, Theory and Applications" (J. Moser, Ed.), Lecture Notes in Physics No. 38, pp. 511–524, Springer-Verlag, New York.

CONLEY, C. C. AND SMOLLER, J. A. [1978], Isolated invariant sets of parametrized systems of differential equations, *in* "The Structure of Attractors in Dynamical Systems" (N. Markley, J. Martin and W. Perrizo, Eds.), Lecture Notes in Mathematics No. 668, pp. 30–47, Springer-Verlag, Berlin.

CONLON, J. G. [1980], A theorem in ordinary differential equations with an application to hyperbolic conservation laws, *Adv. in Math.* **35**, 1–18.

DICKEY, R. W. [1976], Stability theory for the damped sine–Gordon equation, *SIAM J. Appl. Math.* **30**, 248–262.

FOY, L. R. [1964], Steady state solutions of hyperbolic systems of conservation laws with viscosity terms, *Comm. Pure Appl. Math.* **17**, 177–181.

GELFAND, I. M. [1963], Some problems in the theory of quasilinear equations, *Amer. Math. Soc. Transl. Ser.* 2, **29**, 295–381.

GORDON, P. [1974], Path connecting elementary critical points of dynamical systems, *SIAM. J. Appl. Math.* **26**, 35–102.

HOWARD, L. N. AND KOPELL, N. [1975], Bifurcations and trajectories joining critical points, *Adv. in Math.* **18**, 306–358.

PAZY, A. [1974], "Semigroups of Linear Operators and Applications to Partial Differential Equations," Department of Mathematics, University of Maryland, Lecture Notes No. 10.

PAZY, A. [1975], A class of semi-linear equations of evolution, *Israel J. Math.* **20**, 23–36.

SELL, G. R. [1967], Nonautonomous differential equations and topological dynamics, I and II, *Trans. Amer. Math. Soc.* **127**, 241–283.

TIMOSHENKO, S. AND GERE, J. M. [1961], "Theory of Elastic Stability," 2nd ed. McGraw-Hill, New York.

WEBB, G. F. [1979], Compactness of bounded trajectories of dynamical systems in infinite dimensional spaces, *Proc. Roy. Soc. Edinburgh* **84**, 169–179.

WEBB, G. F. [1979a], A bifurcation problem for a nonlinear hyperbolic partial differential equation, *SIAM J. Math. Anal.* **10**, 922–932.