Adaptive Attitude Trajectory Tracking Control of Rigid Body Dynamics

Abdelaziz Benallegue, Yacine Chitour, Abdelhamid Tayebi

***Abstract*—This note presents a new adaptive attitude tracking controller for rigid body systems with unknown inertia. The proposed control scheme does not use any a priori attitude reconstruction; it explicitly incorporates, in the feedback loop, biased angular velocity measurements and body-frame measurements of some known inertial vectors. The proposed control scheme guarantees almost global asymptotic convergence of the attitude and angular velocity to their desired values.**

I. Introduction

The rigid body attitude tracking problem is still relevant, despite having been extensively studied in the literature for several decades. Several solutions have been proposed in the literature in the full state measurement case (*i.e.,* attitude and angular velocity available for feedback) using different attitude representations, see for instance, [1]–[3]. Since there is no sensor that directly measures the orientation, the explicit use of the attitude in the control law calls for efficient attitude estimation algorithms (observers) that reconstruct the attitude from the measurements provided by some appropriate sensors, such as inertial measurements units (IMUs) typically including a gyroscope an accelerometer and a magnetometer. The attitude can be determined using either static reconstruction algorithms [4] which are vulnerable to measurement noise or dynamic attitude estimation algorithms such as Kalman-type filters [5] and nonlinear-complimentary filters [6]. Consequently, it is interesting to design control schemes that bypass the attitude reconstruction through direct incorporation of the available measurements or their filtered versions in the control law. In this case, we don't have to worry about stability issues of the combination of separately designed attitude observer and attitude controller.

The attitude tracking problem with biased angular velocity measurements has been treated in [7] assuming that the attitude is available for feedback. In [8], the attitude control problem has been addressed in the presence of unknown angular velocity bias, using IMU measurements in the state feedback, assuming that the rigid body inertia is known. In [9], [10], for instance, the attitude stabilization problem has been solved without attitude and angular velocity measurements and without the knowledge of the inertia matrix. The proposed control schemes rely directly on measurements in the body frame of some known inertial vectors. The extension to the case of trajectory tracking remains an open problem. In [11], the attitude tracking problem using IMU measurements, with unknown bias and unknown inertia has been addressed. Two control laws were presented in [11]; the first one considers only the case of biased angular velocity measurements, and the second one is an extension to the case of unknown inertia matrix. These control laws are quaternion based and seem to suffer from the unwinding phenomenon[1]. Note also that, the number of parameters to be estimated is 24 and some restrictive conditions on gains are imposed. Moreover, the non-attractiveness of the undesired equilibria has not been proven.

In the present work, we aim to solve the attitude tracking problem in the case where 1) the rigid body inertia is unknown, 2) the measured angular velocity is biased with an unknown constant bias, and 3) the attitude is not directly available for measurement. To handle simultaneously the three above mentioned constraints, we derive an adaptive control scheme that relies only on biased angular velocity measurements and body-frame measurements of some known inertial vectors. The control design relies on a transformation that allows to linearly parameterize some terms in the system's vector field with respect to the unknown inertia matrix [9], [12]–[14]. Thereafter, an almost global adaptive attitude tracking control scheme, involving 21 adaptive parameter, independent of the attitude representation, is derived. Moreover, as it will be shown later, the unwinding phenomenon is avoided in our approach.

II. Background

A. Preliminaries

The quaternion set $\mathbb{Q}$ is a four-dimensional vector space over the reals, which forms a group with the quaternion multiplication denoted by "$\odot$", which is distributive, associative but not commutative. The multiplication of two quaternions $P = (p_0, p)$ and $Q = (q_0, q)$ is defined as

$$P \odot Q = (p_0 q_0 - p^T q\ ,\ p_0 q + q_0 p + p \times q), \tag{1}$$

and has the quaternion $(1, \mathbf{0})$ as the identity element. Note that, for a given quaternion $Q = (q_0, q)$, one has $Q \odot Q^{-1} = Q^{-1} \odot Q = (1, \mathbf{0})$, where $Q^{-1} = \frac{(q_0, -q)}{\|Q\|^2}$.
Note that in the case where $Q = (q_0, q)$ is a unit-quaternion, the inverse is given by $Q^{-1} = (q_0, -q)$.
The unit quaternion $Q = (q_0, q)$, composed of a scalar component $q_0 \in \mathbb{R}$ and a vector component $q \in \mathbb{R}^3$, represents the orientation of the inertial frame $\mathcal{I}$ with respect to the body-attached frame $\mathcal{B}$, and are subject to the constraint $q_0^2 + q^T q = 1$. The rotation matrix, related to the unit-quaternion Q, that brings the inertial frame into the body-attached frame, can be obtained through the Rodrigues formula $R = \mathcal{R}(Q)$ with the mapping $\mathcal{R} : \mathbb{S}^3 \to SO(3)$ is defined as

$$\begin{aligned} \mathcal{R}(Q) &= I_3 + 2q_0 S(q) + 2S^2(q) \\ &= (q_0^2 - q^T q) I_3 + 2qq^T + 2q_0 S(q) \end{aligned} \tag{2}$$

where I_3 is the 3-by-3 identity matrix and $S(x)$ is the skew-symmetric matrix associated to the vector $x \in \mathbb{R}^3$ such that $S(x)V = x \times V$ for any vector $V \in \mathbb{R}^3$, where $\times$ denotes the vector cross product of $\mathbb{R}^3$. Note that $\mathcal{R}(Q) = \mathcal{R}(-Q)$ for every $Q \in \mathbb{Q}$ and $\mathcal{R}$ defines a two-sheet covering of $SO(3)$ by $\mathbb{Q}$, *i.e.,* for every $R \in SO(3)$ there exist exactly two distinct quaternions

A. Benallegue is with Laboratoire d'Ingénierie des Systèmes de Versailles, France and JRL-AIST (Joint Robotics Laboratory), Tsukuba, Ibaraki, Japan. Y. Chitour is with Université Paris-Sud, CentraleSupélec, CNRS, France. A. Tayebi is with the Department of Electrical Engineering, Lakehead University, Thunder Bay, Ontario, Canada. benalleg@lisv.uvsq.fr, Yacine.CHITOUR@lss.supelec.fr, atayebi@lakeheadu.ca

[1]This undesirable phenomena often occurs with the use of quaternion representation, where some trajectories, for some initial conditions close to the desired attitude equilibrium, can undergo an unnecessary homoclinic-like orbit [2].

verifying $\mathcal{R}(Q) = R$. As a consequence every vector field f defined on $\mathbb{Q}$ so that $f(-Q) = -f(Q)$ for every $Q \in \mathbb{Q}$ defines a vector field $\tilde{f}$ on $SO(3)$.

Throughout this paper, we will denote by $(0, X)$ the quaternion associated to the three-dimensional vector X. A vector $x_{\mathcal{I}}$ expressed in the inertial frame $\mathcal{I}$ can be expressed in the body frame $\mathcal{B}$ by $x_{\mathcal{B}} = R^T x_{\mathcal{I}}$ or equivalently in terms of unit-quaternion as $(0, x_{\mathcal{B}}) = Q^{-1} \odot (0, x_{\mathcal{I}}) \odot Q$, where Q is the unit-quaternion associated to R by (2).

Let us define the following mapping $vect : \mathbb{R}^{n\times n} \to \mathbb{R}^{n^2}$, such that for a given matrix $A \in \mathbb{R}^{n\times n}$, we associate the vector $vect(A) = [v_1, \dots, v_n]^T$, where v_i, $i = 1, \dots, n$, are the row vectors of the matrix A.

B. Equations of motion

In this work, we consider a rigid body whose rotational dynamics are governed by

$$\Sigma_R : \begin{cases} \dot{Q} = \frac{1}{2} Q \odot (0, \omega), \\ I_b \dot{\omega} = \tau - S(\omega) I_b \omega, \end{cases} \tag{3}$$

where ω is the angular velocity of the rigid body expressed in the body-attached frame $\mathcal{B}$, τ is the external torque applied to the system expressed in $\mathcal{B}$ and $I_b \in \mathbb{R}^{3\times 3}$ is a symmetric positive definite constant inertia matrix (assumed to be unknown) of the rigid body with respect to $\mathcal{B}$ of the form

$$I_b = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{bmatrix}.$$

III. Main Results

A. Problem statement

Let us define the desired attitude trajectory in terms of the rotation matrix $R_d(t)$ governed by the following dynamics,

$$\dot{R}_d = R_d S(\omega_d),$$

with $\omega_d(t)$ being the desired angular velocity vector.

An equivalent desired unit-quaternion $Q_d(t)$ is defined as $R_d(t) = \mathcal{R}(Q_d(t))$. Its dynamics are governed by

$$\dot{Q}_d = \frac{1}{2} Q_d \odot (0, \omega_d). \tag{4}$$

The following assumptions are used throughout the paper:

- A1. The rigid body is equipped with sensors that provide measurements (in the body-attached frame) of constant and known inertial vectors $r_i \in \mathbb{R}^3$, $i = 1, \dots, n \geq 2$. At least two vectors, among the n inertial vectors, are non-collinear. The vector measurements in the body-attached frame are denoted by $b_i \in \mathbb{R}^3$, $i = 1, \dots, n$. The vectors r_i and b_i are related by $b_i = R^T r_i$.
- A2. The attitude (Q or R) is unknown (*i.e.,* unavailable for feedback).
- A3. The measured angular velocity is assumed to be biased, so that the relation between the actual and measured velocities is given by
$$\omega = \omega_m + \delta,$$
where δ is the unknown constant bias, ω and ω_m are the actual and the measured velocity vectors respectively.
- A4. The inertia matrix I_b is assumed to be unknown.

Our objective is to design a control input τ guaranteeing Almost Global Asymptotic Convergence (AGAC) of the body attitude and angular velocity to their desired values, under the above assumptions. This means that there exists an equilibrium point Eq (in the appropriate state space) such that, for almost every initial condition (with respect to the Lebesgue measure in the state space), the corresponding trajectory of the closed loop system converges to Eq.

B. Linearly parameterized model for the control

Let us consider Assumptions A3 and A4 and define the following parameters

$$\begin{aligned} \theta_1 &= \delta \in \mathbb{R}^3, \\ \theta_2 &= S(\delta) I_b \delta \in \mathbb{R}^3, \\ \theta_3 &= (I_{11}, I_{22}, I_{33}, I_{23}, I_{13}, I_{12})^T \in \mathbb{R}^6, \\ \theta_4 &= vect(S(\delta) I_b - S(I_b \delta)) \in \mathbb{R}^9. \end{aligned}$$

Using the second equation of (3), we can write the following

$$I_b(\dot{\omega} - \dot{\omega}_d) = -(S(\omega_m) F_1(\omega_m) + F_1(\dot{\omega}_d))\theta_3 - F_2(\omega_m)\theta_4 - \theta_2 + \tau, \tag{5}$$

where $F_1(\omega)$ is defined as

$$F_1(\omega) = \begin{bmatrix} \omega_1 & 0 & 0 & 0 & \omega_3 & \omega_2 \\ 0 & \omega_2 & 0 & \omega_3 & 0 & \omega_1 \\ 0 & 0 & \omega_3 & \omega_2 & \omega_1 & 0 \end{bmatrix}$$

and $F_2(\omega)$ as

$$F_2(\omega) = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_1 & \omega_2 & \omega_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega_1 & \omega_2 & \omega_3 \end{bmatrix}.$$

The model given by Equation (5) can be written in a linear parameterization form as

$$I_b(\dot{\omega} - \dot{\omega}_d) = -G(\omega_m, \dot{\omega}_d)\Theta + \tau, \tag{6}$$

with

$$G(\omega_m, \dot{\omega}_d) = \begin{bmatrix} I_3 & S(\omega_m) F_1(\omega_m) + F_1(\dot{\omega}_d) & F_2(\omega_m) \end{bmatrix} \in \mathbb{R}^{3\times 18}$$

and

$$\Theta^T = \begin{bmatrix} \theta_2^T & \theta_3^T & \theta_4^T \end{bmatrix} \in \mathbb{R}^{18}.$$

We also assume that ω_d verifies the following additional assumption.

(A_δ): The desired angular velocity vector ω_d and its six first derivatives are bounded. Moreover, for every $\delta \in \mathbb{R}$, define

$$H_\delta(\omega_d) = \begin{bmatrix} S(\omega_d - \delta) F_1(\omega_d - \delta) + F_1(\dot{\omega}_d) & F_2(\omega_d - \delta) \end{bmatrix},$$

which is an 3×15 matrix-valued function of the time t and the 15×15 matrix-valued function $J_\delta(\omega_d)$ given by

$$J_\delta(\omega_d) = \begin{bmatrix} \frac{d}{dt}[H_\delta(\omega_d))] \\ \vdots \\ \frac{d^5}{dt^5}[H_\delta(\omega_d))] \end{bmatrix}.$$

Then, the following holds true

$$\limsup_{t\to\infty} |\det J_\delta(\omega_d)| > 0. \tag{7}$$

Assumption (A_δ) is tailored to insure the following convergence result.

Lemma 1. *Let and $\delta \in \mathbb{R}$, $\omega_d : [0, \infty) \to \mathbb{R}^3$ satisfying Assumption (A_δ). Assume that there exists a measurable function $\Psi : [0, \infty) \to \mathbb{R}^{15}$ such that $\lim_{t\to\infty} J_\delta(\omega_d)\Psi(t) = 0$. Then $\liminf_{t\to\infty} \Psi(t) = 0$.*

Proof: Assumption (A_δ) implies that $\liminf_{t\to\infty} \|J_\delta^{-1}\|$ is finite and since $\Psi = J_\delta^{-1}(J_\delta \Psi)$ one immediately deduces the conclusion. ∎

Remark 1. *Assumption* (H_δ) *can be seen as a persistence of excitation condition for the biased desired angular velocity* $\omega_d - \delta$ *together with its first six time derivatives.*

C. Control design

In the sequel, we assume that Assumptions A1 to A4 hold true, as well as Assumption (A_δ). Define n vectors b_i^d and n vectors $\hat{b}_i$ corresponding to the desired and estimated vectors such that $b_i^d = R_d^T r_i$ and $\hat{b}_i = \hat{R}^T r_i$, for $i = 1, \ldots, n$.

According to the model given by (6), we propose the following adaptive control law

$$\tau = G(\omega_m, \dot{\omega}_d - \dot{\hat{\theta}}_1)\hat{\Theta} + z_\gamma - \alpha\bar{\omega}, \tag{8}$$

with

$$\bar{\omega} = \omega_m + \hat{\theta}_1 - \omega_d,$$

$$z_\gamma = \sum_{i=1}^{n} \gamma_i S(b_i^d) b_i;\ z_\rho = \sum_{i=1}^{n} \rho_i S(\hat{b}_i) b_i,$$

where $\alpha > 0$, $\gamma_i > 0$ and $\rho_i > 0$ are constant scalar gains.

The attitude estimator is given by

$$\begin{array}{l} \dot{\hat{Q}} = \frac{1}{2}\hat{Q} \odot (0, \hat{\omega}), \\ \hat{\omega} = \omega_m + \hat{\theta}_1 - z_\rho. \end{array}$$

The adaptation algorithms are given by

$$\begin{array}{l} \dot{\hat{\theta}}_1 = \Gamma_1 Proj(-(z_\gamma + z_\rho), \hat{\theta}_1, \theta_m), \\ \dot{\hat{\Theta}} = \Gamma_2 Proj(-G(\omega_m, \dot{\omega}_d - \dot{\hat{\theta}}_1)^T \bar{\omega}, \hat{\Theta}, \Theta_m), \end{array}$$

with $||\hat{\theta}_1(0)|| \leq \theta_m$ and $||\hat{\Theta}(0)|| \leq \Theta_m$. The matrices Γ_1 and Γ_2 are real symmetric positive definite. The positive parameters θ_m and Θ_m are the upper bounds of θ_1 and Θ, *i.e.,* $||\theta_1|| \leq \theta_m$, $||\Theta|| \leq \Theta_m$. For n a positive integer, the projection operator $Proj$ is defined on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$ as follows:

$$Proj(x, \hat{y}, y_0) = x - \frac{\eta_1 \eta_2}{4(\epsilon^2 + 2\epsilon y_0)^{n+1} y_0^2}\hat{y}, \tag{9}$$

with

$$\eta_1 = \begin{cases} (\hat{y}^T\hat{y} - y_0^2)^{n+1} & \text{if } \hat{y}^T\hat{y} > y_0^2, \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

$$\eta_2 = 0.5\hat{y}^T x + \sqrt{(0.5\hat{y}^T x)^2 + \delta^2}, \tag{11}$$

where ϵ and δ are arbitrary positive constants. Let $\bar{y}$ be a constant vector in $B_{y_0} = \{y \in \mathbb{R}^n \mid ||y|| \leq y_0\}$, $\hat{y}(0) \in B_{y_0}$ and $\tilde{y} = \bar{y} - \hat{y}$. Consider the adaptation algorithm $\dot{\hat{y}} = Proj(x, \hat{y}, y_0)$, then the following properties hold [15] for every $t \geq 0$:

P_1) $||\hat{y}(t)|| \leq y_0 + \epsilon$,
P_2) $-\tilde{y}(t)^T Proj(-x, \hat{y}(t), y_0) \leq x^T \tilde{y}(t)$.

D. Convergence analysis

Let us define the estimation error $\bar{R} = R\hat{R}^T$ and the tracking error $\tilde{R} = RR_d^T$ of the attitude that correspond to the unit quaternion errors $\bar{Q} = Q \odot \hat{Q}^{-1} \equiv (\bar{q}_0, \bar{q})$ and $\tilde{Q} = Q \odot Q_d^{-1} \equiv (\tilde{q}_0, \tilde{q})$ respectively. The estimation error dynamics are given by

$$\dot{\bar{Q}} = \begin{bmatrix} \dot{\bar{q}}_0 \\ \dot{\bar{q}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\bar{q}^T\hat{R}(\omega - \hat{\omega}) \\ \frac{1}{2}(\bar{q}_0 I + S(\bar{q}))\hat{R}(\omega - \hat{\omega}) \end{bmatrix}, \tag{12}$$

with

$$\omega - \hat{\omega} = z_\rho + \tilde{\theta}_1.$$

The tracking error dynamics are given by

$$\dot{\tilde{Q}} = \begin{bmatrix} \dot{\tilde{q}}_0 \\ \dot{\tilde{q}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\tilde{q}^T R_d(\omega - \omega_d) \\ \frac{1}{2}(\tilde{q}_0 I + S(\tilde{q}))R_d(\omega - \omega_d) \end{bmatrix}, \tag{13}$$

where

$$\omega - \omega_d = \bar{\omega} + \tilde{\theta}_1,$$

with $\tilde{\theta}_1 = \theta_1 - \hat{\theta}_1$. Before stating our main results, we recall the following useful lemma given in [16] that will be used throughout the paper.

Lemma 2. *Assume that there are* n *vectors* b_i, $i = 1, \ldots, n$ *measured in the body attached frame, corresponding to* n *known inertial vectors* r_i, $i = 1, \ldots, n$. *Assume that the constant parameters* γ_i *and* ρ_i *are strictly positive and at least two vectors among the* r_i *vectors are non-collinear. Then, the following properties hold*

1) *The vectors* z_γ *and* z_ρ *verify*

$$z_\gamma \equiv \sum_{i=1}^{n} \gamma_i S(b_i^d) b_i = -2R_d^T(\tilde{q}_0 I - S(\tilde{q}))W_\gamma \tilde{q}, \tag{14}$$

$$z_\rho \equiv \sum_{i=1}^{n} \rho_i S(\hat{b}_i) b_i = -2\hat{R}^T(\bar{q}_0 I - S(\bar{q}))W_\rho \bar{q}, \tag{15}$$

where the matrices $W_\gamma = -\sum_{i=1}^n \gamma_i S(r_i)^2$ *and* $W_\rho = -\sum_{i=1}^n \rho_i S(r_i)^2$ *are real symmetric and positive definite. If the gains* γ_i, ρ_i, $i = 1, \ldots, n$, *are chosen such that* W_γ *and* W_ρ *have two by two distinct eigenvalues, the following holds true.*

2) $z_\gamma = 0$ *is equivalent to* $(\tilde{q}_0 = 0, \tilde{q} = v_\gamma)$ *or* $(\tilde{q}_0 = \pm 1, \tilde{q} = 0)$, *where* v_γ *is a unit eigenvector of* W_γ.
3) $z_\rho = 0$ *is equivalent to* $(\bar{q}_0 = 0, \bar{q} = v_\rho)$ *or* $(\bar{q}_0 = \pm 1, \bar{q} = 0)$, *where* v_ρ *is a unit eigenvector of* W_ρ.

The closed loop attitude error dynamics are given by

$$\dot{\tilde{Q}} = \begin{bmatrix} \dot{\tilde{q}}_0 \\ \dot{\tilde{q}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\tilde{q}^T R_d(\bar{\omega} + \tilde{\theta}_1) \\ \frac{1}{2}(\tilde{q}_0 I + S(\tilde{q}))R_d(\bar{\omega} + \tilde{\theta}_1) \end{bmatrix}, \tag{16}$$

$$\dot{\bar{Q}} = \begin{bmatrix} \dot{\bar{q}}_0 \\ \dot{\bar{q}} \end{bmatrix} = \begin{bmatrix} \bar{q}_0\bar{q}^T W_\rho \bar{q} - \frac{1}{2}\bar{q}^T\hat{R}\tilde{\theta}_1 \\ -(I - \bar{q}\bar{q}^T)W_\rho\bar{q} + \frac{1}{2}(\bar{q}_0 I + S(\bar{q}))\hat{R}\tilde{\theta}_1 \end{bmatrix}, \tag{17}$$

$$I_b\dot{\bar{\omega}} = -\alpha\bar{\omega} + z_\gamma - G(\omega_m, \dot{\omega}_d - \dot{\hat{\theta}}_1)\tilde{\Theta}, \tag{18}$$

$$\dot{\tilde{\theta}}_1 = -\Gamma_1 Proj(-(z_\gamma + z_\rho), \hat{\theta}_1, \theta_m), \tag{19}$$

$$\dot{\tilde{\Theta}} = -\Gamma_2 Proj(-G(\omega_m, \dot{\omega}_d - \dot{\hat{\theta}}_1)^T\bar{\omega}, \hat{\Theta}, \Theta_m). \tag{20}$$

where $\tilde{\theta}_1 = \theta_1 - \hat{\theta}_1$ and $\tilde{\Theta} = \Theta - \hat{\Theta} = [\ \tilde{\theta}_2^T \quad \tilde{\theta}_3^T \quad \tilde{\theta}_4^T\ ]^T$. Note that these dynamics are non-autonomous. Define $X = (\tilde{Q}, \bar{Q}, \bar{\omega}, \tilde{\theta}_1, \tilde{\Theta})$ in the state space $\mathcal{X} := \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{18}$. Note that $\mathcal{X}$ has dimension 30. The above dynamics can be written as

$$\dot{X} = f(X, R_d(t), \omega_d(t), \dot{\omega}_d(t)), \tag{21}$$

where f is a time-varying vector field defined on $\mathcal{X}$.

Let us define the following Lyapunov function candidate

$$\begin{array}{rcl} V & = & 2\tilde{q}^T W_\gamma \tilde{q} + 2\bar{q}^T W_\rho \bar{q} + \frac{1}{2}\bar{\omega}^T I_b \bar{\omega} \\ & + & \frac{1}{2}\tilde{\theta}_1^T \Gamma_1^{-1}\tilde{\theta}_1 + \frac{1}{2}\tilde{\Theta}^T\Gamma_2^{-1}\tilde{\Theta} \end{array}. \tag{22}$$

The time derivative of (22), in view of (16), (17), (18) and property P_2 of the projection operator, is given by

$$\dot{V} \leq -\alpha\bar{\omega}^T\bar{\omega} - z_\rho^T z_\rho. \tag{23}$$

According to (23), V is non-increasing along trajectories of the dynamical system, implying that $\bar{\omega}$, $\bar{q}$, $\tilde{q}$, $\tilde{\theta}_1$ and $\tilde{\Theta}$ are bounded and V converges to a non negative limit. One checks easily that $\ddot{V}$ is bounded for every trajectory of the system, implying that $\dot{V}$ is uniformly continuous, hence that $\dot{V} \to 0$. On the other hand, the time derivative of z_ρ and z_γ are given by

$$\dot{z}_\rho = -S(\omega - \tilde{\theta}_1)z_\rho + \left(\sum_{i=1}^{n} \rho_i S(\hat{b}_i)S(b_i)\right)(z_\rho + \tilde{\theta}_1),$$

$$\dot{z}_\gamma = -S(\omega_d)z_\gamma + \left(\sum_{i=1}^{n} \rho_i S(b_i^d)S(b_i)\right)(\omega - \omega_d),$$

which are clearly bounded. Since, for any given trajectory of (21), $\ddot{V}$ is bounded, one deduces that $\bar{\omega} \to 0$ and $z_\rho \to 0$ as time tends to infinity. Using Lemma 2, one sees that $(\bar{q}_0 I - S(\bar{q}))W_\rho \bar{q} \to 0$ as time tends to infinity, which implies that either $(\bar{q}_0, \bar{q}) \to (\pm 1, 0)$ or $(\bar{q}_0, \bar{q}) \to (0, v_\rho)$ where v_ρ is a unit eigenvector of W_ρ. Since $\ddot{\bar{q}}_0$ and $\ddot{\bar{q}}$ are bounded, (17) implies that $\tilde{\theta}_1 \to 0$. Therefore, using (19), and the fact that $z_\rho \to 0$ and $\dot{z}_\gamma$ is bounded, it can be concluded that $\dot{\tilde{\theta}}_1 \to 0$ and therefore $z_\gamma \to 0$. Using Lemma 2, one concludes that either $(\tilde{q}_0, \tilde{q}) \to (\pm 1, 0)$ or $(\tilde{q}_0, \tilde{q}) \to (0, v_\gamma)$ where v_γ is a unit eigenvector of W_γ.

Lemma 3. *With the notations above and assuming that ω_d verifies Assumption (A_δ), one deduces that $\tilde{\Theta}$ converges to zero as t tends to infinity.*

Proof: Notice that $\tilde{\Theta}^T \Gamma_2^{-1} \tilde{\Theta}$ admits a limit as t tends to infinity and therefore, to prove the lemma it is enough to prove that $\liminf_{t\to\infty} \|\tilde{\Theta}(t)\| = 0$.

Define

$$G_d := G(\omega_d - \theta_1, \dot{\omega}_d) = [\ I_3 \quad H_\delta(\omega_d - \theta_1)\ ]. \tag{24}$$

Then one has

$$G(\omega_m, \dot{\omega}_d - \dot{\tilde{\theta}}_1) - G_d \to 0,$$

as t tends to infinity. Since $\bar{\omega}$ tends to zero and $\ddot{\bar{\omega}}$ is bounded, one deduces that $\dot{\bar{\omega}}$ tends to zero at t tends to infinity and therefore so does $G_d\tilde{\Theta}$. By an easy induction argument, it can be concluded that, for $0 \le k \le 5$,

$$\lim_{t\to\infty} \frac{d^{(k)}}{dt^k} G_d \tilde{\Theta} = 0.$$

Setting $\Psi(t) := (\tilde{\theta}_3^T, \tilde{\theta}_4^T)$, one has $G_d\tilde{\Theta} = \tilde{\theta}_2 + H_\delta(\omega_d - \theta_1)\Psi(t)$ and $\lim_{t\to\infty} J_\delta(\omega_d)\Psi(t) = 0$, where $J_\delta(\omega_d)$ has been defined in Assumption (A_δ). According to Lemma 1, this implies that $\liminf_{t\to\infty} \|\tilde{\Theta}(t)\| = 0$ and Lemma 3 is proved. ∎

Using Lemma 2 and Lemma 3, we have proved that trajectories of (21) converge to the following subsets of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{18}$ given by

$$\begin{aligned}
\widetilde{\Omega}_1 &= \{((\pm 1, 0), (\pm 1, 0), 0, 0, 0)\},\\
\widetilde{\Omega}_2 &= \{((\pm 1, 0), (0, \pm v_{j\rho}), 0, 0, 0),\ j = 1, 2, 3\},\\
\widetilde{\Omega}_3 &= \{((0, \pm v_{i\gamma}), (\pm 1, 0), 0, 0, 0),\ i = 1, 2, 3\},\\
\widetilde{\Omega}_4 &= \{((0, \pm v_{i\gamma}), (0, \pm v_{j\rho}), 0, 0, 0),\ i = 1, 2, 3;\ j = 1, 2, 3\},
\end{aligned}$$

with $v_{i\gamma}$ and $v_{j\rho}$ are unit eigenvectors of W_γ and W_ρ respectively for $1 \le i \le j \le 3$.

Let X_{eq} be an element in some $\widetilde{\Omega}_i$ and write a trajectory as

$$x(\cdot) = X_{eq} + Z(\cdot)$$

where

$$Z = (Z_{\tilde{q}_0}, Z_{\tilde{q}}, Z_{\bar{q}_0}, Z_{\bar{q}}, Z_{\bar{\omega}}, Z_{\tilde{\theta}_1}, Z_{\tilde{\Theta}})^T.$$

First note that $\bar{\omega} = Z_{\bar{\omega}}$ and we set $z_\gamma = Z_\gamma$ and $z_\rho = Z_\rho$ with

$$Z_\gamma = -2R_d^T \left[\lambda_\gamma Z_{\tilde{q}_0} v_\gamma - S(v_\gamma)W_\gamma Z_{\tilde{q}} + Z_{\tilde{q}_0} W_\gamma Z_{\tilde{q}} - S(Z_{\tilde{q}})W_\gamma Z_{\tilde{q}}\right],$$

if $\tilde{Q} = (0, v_\gamma)$, and

$$Z_\gamma - 2R_d^T \left[W_\gamma Z_{\tilde{q}} + (Z_{\tilde{q}_0} I_3 - S(Z_{\tilde{q}}))\, W_\gamma Z_{\tilde{q}}\right],$$

if $\tilde{Q} = (1, 0)$ and

$$Z_\rho = \begin{array}{l} -2\hat{R}^T[\lambda_\rho(Z_{\bar{q}_0} I_3 - S(Z_{\bar{q}}))v_\rho - S(v_\rho)W_\rho Z_{\bar{q}} \\ +(Z_{\bar{q}_0} I_3 - S(Z_{\bar{q}}))W_\rho Z_{\bar{q}}], \end{array}$$

if $\bar{Q} = (0, v_\rho)$, and

$$Z_\rho = -2\hat{R}^T \left(W_\rho Z_{\bar{q}} + (Z_{\bar{q}_0} I_3 - S(Z_{\bar{q}}))\, W_\rho Z_{\bar{q}}\right),$$

if $\bar{Q} = (1, 0)$.

If $\tilde{Q}_{eq} = (0, v_\gamma)$, the corresponding quaternion constraint yields

$$Z_{\tilde{q}_0}^2 + \| v_\gamma + Z_{\tilde{q}} \|^2 = 1, \tag{25}$$

and then

$$\left(Z_{\tilde{q}_0}^2 + \| Z_{\tilde{q}} \|^2\right) + 2v_\gamma^T Z_{\tilde{q}} = 0. \tag{26}$$

Similarly, if $\bar{Q}_{eq} = (0, v_\rho)$, one deduces from the corresponding quaternion constraint that

$$\left(Z_{\bar{q}_0}^2 + \| Z_{\bar{q}} \|^2\right) + 2v_\rho^T Z_{\bar{q}} = 0. \tag{27}$$

If $\tilde{Q}_{eq} = (1, 0)$, the corresponding quaternion constraint yields

$$\left(Z_{\tilde{q}_0}^2 + \| Z_{\tilde{q}} \|^2\right) + 2Z_{\tilde{q}_0} = 0, \tag{28}$$

and similarly, if $\bar{Q}_{eq} = (1, 0)$, one deduces from the corresponding quaternion constraint that

$$\left(Z_{\bar{q}_0}^2 + \| Z_{\bar{q}} \|^2\right) + 2Z_{\bar{q}_0} = 0. \tag{29}$$

We now state in the following theorem the convergence result obtained for the closed loop attitude error dynamics.

Theorem 1. *Consider the closed loop attitude error dynamics given by Eqs.* (16)-(20)*, where the desired angular velocity vector ω_d verifies Assumption (A_δ) and the gain matrices W_γ and W_ρ have two by two distinct eigenvalues and verifie the following assumption*

(W): one has

$$4\lambda_{min}(M) - TrM - \alpha Tr I_b^{-1} > 0, \text{ for } M \in \{W_\gamma, W_\rho\}.$$

Then, for almost any initial condition $X_0 \in \mathcal{X}$, the corresponding trajectory of (21) *converges to a point of Ω_1.*

Proof: We will actually prove that the points of the state space converging to the equilibrium points in $\widetilde{\Omega}_2$, $\widetilde{\Omega}_3$ and $\widetilde{\Omega}_4$, form a set of measure zero.

Consider, for instance, a point in $\tilde{\Omega}_2$, let say $X_{eq} = ((1, 0), (0, v_\rho), 0, 0, 0)$ where v_ρ is a unit-length eigenvector of W_ρ. If $x \in \mathbb{R}^3$, we use $x^\perp$ to denote the vector in the two-dimensional plane $v_\rho^\perp$ given by $x^\perp = x - (v_\rho^T x)v_\rho$ and $W_\rho^\perp$ the restriction of W_ρ to $v_\rho^\perp$. Recall first that the dimension of the state space $\mathcal{X}$ is equal to 30 and, by using the equations (27) and (28), one deduces that

$$Z_{\tilde{q}_0} = -\frac{\| Z_{\tilde{q}} \|^2}{1 + \sqrt{1 - \| Z_{\tilde{q}} \|^2}}, \quad v_\rho^T Z_{\bar{q}} = -\frac{Z_{\bar{q}_0}^2 + \| Z_{\bar{q}}^\perp \|^2}{1 + \sqrt{1 - (Z_{\bar{q}_0}^2 + \| Z_{\bar{q}}^\perp \|^2)}}.$$

The reduced variable Z_{red} is given by

$$Z_{red} = (Z_{\tilde{q}}, Z_{\bar{q}_0}, Z_{\bar{q}}^\perp, Z_{\bar{\omega}}, Z_{\tilde{\theta}_1}, Z_{\tilde{\Theta}})^T,$$

belongs to a smooth manifold $\mathcal{M}_{eq}$ of dimension 30. Fix a neighborhood $\mathcal{N}$ of the origin for the reduced variable so that the projection operators are equal to the corresponding identity operators in $\mathcal{N}$. Then, as long as the corresponding trajectory lies in $\mathcal{N}$ it obeys to the following dynamics

$$\dot{Z}_{red} = A(t)Z_{red} + F(t, Z_{red}), \tag{30}$$

where $A(t)$ and $F(t, Z_{red})$ are given in the appendix.

We decomposed the error dynamics in (30) into a linear part and a super linear one, *i.e,* there exists a positive constant C_0 such that F verifies an estimate of the type

$$\parallel F(t, Z_{red}) \parallel \leq C_0 \parallel Z_{red} \parallel^2, \quad |\text{div}F(t, Z_{red})| \leq C_0 \parallel Z_{red} \parallel, \tag{31}$$

for every $(t, Z_{red}) \in \mathbb{R}_+ \times \mathcal{M}_{eq}$. Note also that the time-varying matrix $A(\cdot)$ does not depend on Z_{red} and its trace is constant and equal to

$$\text{Tr}A(t) \equiv 4\lambda_\rho - \text{Tr}W_\rho - \alpha\text{Tr}I_b^{-1} := \xi. \tag{32}$$

Since the matrix W_ρ verifies Assumption (W), the right-hand side ξ of (32) is strictly positive. Assume now that conclusion of the theorem does not hold true and more particularly, that there exist a measurable subset set J of $\mathcal{X}$ with positive measure such that all trajectories of (21) starting in J converge to X_{eq}. Let $J(t)$, the image of J at time t by the flow $\psi(t, 0)$ of the reduced dynamics. Since $J(t)$ converges to $\{X_{eq}\}$ as t tends to infinity, one can assume, with no loss of generality, that J is chosen close enough to X_{eq} so that $J(t)$ lies in the neighborhood $\mathcal{N}$ for every $t \geq 0$ and therefore $\psi(t, 0)$ is the flow associated with he time-varying equation (30). Moreover, if $m(J(t))$ denotes the measure of $J(t)$, then $m(J(t))$ must tend to zero as t tends to infinity.

On the other hand, one has for $t \geq 0$,

$$m(J(t)) = \int_{J(t)} dZ_t = \int_J |\det(t, Z)| dZ,$$

where $\det(t, Z)$ denotes the determinant of $D\psi(t, 0, Z)$, the differential of $\psi(t, 0, Z)$ with respect to the initial condition $Z \in J$. Recall that, for every $t \geq 0$, one has

$$\frac{\partial d(t, Z)}{\partial t} = (\text{Tr}A(t) + \text{div}F(t, \psi(t, 0, Z)))\, d(t, Z), \quad d(0, Z) = 1.$$

By taking into account (31) and (32), one deduces that $\det(t, Z) \geq e^{\xi t/2}$ for t large enough, hence $m(J(t)) \geq e^{\xi t/2} m(J)$ which tends to infinity as t tends to infinity. We reached a contradiction.

For the other equilibrium points of $\widetilde{\Omega}_2$, $\widetilde{\Omega}_3$ and $\widetilde{\Omega}_4$, one proceeds similarly to show that there does not exist a measurable subset J of $\mathcal{X}$ with positive measure and such that trajectories of (21) starting in J would converge to that equilibrium point. Therefore, for almost any initial condition $X_0 \in \mathcal{X}$, the corresponding trajectory of (21) converges to a point of Ω_1. ■

IV. Simulation Results

In this section, we present simulation results showing the effectiveness of the proposed adaptive attitude trajectory tracking controller. We have considered for the simulations the inertia matrix $J = diag(0.5, 0.5, 1)$, the gyro bias $\delta = [0.1, 0.8, -0.6]^T$ and the inertial vectors $r_1 = [0, 0, 1]^T$ and $r_2 = [0.4340, -0.0091, 0.9009]^T$. The simulation sampling time is $0.01sec$. The control parameters have been chosen as $\gamma_1 = \gamma_2 = 10$ and $\rho_1 = \rho_2 = 10$. The desired angular velocity vector ω_d has been chosen to verify the persistency excitation condition mentioned in assumption (A_δ). We performed two simulation tests Test1 and Test2 to show the performance of the proposed control scheme and confirm the avoidance of the unwinding phenomenon. In the first simulation test, we considered the following initial conditions: $\omega(0) = [0, 0, 0]^T$, $Q_d(0) = [0.8, 0, 0.6, 0]^T$, $\hat{Q}(0) = [-0.8, 0, 0.6, 0]^T$ and $Q(0) = [1, 0, 0, 0]^T$. In the second simulation test, we considered the same initial conditions except for Q, where we started the scalar part of the unit quaternion from a negative value, *i.e.,* $Q(0) = [-1, 0, 0, 0]^T$. Figure 1 shows the evolution of the four components of the unit-quaternion tracking errors with respect to time for Test1 and Test2, respectively. Figure 2 shows the evolution of the unit-quaternion estimation error with respect to time for Test1 and Test2 respectively. We can clearly see that the unwinding phenomenon is avoided since both equilibria given by $\tilde{\Omega}_1$ are asymptotically stable. Figures 3 and 4 show the input signals and the angular velocity tracking signals respectively. Figures 5 shows that the parameter errors $\tilde{\theta}_1$ converging to zero relatively fast, while in figures 6, 7 and 8 the rest of the parameter errors $\tilde{\Theta}$ converge to zero relatively slow. This is due to the fact that only $\tilde{\Theta}$ depends on the richness of the reference signal.

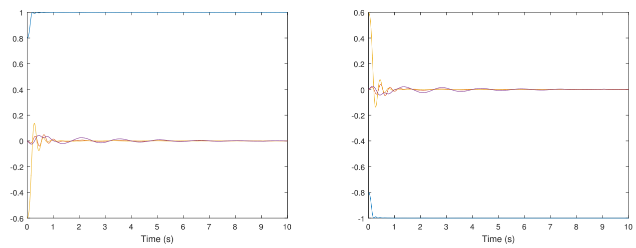

Fig. 1: Unit-quaternion tracking error $\tilde{Q}$ for Test1 and Test2

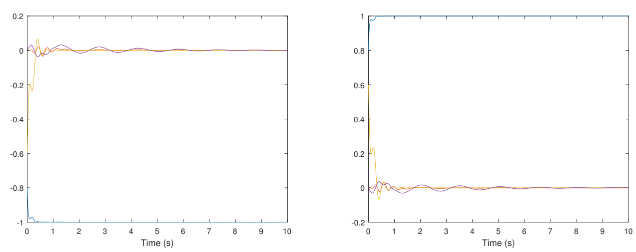

Fig. 2: Unit-quaternion estimation error $\bar{Q}$ for Test1 and Test2

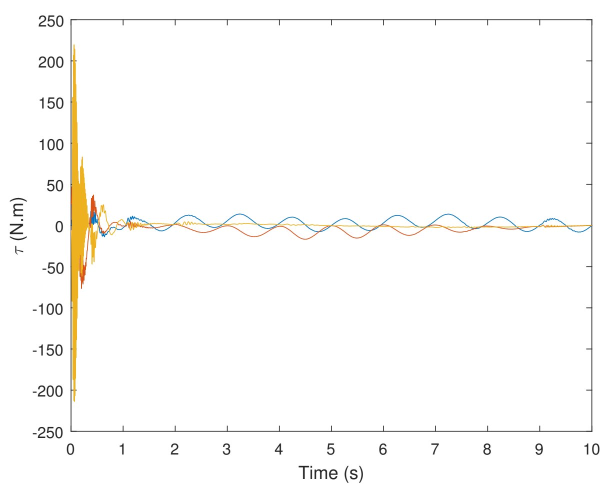

Fig. 3: Input torques τ

V. Conclusion

A new adaptive attitude tracking control scheme, relying on inertial vector measurements, has been proposed for rigid body systems with

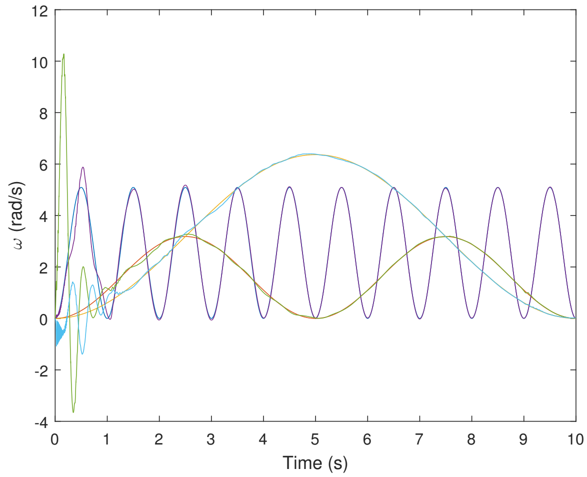

Fig. 4: Angular velocities ω and ω_d

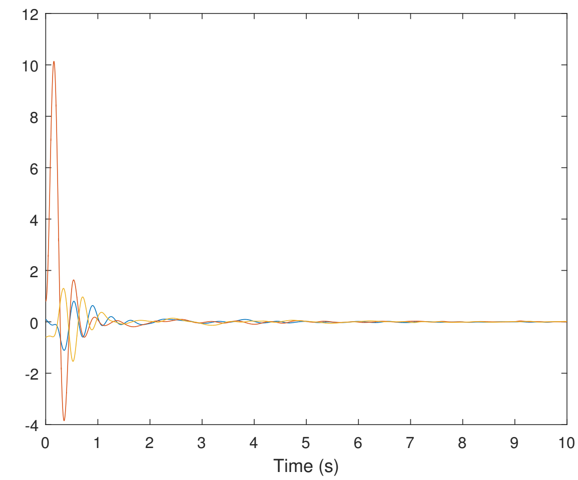

Fig. 5: Parameter estimation errors $\tilde{\theta}_1$

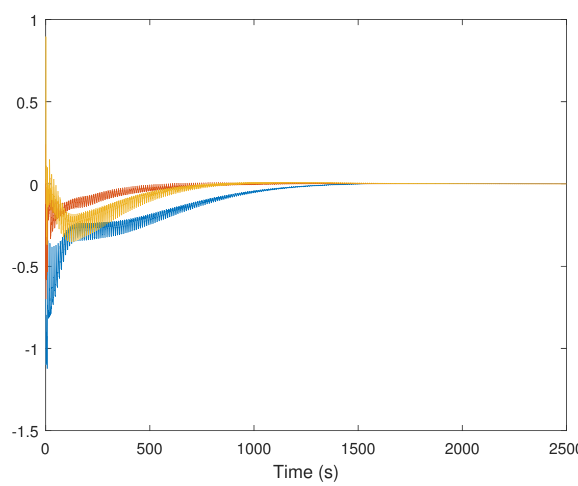

Fig. 6: Parameter estimation errors $\tilde{\theta}_2$

unknown inertia and unknown angular velocity bias. Global boundedness of the system state variables and almost global asymptotic

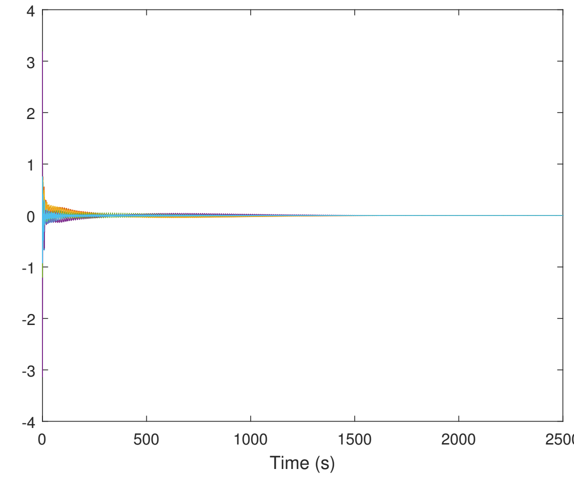

Fig. 7: Parameter estimation errors $\tilde{\theta}_3$

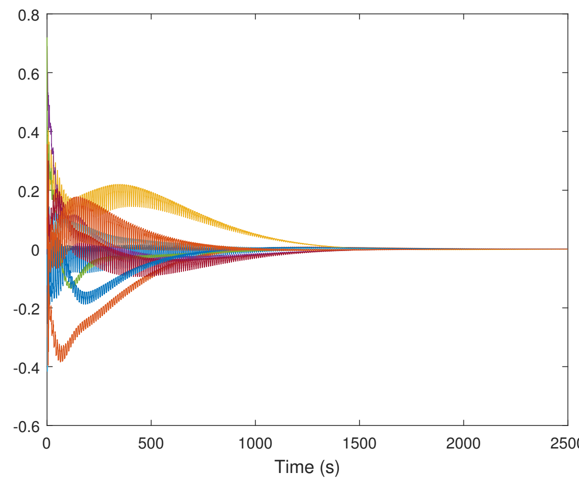

Fig. 8: Parameter estimation errors $\tilde{\theta}_4$

convergence of the body attitude and angular velocity to their desired values are proven. The convergence of the adaptive parameters to their true values is guaranteed under some kind of persistency of excitation condition on the reference trajectories. Compared to [11], the proposed control scheme involves fewer parameter adaptations and avoids the unwinding phenomenon. The performance of the proposed controller is illustrated through some simulation results.

References

[1] B. Wie, H. Weiss, and A. Arapostathis, "Quarternion feedback regulator for spacecraft eigenaxis rotations," *Journal of Guidance, Control, and Dynamics*, vol. 12, no. 3, pp. 375–380, 2017/02/14 1989. [Online]. Available: http://dx.doi.org/10.2514/3.20418

[2] N. A. Chaturvedi, A. K. Sanyal, and N. H. McClamroch, "Rigid-body attitude control," *IEEE Control Systems*, vol. 31, no. 3, pp. 30–51, June 2011.

[3] J. T. Y. Wen and K. Kreutz-Delgado, "The attitude control problem," *IEEE Transactions on Automatic Control*, vol. 36, no. 10, pp. 1148–1162, Oct 1991.

[4] M. D. Shusters and D. Oh, "Three-axis attitude determination from vector observations," *Journal of Guidance, Control, and Dynamics*, vol. 4, no. 1, pp. 70–77, 2017/02/14 1981. [Online]. Available: http://dx.doi.org/10.2514/3.19717

[5] J. L. Crassidis, F. L. Markley, and Y. Cheng, “Survey of nonlinear attitude estimation methods,” *Journal of Guidance, Control, and Dynamics*, vol. 30, no. 1, pp. 12–28, 2017/02/14 2007. [Online]. Available: http://dx.doi.org/10.2514/1.22452

[6] R. Mahony, T. Hamel, and J. M. Pflimlin, “Nonlinear complementary filters on the special orthogonal group,” *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1203–1218, June 2008.

[7] J. Thienel and R. M. Sanner, “A coupled nonlinear spacecraft attitude controller and observer with an unknown constant gyro bias and gyro noise,” *IEEE Transactions on Automatic Control*, vol. 48, no. 11, pp. 2011–2015, Nov 2003.

[8] P. Pounds, T. Hamel, and R. Mahony, “Attitude control of rigid body dynamics from biased imu measurements,” in *2007 46th IEEE Conference on Decision and Control*, Dec 2007, pp. 4620–4625.

[9] W. Luo, Y.-C. Chu, and K.-V. Ling, “Inverse optimal adaptive control for attitude tracking of spacecraft,” *IEEE Transactions on Automatic Control*, vol. 50, no. 11, pp. 1639–1654, Nov 2005.

[10] L. Benziane, A. Benallegue, Y. Chitour, and A. Tayebi, “Velocity-free attitude stabilization with inertial vector measurements,” *Int. J. Robust. Nonlinear Control*, vol. 26, pp. 2478–2493, 2016.

[11] T. Mercker and M. Akella, “Rigid-body attitude tracking with vector measurements and unknown gyro bias,” *Journal of Guidance, Control, and Dynamics*, vol. 34, no. 5, pp. 1474–1484, 2017/02/14 2011. [Online]. Available: http://dx.doi.org/10.2514/1.53111

[12] J. Ahmed, V. Coppola, and D.S.Bernstein, “Adaptive asymptotic tracking of spacecraft attitude motion with inertia matrix identification,” *Journal of Guidance, Control, and Dynamics*, vol. 21, no. 5, pp. 684–691, 2017/02/14 1998. [Online]. Available: http://dx.doi.org/10.2514/2.4310

[13] A. H. J. de Ruiter, “Observer-based adaptive spacecraft attitude control with guaranteed performance bounds,” *IEEE Transactions on Automatic Control*, vol. 61, no. 10, pp. 3146–3151, Oct 2016.

[14] A. Khosravian and M. Namvar, “Rigid body attitude control using a single vector measurement and gyro,” *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1273–1279, May 2012.

[15] Z. Cai, M. S. de Queiroz, and D. M. Dawson, “A sufficiently smooth projection operator,” *IEEE Transactions on Automatic Control*, vol. 51, no. 1, pp. 135–139, Jan 2006.

[16] A. Tayebi, A. Roberts, and A. Benallegue, “Inertial vector measurements based velocity-free attitude stabilization,” *IEEE Transactions on Automatic Control*, vol. 58, no. 11, pp. 2893–2898, Nov 2013.

APPENDIX

The expressions of $A(t)$ and $F(t, Z_{red})$ are given by

$$A(t) = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2}R_d & \frac{1}{2}R_d & 0 \\ 0 & \lambda_\rho & 0 & 0 & -\frac{1}{2}v_\rho^T\hat{R} & 0 \\ 0 & 0 & \lambda_\rho I_2 - W_\rho^\perp & 0 & \frac{1}{2}S(v_\rho)\hat{R} & 0 \\ -2I_b^{-1}R_d^T W_\gamma & 0 & 0 & -\alpha I_b^{-1} & 0 & -I_b^{-1}G_d \\ -2\Gamma_1 R_d^T W_\gamma & -2\lambda_\rho\Gamma_1\hat{R}^T v_\rho & 2\Gamma_1\hat{R}^T S(v_\rho)(\lambda_\rho I_2 - W_\rho^\perp) & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_2 G_d^T & 0 & 0 \end{bmatrix}, \tag{33}$$

$$F(t, Z_{red}) = \begin{pmatrix} \frac{1}{2}[(Z_{\tilde{q}_0} Id_3 + S(Z_{\tilde{q}}))R_d(Z_{\bar{\omega}} + Z_{\tilde{\theta}})] \\ Z_{\bar{q}_0}(2\lambda_\rho v_\rho^T Z_{\bar{q}} + Z_{\bar{q}}^T W_\rho Z_{\bar{q}}) - \frac{1}{2}Z_{\bar{q}}^T\hat{R}Z_{\tilde{\theta}} \\ \lambda_\rho v_\rho^T Z_{\bar{q}} Z_{\bar{q}}^\perp + \frac{1}{2}[(Z_{\bar{q}_0} Id_3 + S(Z_{\bar{q}}))\hat{R}Z_{\tilde{\theta}})]^\perp \\ -2I_b^{-1}R_d^T(Z_{\tilde{q}_0} Id_3 - S(Z_{\tilde{q}}))W_\gamma Z_{\tilde{q}} - I_b^{-1}G(Z_{\bar{\omega}} + Z_{\tilde{\theta}}, \Gamma_1(Z_\gamma + Z_\rho))Z_{\tilde{\Theta}} \\ -2\Gamma_1 R_d^T[Z_{\tilde{q}_0} Id_3 - S(Z_{\tilde{q}}))W_\gamma Z_{\tilde{q}}] - 2\Gamma_1\hat{R}^T[(Z_{\bar{q}_0} I_3 - S(Z_{\bar{q}}))\,W_\rho Z_{\bar{q}}] \\ \Gamma_2 G^T(Z_{\bar{\omega}} + Z_{\tilde{\theta}}, \Gamma_1(Z_\gamma + Z_\rho))Z_{\bar{\omega}} \end{pmatrix}. \tag{34}$$