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DUBINS' PROBLEM ON SURFACES II: NONPOSITIVE CURVATURE*

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Abstract. Let M be a complete, connected, two-dimensional Riemannian manifold with nonpositive Gaussian curvature K. We say that M satisfies the *unrestricted complete controllability* (UCC) property for the Dubins problem if the following holds: given any (p_1, v_1) and (p_2, v_2) in TM, there exists, for every $\varepsilon > 0$, a curve γ in M, with geodesic curvature smaller than ε, such that γ connects p_1 to p_2 and, for $i = 1, 2$, $\dot\gamma$ is equal to v_i at p_i. Property UCC is equivalent to the complete controllability of a family of control systems of Dubins' type, parameterized by ε. It is well known that the Poincaré half-plane does not verify property UCC. In this paper, we provide a complete characterization of the two-dimensional nonpositively curved manifolds M, with either uniformly negative or bounded curvature, that satisfy property UCC. More precisely, if $\sup_M K < 0$ or $\inf_M K > -\infty$, we show that UCC holds if and only if (i) M is of the first kind or (ii) the curvature satisfies a suitable integral decay condition at infinity.

Key words. controllability, geometric control, geodesic curvature, nonpositive curvature

AMS subject classifications. 14H55, 53C21, 93B05, 93B27

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1. Introduction.

Let (M, m) be a connected, oriented, complete Riemannian manifold and let $N = UM$ be its unit tangent bundle. Points of N are pairs (p, v), where $p \in M$, $v \in T_pM$, and $m(v, v) = 1$. Given $\varepsilon > 0$, Dubins' problem consists of finding, for every $(p_1, v_1), (p_2, v_2) \in N$, a curve $\gamma : [0, T] \to M$, parameterized by arc-length and whose geodesic curvature is bounded by ε, such that $\gamma(0) = p_1$, $\dot\gamma(0) = v_1$, $\gamma(T) = p_2$, $\dot\gamma(T) = v_2$, and which minimizes T. Although Dubins' problem makes sense in any dimension, the present paper deals only with the two-dimensional case, which can be seen as the time-optimal control problem for the following control system:

$$(D_\varepsilon): \qquad \dot q = f(q) + ug(q),\ q \in N,\ u \in [-\varepsilon, \varepsilon],$$

where f is the geodesic spray on N (i.e., the infinitesimal generator of the geodesic flow on M), g is the smooth vector field generating the fiberwise rotation with angular velocity equal to one, and the admissible controls are measurable functions $u : [0, T] \to [-\varepsilon, \varepsilon]$.

In the robotics literature (cf. [20]), the dynamics defined by (D_ε) when M is the Euclidean plane represent the motion of a unicycle (or a rolling penny) and the projections of the trajectories of (D_ε) on the plane are planar curves parameterized by arc-length with curvature bounded by ε. In 1957, Dubins [9] determined the global structure of the corresponding time-optimal trajectories: he showed that they are concatenations of at most three pieces made of circles of radius $1/\varepsilon$ or straight lines. Further restrictions on the length of the arcs of an optimal concatenation have been obtained by Sussmann and Tang [30].

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Dubins'-like problems have been proposed by considering more general manifolds M. For instance, the case where M is a two-dimensional manifold of constant Gaussian curvature was investigated in [5, 6, 12, 17], and the case where $M = \mathbf{R}^n, S^n$, $n \geq 3$, was studied in [18, 19, 29]. The controllability issue is not difficult to solve, because Dubins' problem can be reformulated as a control system (left-)invariant under the action of a Lie group. Let us mention another line of generalization, which consists of considering the distributional version of (D_ε), whose best-known example is the so-called Reeds–Shepp car [22]. In dimension two, the distributional dynamics can be represented by the two-input control system $(DD_\varepsilon) : \dot{q} = uf(q) + vg(q)$ with $|u|, |v| \leq \varepsilon$. The controllability of (DD_ε) simply results from the fact that the distribution (f, g) is strong bracket generating, i.e., for every $q \in N$, the triple $(f(q), g(q), [f, g](q))$ spans T_qN.

A motivation for the present work is the remark that when generalizing Dubins' problem from the Euclidean plane to a general Riemannian surface M, the complete controllability of (D_ε) stops to be a trivial issue. Indeed, when M has no particular symmetry, it is not clear how to design a trajectory starting from and ending at the fiber of a given point $p \in M$, with "small" geodesic curvature. We are naturally faced with an intrinsic property of M, independent of global rescaling of its metric m, that is, whether (D_ε) is completely controllable for every $\varepsilon > 0$. We refer to such property as the *unrestricted complete controllability* (UCC) of Dubins' problem on M. The present paper primarily focuses on the controllability of Dubins' problem, and our main scope is to find geometric or topological conditions on M characterizing UCC.

The Lie algebraic structure of (D_ε) reveals quickly that the Gaussian curvature K plays a crucial role in the characterization of controllable Dubins' problems. Indeed, as it follows from the structural equations satisfied by Cartan's connections (cf. Spivak [28, Volume II]),

$$[f, [f, g]](q) = -K(\pi(q))g(q) \tag{1}$$

for every $q \in N$, where $\pi : N \to M$ denotes the bundle projection. Notice that the bracket relation (1) identifies K uniquely. A generalized version of it was proposed by Agrachev and Sachkov ([1]; see also [26]) in order to introduce a notion of curvature for two-dimensional optimal control systems.

The role of K in the study of UCC of Dubins' problem is further suggested by the following standard example of noncontrollability: if M is the Poincaré half-plane H^2, then (D_ε) is completely controllable if and only if $\varepsilon > 1$ (cf. [6, 17]). Roughly speaking, the negativeness of K is an obstacle for the controlled turning action (modeled by g) to overcome the spreading of the geodesics. When $|u| \leq \varepsilon \leq 1$, the effect of ug is not strong enough and complete controllability fails to hold.

In a previous paper [7], we proposed some geometric and topological conditions on M, which ensure that the UCC property holds true. In particular, it was proved that any compact M satisfies the UCC property. In the unbounded case, the same conclusion was obtained when M is *asymptotically flat*, i.e., K tends to zero at infinity. The control strategy goes as follows. It is first shown that the controllability of (D_ε) is equivalent to the fact that every $q = (p, v) \in N$ can be steered to $q^- = (p, -v)$ by an admissible trajectory of (D_ε). The latter property, called *weak controllability* of (D_ε), is proved by tracking a teardrop loop (see Figure 1) in a covering domain over a piece of M at infinity. A suitable covering manifold can be globally described by a single appropriate geodesic chart.

If one keeps in mind the noncontrollability of Dubins' problem on H^2, it is reasonable to consider the situation where $K \geq 0$. Indeed, in that case, no local spreading effect due to the drift term has to be compensated. Moreover, if M is an open manifold with nonnegative curvature, then the theorems of Cohn-Vossen [8] and Huber [11] imply that

$$\int_M K dA < \infty, \tag{2}$$

where dA is the surface element in M (see [7] for details). The integral decay of K to zero at infinity can be interpreted as a sort of asymptotic flatness condition and it suggests that (D_ε) should be completely controllable for every $\varepsilon > 0$. In [7], we were able to confirm that intuition under the additional assumption that K is bounded over M, i.e., $\sup_M K$ is finite, and also to extend that result to the case where $K \geq 0$ outside a compact set. The strategy of the proof still consists in tracking a teardrop loop in a covering domain D but the construction of D becomes much more delicate than that of the asymptotic flat case.

Recently, Pansu pointed out how the existence of a closed geodesic on a surface M with constant negative curvature implies that for every point $p \in M$ and every $\varepsilon > 0$, there exists a noncontractible admissible trajectory of (D_ε) which steers some point $q \in U_pM$ to the point q^-. Such a feature is particularly interesting, taking into account that UCC does not hold for the Poincaré half-plane H^2. An immediate question is whether UCC can be recovered for a quotient manifold $M = H^2/\Gamma$, provided that we add enough topology to H^2 (i.e., provided that the group Γ is large enough), or if, on the contrary, Dubins' problem on any quotient of H^2 is not UCC. It turns out that the answer can be easily stated: $M = H^2/\Gamma$ has the UCC property if and only if it is of the first kind, i.e., its limit set (see Definition 3.1) is the entire boundary at infinity of H^2.

In the present work, we investigate the more general case of Riemannian surfaces with nonpositive curvature. It is well known that a surface M of such kind can be identified with the quotient space X/Γ, where X is a Hadamard surface (i.e., a simply connected, complete Riemannian manifold of nonpositive curvature) and Γ is a group of orientation-preserving isometries which acts freely and discontinuously on X. (For all the definitions, see section 3.) We provide conditions on M, sufficient for UCC to hold. The first one generalizes the hyperbolic case. It says that (i) $M = X/\Gamma$ is of the first kind, i.e., its limit set $\mathcal{L}(\Gamma)$ is equal to the ideal boundary $X(\infty)$ of X. The second condition is reminiscent of the nonnegative curvature case, namely, it states that (ii) for every $r > 0$ and every sector S in X, $\sup_{p\in S} \int_{B_X(p,r)} K dA = 0$, where $B_X(p,r)$ denotes the ball of center p and radius r. Then, we present necessary conditions on M for property UCC to hold. A quite surprising result is the following. Under the assumption that either K is bounded or $\sup_M K < 0$, if M verifies property UCC, then either condition (i) or condition (ii) must hold true. In that way, we exactly characterize the surfaces M, with nonpositive curvature K either bounded or such that $\sup_M K < 0$, verifying property UCC. In addition, some controllability and noncontrollability results in the case where K is nonpositive outside a compact subset of M are given. We also provide sufficient conditions ensuring that M has the UCC property, with no sign assumption on K, namely, (a) M has finite area, (b) the geodesic flow on M is topologically transitive. We conclude with some remarks on the structure of time-optimal trajectories for (D_ε). If $\sup_M K \leq -\varepsilon$, we show that they follow Dubins' pattern, namely, that they are concatenation of a bang, a singular, and a bang arc (where some arc can possibly have zero length), with the (possible) singular arc being a geodesic of the surface.

The paper is organized as follows. In section 2, we formulate the control problem at hand and describe its basic features. In section 3, general facts on Riemannian surfaces of nonpositive curvature are recalled. The main results of the paper are stated in section 4 and their proofs are provided in section 5. Finally, section 6 contains some remarks on time-optimal trajectories.

2. Formulation of the problem. Let (M,m) be a two-dimensional Riemannian manifold, or, as we will equivalently call it, a *Riemannian surface*. Notice that M is not required to be embedded nor immersed in $\mathbf{R}^3$. Assume that M is oriented and that its metric m is complete. Denote by N the unit tangent bundle $UM = \{q \in TM \mid m(q,q) = 1\}$ and by $\pi : N \to M$ the canonical bundle projection of N onto M. Let K be the Gaussian curvature on M. We will use the symbol K also for the trivial extension (constant on fibers) of K on N. The distance on M induced by m is denoted by $d_M(\cdot,\cdot)$ (or $d(\cdot,\cdot)$ when no confusion is possible), and, for every $p \in M$ and $r > 0$, $B_M(p,r)$ stands for the ball of center p and radius r.

Let f be the *geodesic spray* on TM, whose restriction to N (still denoted by f) is a well-defined vector field on N. Recall that f is characterized by the following property: $p(\cdot)$ is a geodesic on M if and only if $(p(\cdot),\dot p(\cdot))$ is an integral curve of f. In particular, f satisfies the relation

$$\pi_\star(f(q)) = q \tag{3}$$

for every $q \in N$. The *exponential map on M* is defined by

$$\exp(t,q) = \pi(e^{tf}(q)), \tag{4}$$

where $e^{tf} : N \to N$ denotes the flow of the vector field f at time t.

Let g be the smooth vector field on N, whose corresponding flow at time t is the fiberwise rotation of angle t. For every $q \in N$, we set $q^- = e^{\pi g}(q)$ and $\mathcal{R}q = e^{-\frac{\pi}{2}g}(q)$.

For $\varepsilon > 0$, let $(\mathcal{D}_\varepsilon^M)$ be the control system

$$(\mathcal{D}_\varepsilon^M): \qquad \dot q = f(q) + ug(q), \quad q \in N, \quad u \in [-\varepsilon,\varepsilon].$$

An *admissible control* is a measurable function $u(\cdot)$, defined on some interval of $\mathbf{R}$, with values in $[-\varepsilon,\varepsilon]$. The solutions of $(\mathcal{D}_\varepsilon^M)$ corresponding to admissible controls are called *admissible trajectories*. (Sometimes, to prevent any confusion, we will speak also of *ε-admissible controls* and *ε-admissible trajectories*).

For every $q \in N$ and $T > 0$, the *attainable set from q up to time T* is the set $A_q^T = A_q^T(M,\varepsilon)$ consisting of the endpoints of all admissible trajectories for $(\mathcal{D}_\varepsilon^M)$, starting from q, of length smaller than or equal to T. We also write

$$A_q = A_q(M,\varepsilon) = \cup_{T>0} A_q^T(M,\varepsilon).$$

The control system $(\mathcal{D}_\varepsilon^M)$ is called *completely controllable* if $A_q = N$ for every $q \in N$.

DEFINITION 2.1. *We say that the Dubins problem on M has the* unrestricted complete controllability (UCC) property *(or, equivalently, that it is* UCC*) if, for every $\varepsilon > 0$, $(\mathcal{D}_\varepsilon^M)$ is completely controllable.*

We stress that the UCC notion corresponds to the property of existence of solutions to the original, control-free Dubins problem on M: UCC holds if and only if, for every $\varepsilon > 0$ and every (p_1,v_1), (p_2,v_2) in TM, there exists a curve $\gamma : [0,T] \to M$ with geodesic curvature smaller than ε such that $\gamma(0) = p_1$, $\gamma(T) = p_2$ and $\dot\gamma(0) = v_1$,

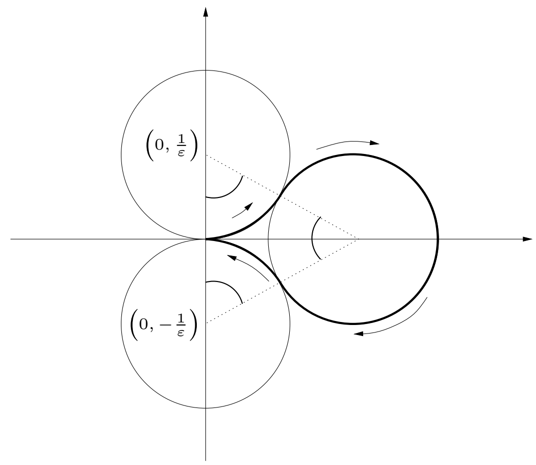

FIG. 1. *The teardrop trajectory of size* $1/\varepsilon$.

$\dot{\gamma}(T) = v_2$. Up to a reparameterization by arc-length, $(\gamma, \dot{\gamma})$ is an ε-admissible trajectory in N, whose corresponding control $u(t)$ is equal, at almost every $t \in [0, T]$, to the geodesic curvature of γ at the point $\gamma(t)$.

It is easy to check that the distribution $\{f, g\}$ is *bracket generating*, i.e., that, for every $q \in N$, $T_qM = \mathrm{span}\{X(q) |\ X \in \mathrm{Lie}(f, g)\}$, where $\mathrm{Lie}(f, g)$ denotes the Lie algebra of vector fields on N generated by f and g. Indeed, for every $q \in N$, we have that $\pi_*([f, g](q)) = \pi_*(f(\mathcal{R}q))$. Therefore, f, g, and $[f, g]$ span T_qM at every $q \in N$, proving not only that $\{f, g\}$ is bracket generating but also that it is a *contact distribution* on N. As a consequence, for every $0 < t < T$ and every $q \in N$,

$$e^{tf}(q) \in \mathrm{Int}(A_q^T), \tag{5}$$

where $\mathrm{Int}(A_q^T)$ denotes the interior of A_q^T. This follows, for instance, from the description of small-time attainable sets for single-input nondegenerate three-dimensional control systems given by Lobry in [15].

Since f, g, and $[f, g]$ are linearly independent at every point, they can be used to introduce a metric on N, requiring $(f(q), g(q), [f, g](q))$ to be an orthonormal basis of T_qN, for every $q \in N$. Such metric endows N with a complete Riemannian structure (see, for instance, [23]). In accordance with the notations introduced above, we will denote by $d_N(\cdot, \cdot)$ the induced distance on N and, for every $q \in N$ and $\rho > 0$, by $B_N(q, \rho)$ the ball of center q and radius ρ.

Remark 2.2. In [7] we noticed that, since (D_ε^M) has the property of being bracket generating, Dubins' problem on M is UCC if and only if for every $\varepsilon > 0$ and every $q \in N$, there exists $\hat{q} \in A_q(M, \varepsilon)$ such that $\hat{q}^- \in A_{\hat{q}}(M, \varepsilon)$. In order to prove this last property, we will often mimic on M the behavior of a teardrop trajectory on the Euclidean plane. We call teardrop trajectory of size $1/\varepsilon$ (see Figure 1) the bang-bang trajectory of $(D_\varepsilon^{\mathbf{R}^2})$ whose control u, defined by

$$u(t) = \begin{cases} \varepsilon & \text{if} \quad 0 \leq t \leq \frac{\pi}{3\varepsilon}, \\ -\varepsilon & \text{if} \quad \frac{\pi}{3\varepsilon} < t \leq \frac{2\pi}{\varepsilon}, \\ \varepsilon & \text{if} \quad \frac{2\pi}{\varepsilon} < t \leq \frac{7\pi}{3\varepsilon}, \end{cases}$$

steers $(1, 0) \in U_{(0,0)}\mathbf{R}^2$ to $(-1, 0) \in U_{(0,0)}\mathbf{R}^2$.

Remark 2.3. If M_1, M_2 are two Riemannian manifolds and $P : M_1 \to M_2$ is a local isometry at every point of M_1, then every admissible trajectory for $(D_\varepsilon^{M_1})$ is transformed by $P_* : UM_1 \to UM_2$ in an admissible trajectory for $(D_\varepsilon^{M_2})$. In particular, if P is onto and the Dubins problem on M_1 is UCC, then the same is true for the Dubins problem on M_2.

3. General facts on Riemannian surfaces of nonpositive curvature. Of special interest for the present study are Riemannian surfaces of nonpositive curvature. We collect in this section some definitions and known results about them, which will be used in what follows. When no explicit source is given, see [4].

A simply connected, complete Riemannian manifold of nonpositive sectional curvature is called a *Hadamard manifold*. If M is a complete, connected, oriented Riemannian manifold of nonpositive sectional curvature, then M can be described as X/Γ, where X is a Hadamard manifold and Γ is a group of orientation-preserving isometries which acts freely and discontinuously on X. We will denote by $\Pi : X \to M$ the canonical projection of X onto M. When the sectional curvature is constant on M, then M is said a *hyperbolic manifold* and X a *hyperbolic space*.

3.1. Hadamard surfaces. From now on, X will denote a Hadamard surface, that is, a Hadamard two-dimensional manifold. If $M = X/\Gamma$ is a complete Riemannian surface, then we will denote by the same letters f, g, K, m, and π the corresponding objects on X and M. This is motivated by the fact that $\Pi : X \to M$, being a local isometry, identifies such objects, at least at a local level.

It is well known that X is diffemorphic to $\mathbf{R}^2$. Any geodesic segment in X is the unique length-minimizing trajectory between its extremes. In particular, all complete geodesics are simple.

A *ray* is a half-geodesic on X. A *sector* is a region of X bounded by two distinct rays starting at the same point, which is called the *vertex of S*.

The *ideal boundary* of X, denoted by $X(\infty)$, is defined as the quotient of the set of all rays parameterized by arc-length by the equivalence relation

$$c_1 \sim c_2 \iff \limsup_{t\to\infty} d_X(c_1(t), c_2(t)) < \infty.$$

The equivalence class of a parameterized ray c is denoted by $c(+\infty)$ and it is called the *endpoint of* c. If $c : \mathbf{R} \to X$ is a parameterized complete geodesic, then $c(-\infty)$ denotes the equivalence class of $[0, \infty) \ni t \mapsto c(-t)$.

For a given point $p \in X$ there is a one-to-one correspondence ψ_p between U_pX and $X(\infty)$, which assigns to $v \in U_pX$ the equivalence class of $[0, \infty) \ni t \mapsto \exp(t, v)$. The correspondence defines a topology on $X(\infty)$, which is called the *sphere topology*. The sphere topology extends to the so-called cone topology on $\overline{X} = X \cup X(\infty)$. The cone topology is generated by the open sets of X and the sets

$$\psi_p(U) \cup (\cup_{t>0} \exp(t, U)),$$

where $p \in X$ and U is an open set of U_pX. Notice that the action on X of an element of Γ has a natural continuous extension on $\overline{X}$.

Given a set Ω in X, we write $\partial\Omega$ for the boundary of Ω in X, while $\Omega(\infty)$ will denote the intersection between $X(\infty)$ and the closure of Ω in $\overline{X}$.

The isometric transformations of X can be classified in terms of the so-called *displacement function* $X \ni p \mapsto d_X(p, \gamma(p))$. An isometry $\gamma : X \to X$ is called *elliptic* if it has at least one fixed point in X; *hyperbolic* if the displacement function attains its minimum in X, and such minimum is strictly positive; *parabolic* otherwise.

3.2. Closed geodesics and limit sets. If $M = X/\Gamma$ is a complete Riemannian surface, then the only elliptic element of Γ is the identity. Let G be a closed geodesic in M. Fix one of its lifts $\widetilde{G}$ in X and let $c : \mathbf{R} \to X$ be a parameterization of $\widetilde{G}$ by arc-length. There is one isometry $\gamma \in \Gamma$ such that $\gamma(c(0)) = c(T)$, where T is the length of G. Then $c(t+T) = c(t)$ for every $t \in \mathbf{R}$. (The proof goes as in the hyperbolic case; see [21, Theorem 9.6.2].) The isometry γ is hyperbolic (see [4, section 6]) and $\widetilde{G}$ is called an *axis* of γ. Actually, every hyperbolic isometry γ has at least one axis. (Given any point p at which the displacement function attains its minimum, the geodesic line between p and $\gamma(p)$ is indeed an axis.) If none of the half-planes bounded by $\widetilde{G}$ is flat, then, given a neighborhood U of $c(-\infty)$ and a neighborhood V of $c(+\infty)$ in $\overline{X}$, there exists $\bar{n} \in \mathbf{N}$ such that

$$\gamma^n \left(\overline{X} \setminus U\right) \subset V, \qquad \gamma^{-n} \left(\overline{X} \setminus V\right) \subset U, \tag{6}$$

for every $n \geq \bar{n}$, as proved by Ballmann in [3].

The existence of a closed geodesic on Riemannian surfaces is established by some classical results under very general assumptions. Lyusternik and Fet [16] proved that all complete compact Riemannian surfaces contain at least one closed geodesic. After that, Thorbergsson [31] extended the result to all complete, connected Riemannian surfaces neither homeomorphic to the plane nor to the cylinder. As a consequence, if Γ is not cyclic, then $M = X/\Gamma$ contains a closed geodesic.

Definition 3.1. *Let Γ be a group of isometries of X. Fix $p \in X$ and consider $\overline{\Gamma(p)}$, the closure in $\overline{X}$ of the Γ-orbit of p. The set $\mathcal{L}(\Gamma) = \overline{\Gamma(p)} \cap X(\infty)$ is called the limit set of Γ.*

The definition of $\mathcal{L}(\Gamma)$ is actually independent of the choice of the point p. Fix, indeed, p, p' in X, and let $\gamma_n(p) \to z \in X(\infty)$ as $n \to \infty$, with $\{\gamma_n\}_{n \in \mathbf{N}} \subset \Gamma$. Notice that for every $n \in \mathbf{N}$, $d(\gamma_n(p), \gamma_n(p')) = d(p, p')$. Moreover, every sector S of X such that $z \in S(\infty)$ contains a subsector S' such that $z \in S'(\infty)$ and whose distance from the boundary of S is larger than $d(p, p')$. Therefore, in the cone topology of $\overline{X}$, $\gamma_n(p') \to z$ as $n \to \infty$.

Definition 3.2. *Let X be a Hadamard surface and $M = X/\Gamma$ be a complete Riemannian surface. We say that M is of the first kind if $\mathcal{L}(\Gamma) = X(\infty)$; otherwise we say that M is of the second kind.*

Remark 3.3. If Γ is cyclic, then M is of the second kind. Indeed, if $\Gamma = \{\mathrm{Id}\}$, then $\mathcal{L}(\Gamma)$ is empty. If Γ is nontrivial, then it is generated by a hyperbolic or a parabolic isometry, denoted by γ. If γ is hyperbolic, then it translates at least one geodesic, hence $\mathcal{L}(\Gamma)$ is made of the two endpoints of such geodesic (just consider the orbit of a point on the geodesic). If γ, instead, is parabolic, then there exists a point $z \in X(\infty)$ which is invariant under the action of γ, together with the horospheres centered at z (see [4, Lemma 6.6]). Fix any point $x \in X$ and consider a horosphere U centered at z which contains x. Then $\mathcal{L}(\Gamma)$ is contained in $U(\infty)$. In both cases, M is not of the first kind. Therefore, if M is of the first kind, then it contains a closed geodesic.

Definition 3.4. *Let M be a Riemannian surface. We say that its Gaussian curvature K is uniformly negative if $\sup_M K < 0$.*

If K is uniformly negative on X, then, for every two elements x, y of $X(\infty)$, there exists a parameterized geodesic $c : \mathbf{R} \to X$ such that $c(+\infty) = x$ and $c(-\infty) = y$. That is, using the terminology introduced by Eberlein and O'Neill in [10], X is a *visibility manifold.*

3.3. Geodesic coordinates. Geodesic coordinates can be globally defined on X. They depend on the choice of an element q of UX and are defined through the map

$$\begin{array}{rccl} \phi_q : & \mathbf{R}^2 & \longrightarrow & X, \\ & (x,y) & \longmapsto & \pi(e^{yf} \circ e^{\frac{\pi}{2}g} \circ e^{xf}(q)). \end{array} \tag{7}$$

Let $B : \mathbf{R}^2 \to \mathbf{R}$ be the solution of the system

$$\begin{cases} B_{yy} + KB = 0, \\ B(x,0) \equiv 1, \\ B_y(x,0) \equiv 0, \end{cases} \tag{8}$$

in which the index y appearing in B_y, B_{yy} stands for the partial differentiation with respect to y. Notice that since $K \leq 0$, B is globally defined and

$$B(x,y) \geq 1, \tag{9}$$

$$yB_y(x,y) \geq 0 \tag{10}$$

for every $(x,y) \in \mathbf{R}^2$. The coordinate expression for the metric m on X is given by

$$m(x,y) = B^2(x,y)dx^2 + dy^2.$$

(See, for instance, [13].) The unit bundle UX can be identified with

$$\left\{ \left(x, y, \frac{\cos\theta}{B(x,y)}, \sin\theta \right) \in \mathbf{R}^4 \middle| \ (x,y) \in \mathbf{R}^2,\ \theta \in \mathcal{S}^1 \right\}.$$

In the coordinates (x,y,θ), f and g are given by

$$f(x,y,\theta) = \left(\frac{\cos\theta}{B(x,y)}, \sin\theta, F(x,y)\cos\theta \right)^T, \qquad g(x,y,\theta) = (0,0,1)^T, \tag{11}$$

where

$$F(x,y) = \frac{B_y(x,y)}{B(x,y)}. \tag{12}$$

Equivalently, (D_ε^X) can be written as follows:

$$\dot{x} = \frac{\cos\theta}{B}, \tag{13}$$

$$\dot{y} = \sin\theta, \tag{14}$$

$$\dot{\theta} = u + F\cos\theta. \tag{15}$$

Notice that, as it can be easily deduced from (8), F satisfies the Cauchy problem

$$F_y = -K - F^2, \qquad F(x,0) \equiv 0. \tag{16}$$

If $M = X/\Gamma$ is a complete Riemannian surface, then, for any fixed $q \in N$, the map ϕ_q, defined as in (7), is a local diffeomorphism at every point of $\mathbf{R}^2$. The pullback of the metric m through ϕ_q endows $\mathbf{R}^2$ with a Riemannian structure, which renders $\mathbf{R}^2$ isomorphic to X.

Remark 3.5. The previous formulation of Dubins' problem in coordinates still makes sense when only local, instead of global, geodesic coordinates are defined. Consider the case of a Riemannian surface M with possibly sign-varying curvature. Recall that a *half-plane* of M is a simply connected open subset of M bounded by a simple open complete geodesic. If M contains a half-plane H and if K is nonpositive on H, then H admits a system of geodesic coordinates. Indeed, for any $q \in N$ such that $\phi_q(\mathbf{R}\times[0,\infty)) = H$, $\mathbf{R}\times[0,\infty)$ can be endowed with the metric $\phi_q^* m$, and Dubins' problem on H can be described by equations (13)–(15). Similarly, if $q = (p, v) \in N$ and $a, b, r > 0$ are such that $K \leq 0$ on $B_M(p, r)$ and $a + b < r$, then the rectangle $[-a, a] \times [-b, b] \subset \mathbf{R}^2$, endowed with $\phi_q^* m$, is a nonpositively curved covering domain of a neighborhood of p. Again, system (13)–(15) describes Dubins' problem on such nonflat rectangle.

4. Statement of the results. We collect here the statements of the main results proved in the next section, in order to stress the interrelations between the proposed necessary and sufficient conditions for the UCC property to hold.

PROPOSITION 4.1. *Let M be a complete, connected Riemannian surface. The Dubins problem on M has the* UCC *property if at least one of the following conditions is satisfied:* (a) *M has finite area or* (b) *the geodesic flow on M is topologically transitive.*

THEOREM 4.2. *Let M be a complete, connected Riemannian surface with nonpositive Gaussian curvature K. Denote by X the universal covering of M. The Dubins problem on M has the* UCC *property if at least one of the following conditions is satisfied:* (i) *M is of the first kind or* (ii) *for every $r > 0$ and every sector S in X,* $\sup_{p\in S} \int_{B_X(p,r)} K dA = 0$.

Conditions which are both sufficient and necessary for UCC can be formulated under more restrictive hypotheses on K.

THEOREM 4.3. *Let M be a complete, connected Riemannian surface and assume that its Gaussian curvature K is uniformly negative. Then the Dubins problem on M has the* UCC *property if and only if M is of the first kind.*

THEOREM 4.4. *Let M be a complete, connected Riemannian surface and assume that its Gaussian curvature K is nonpositive and bounded. Assume, moreover, that M is of the second kind and denote by X its universal covering. The Dubins problem on M has the* UCC *property if and only if, for every $r > 0$, for every sector S in X,* $\sup_{p\in S} \int_{B_X(p,r)} K dA = 0$.

The results stated in Theorem 4.2 are proved in Propositions 5.9 and 5.10. For what concerns Theorem 4.3, it is a combination of Proposition 5.9 and Corollary 5.15. Finally, the proof of Theorem 4.4 is split into the ones of Proposition 5.10 and Corollary 5.18. For results concerning manifolds with possibly sign-varying curvature, we refer to the detail of sections 5.2, 5.3, and 5.4.

5. Proofs of the results.

5.1. Additional properties of Riemannian surfaces of nonpositive curvature. In this paragraph, we prove some geometric properties of Riemannian surfaces of nonpositive curvature, which are at the heart of the arguments of the theorems stated above. In particular, we show how certain properties of limit sets which hold in the hyperbolic case can be recovered in the nonconstant curvature case, under suitable nonflatness assumptions. To this extent we propose the following definition.

Definition 5.1. *A sector S of a Hadamard surface X is said to be heavy if, for every sector S' contained in S, $\int_{S'} KdA = -\infty$.*

For example, as we will often use, if $r, \delta > 0$ exist such that $\int_{B_X(p,r)} KdA \leq -\delta$ for every $p \in S$, then S is heavy. Heavy sectors enjoy the following visibility-type property.

Lemma 5.2. *Let S be a heavy sector of a Hadamard surface X. Then, for every z_1 in the interior of $S(\infty)$ and every $z_2 \in X(\infty)$ different from z_1, there exists a geodesic $c : \mathbf{R} \to X$ such that $c(-\infty) = z_1$ and $c(+\infty) = z_2$.*

The lemma is basically a reformulation of [4, Exercise (i), p. 57]. For sake of completeness, let us provide its proof. Consider a convex sector S' bounded by two rays having z_1 and z_2 as endpoints. Since z_1 belongs to the interior of $S(\infty)$, then S' contains a subsector which is a included in S. Therefore, by definition of heavy sector, $\int_{S'} KdA = -\infty$. Denote by x the vertex of S' and let p_n, q_n be two sequences of points in $\partial S'$, tending, respectively, to z_1, z_2, as n tends to infinity. Denote by T_n the geodesic triangle with vertices x, p_n, and q_n. The Gauss–Bonnet theorem implies that $\int_{T_n} KdA > -\pi$ for every n. Therefore, there exists a compact subset C of S' such that, possibly passing to a subsequence, the geodesic segment between p_n and q_n intersects C for every n. By compactness of $UC = \{q \in UX | \ \pi(q) \in C\}$, there exists $q \in UC$ such that the geodesic $s \mapsto \exp(s, q)$ connects, as required, z_1 and z_2.

Given a Hadamard surface X and a group Γ of isometries of X, let us denote by $\mathcal{A}(\Gamma)$ the closure in $X(\infty)$ of the set of endpoints of axis of hyperbolic isometries contained in Γ.

Lemma 5.3. *Let X be a Hadamard surface and Γ a group of isometries of X. If $\mathcal{A}(\Gamma)$ is nonempty, then, for every heavy sector S of X, $\mathcal{L}(\Gamma) \cap S(\infty) = \mathcal{A}(\Gamma) \cap S(\infty)$.*

Proof. Clearly $\mathcal{A}(\Gamma) \subset \mathcal{L}(\Gamma)$. Fix an axis G of a hyperbolic transformation of Γ, a point z in $S(\infty)$, and assume that there exists a neighborhood J of z which does not contain any endpoint of complete geodesics of the type $\gamma(G)$, $\gamma \in \Gamma$. Let H be a half-plane of X such that $z \in H(\infty) \subset J$, whose existence is guaranteed by Lemma 5.2. Then, the entire Γ-orbit of G is contained in $X \setminus H$. Therefore, z is not a density point of the Γ-orbit of any point of G, i.e., $z \notin \mathcal{L}(\Gamma)$. □

The following lemma concerns the existence of half-planes in nonpositively curved Riemannian surfaces. Its role will be crucial while proving the noncontrollability results contained in section 5.3.

Lemma 5.4. *Let M be a complete, connected Riemannian surface with nonpositive curvature and let X be its universal covering. Assume that M is of the second kind and that X contains a heavy sector S. Then the canonical projection $\Pi : X \to M$ embeds in M a half-plane of X, contained in S.*

Proof. We start the proof by noticing that if there exists an open, nonempty, connected subset U of $S(\infty) \setminus \mathcal{L}(\Gamma)$ such that $\gamma(U) \cap U = \emptyset$ for every $\gamma \in \Gamma \setminus \{\mathrm{Id}\}$, then every half-plane H bounded by U and a geodesic with endpoints in U is embedded by Π in M. Indeed, $H \cap \gamma(H) \neq \emptyset$ implies that $H(\infty) \cap (\gamma(H))(\infty) = H(\infty) \cap \gamma(H(\infty)) \neq \emptyset$ and γ must therefore be the identity. The existence of a geodesic with endpoints in U follows from Lemma 5.2.

We distinguish two cases, depending on whether Γ contains a hyperbolic isometry.

Assume first that $\Gamma \setminus \{\mathrm{Id}\}$ is made of parabolic isometries only. In particular, Γ is cyclic, since M contains no closed geodesic. Denote by γ a generator of Γ. The set of fixed points of γ on $X(\infty)$ is closed and nonempty. If it does not contain $S(\infty)$, then there exists an open, nonempty, connected subset U of $S(\infty)$ such that $\gamma^j(U) \cap U = \emptyset$ for every $j \in \mathbf{Z} \setminus \{0\}$. Then the lemma follows from the above remark. We are

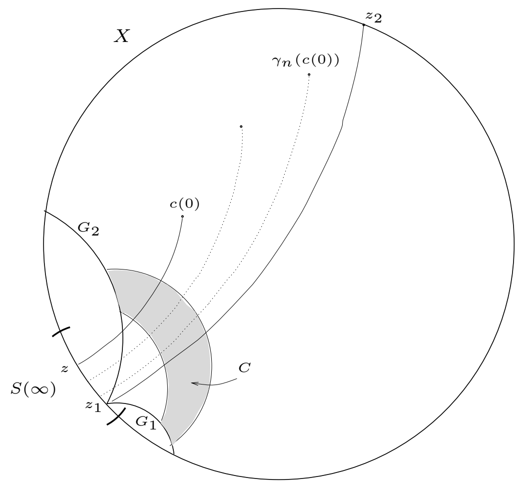

FIG. 2.

left to deal with the case in which every point of $S(\infty)$ is fixed by γ. Consider a parameterized geodesic $c : \mathbf{R} \to X$ such that $c(+\infty)$ and $c(-\infty)$ are both in $S(\infty)$. Since γ fixes the endpoints of c, then P_c, the set spanned by all the geodesics parallel to c, is invariant under the action of γ. The set P_c is isometric to $\mathbf{R} \times \Sigma$, where Σ is a closed geodesic line in X (see [4, Lemma 2.4]). By hypothesis, c bounds no flat half-plane; therefore Σ is bounded. Since γ induces a continuous transformation of Σ, there must exist at least one complete geodesic of X fixed by γ. Thus γ, which is a nonhyperbolic isometry of P_c, is the identity transformation, and the statement of the lemma is trivially true. (Notice that the argument, which follows the proof of Lemma 6.8 in [4], shows that γ has at most one fixed point in the interior of $S(\infty)$, unless $\Gamma = \{\mathrm{Id}\}$.)

Assume now that Γ contains at least one hyperbolic isometry. According to Lemma 5.3, then, $\mathcal{L}(\Gamma) \cap S(\infty) = \mathcal{A}(\Gamma) \cap S(\infty)$. Notice that by definition of heavy sector, for every complete geodesic G with at least one endpoint in the interior of $S(\infty)$, the half-planes bounded by G are not flat. Therefore, $\mathcal{L}(\Gamma) \cap S(\infty)$ has a nonempty interior, since otherwise (6) would imply that M is of the first kind. In particular, $S(\infty)$ is not contained in $\mathcal{L}(\Gamma)$. Fix $z \in S(\infty) \setminus \mathcal{L}(\Gamma)$. According to the remark at the beginning of the proof, it is enough to prove that z is an isolated point of the Γ-orbit of z in $X(\infty)$. To this extent, let us show that the Γ-orbit of z in $X(\infty)$ has no density point in the interior of $S(\infty) \setminus \mathcal{L}(\Gamma)$. Assume, by contradiction, that there exist z_1 in the interior of $S(\infty) \setminus \mathcal{L}(\Gamma)$ and a sequence γ_n of isometries in Γ such that $\gamma_n(z)$ tends to z_1 as n goes to infinity. Let $c : [0, \infty) \to X$ be a ray with $c(+\infty) = z$. Passing possibly to a subsequence, $\gamma_n(c(0))$ converges to a point z_2 of $\mathcal{L}(\Gamma)$. According to Lemma 5.2, there exist two geodesics G_1 and G_2, each of them having one endpoint in z_1, such that the region of X bounded by G_1 and G_2 contains $\gamma_n(c(0))$ for every n. Thus, there exists a compact subset C of X which intersects each ray $c_n(\cdot) = \gamma_n(c(\cdot))$, $n \in \mathbf{N}$ (see Figure 2). Therefore, we can fix a sequence t_n of positive real numbers such that for every $x \in X$, $c_n(t_n)$ has distance from x uniformly bounded with respect to n. Fix $x \in X$ and notice that

$$t_n = d(c_n(0), c_n(t_n)) \geq d(c_n(0), x) - d(x, c_n(t_n)),$$

which implies that t_n tends to infinity with n. Thus, $c(t_n)$ tends to z as n goes to infinity. On the other hand, $\gamma_n^{-1}(x)$ has distance from $c(t_n)$ uniformly bounded with respect to n. It follows that as n goes to infinity, $\gamma_n^{-1}(x)$ converges to z, which contradicts the fact that z does not belong to $\mathcal{L}(\Gamma)$. ∎

5.2. Controllability results. Let us begin the section with the following elementary remark. Consider a complete Riemannian surface M, with no restriction on the sign of its curvature K. Setting $t = 1$ and $T = 2$ in (5), we have that for every $q \in N$ and $\varepsilon > 0$, $e^f(q) \in \mathrm{Int}(A_q^2(M,\varepsilon))$. Fix q and let q' vary in $B_N(q,2)$, which is a relatively compact neighborhood of q. The continuous dependence of $A_{q'}^2(M,\varepsilon)$ on q' implies that there exists $\rho(\varepsilon,q) \in (0,1)$ such that for every $q' \in B_N(q,2)$,

$$B_N(e^f(q'),\rho(\varepsilon,q)) \subset A_{q'}(M,\varepsilon).$$

Therefore,

$$q \in \mathrm{Int}(A_{q'}(M,\varepsilon)) \quad \text{if} \quad e^f(q') \in B(q,\rho(\varepsilon,q)). \tag{17}$$

Indeed, for every $q' \in N$ such that $e^f(q') \in B(q,\rho(\varepsilon,q))$, we have

$$d_N(q,q') \le d_N(q,e^f(q')) + d_N(q',e^f(q')) < 2.$$

LEMMA 5.5. *Let M be a complete, connected Riemannian surface. If $q \in N$ and $\varepsilon > 0$ exist such that $A_q(M,\varepsilon)$ has finite volume, then M has finite area and $A_q(M,\varepsilon) = N$.*

Proof. Fix $q \in N$ and $\varepsilon > 0$ such that $A_q = A_q(M,\varepsilon)$ has finite volume. We want to prove that ∂A_q is empty. Let, by contradiction, $\bar q \in \partial A_q$, and define $\bar\rho = \rho(\varepsilon,\bar q) > 0$, where the function ρ satisfies (17). A well-known theorem by Krener [14] states that any attainable set of a bracket-generating system is contained in the closure of its interior. Therefore, $V = A_q \cap B_N(\bar q,\bar\rho)$ has a nonempty interior and, in particular, positive volume. Since e^f is a volume-preserving diffeomorphism of N (see, for instance, [23]) and A_q has finite volume, then the sets $e^{nf}(V)$, for $n \in \mathbf{N}$, cannot be pairwise disjoint, being $e^{nf}(V) \subset A_q$ for every $n \in \mathbf{N}$. Therefore, there exist $n_1 < n_2$ such that $e^{n_1 f}(V) \cap e^{n_2 f}(V)$ is nonempty. Equivalently, there exists a point q' in $e^{(n_2-n_1-1)f}(V)$ whose image by e^f lies in V. Due to (17), the contradiction is reached. ∎

COROLLARY 5.6. *Let M be a complete, connected Riemannian surface with finite area. Then the Dubins problem on M has the* UCC *property.*

Notice that in the nonpositive curvature case, if M has finite area, then it is of the first kind (see, for instance, [3]). The converse, clearly, is not true in general. It is, however, when M is a finitely connected hyperbolic surface (see, for instance, [21]). In particular, we proved that the Dubins problem on a finitely connected hyperbolic surface of the first kind is UCC. Actually, first kind implies UCC in complete generality, as will be shown in Proposition 5.9.

Let us now prove the second part of Proposition 4.1. Recall that the geodesic flow on M is said to be *topologically transitive* if there exists $q \in N$ such that the orbit of q for the geodesic flow is dense in N.

PROPOSITION 5.7. *Let M be a complete, connected Riemannian surface such that the geodesic flow on M is topologically transitive. Then the Dubins problem on M has the* UCC *property.*

Proof. Fix $q_0 \in N$ such that

$$\mathcal{O}_{q_0} = \{e^{tf}(q_0) \mid t \in \mathbf{R}\}$$

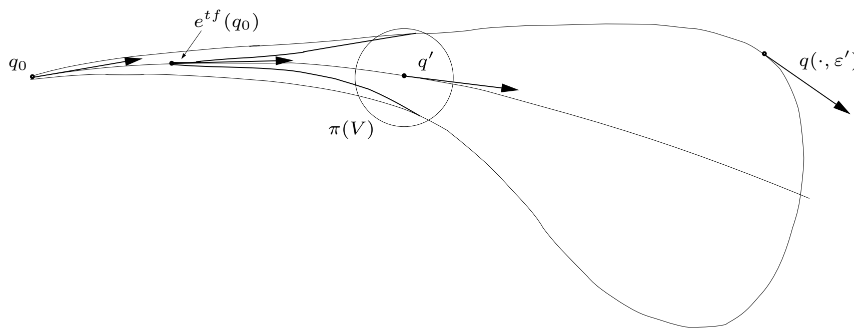

FIG. 3. *Construction of a teardrop trajectory from* $e^{tf}(q_0)$.

is dense in N. The property formulated in (17) implies that for every $\varepsilon > 0$,

$$N = A_{q_0}(M,\varepsilon) \cup \left(A_{q_0^-}(M,\varepsilon)\right)^-,$$

where $(A_{q_0^-}(M,\varepsilon))^-$ denotes the image of $A_{q_0^-}(M,\varepsilon)$ through the involutive diffeomorphism $q \mapsto q^-$. Hence, there exists a decreasing sequence ε_n, converging to zero as n tends to infinity, such that

$$q_0^- \in A_{q_0}(M,\varepsilon_n) \quad \text{for every } n \geq 1 \tag{18}$$

or

$$q_0 \in A_{q_0^-}(M,\varepsilon_n) \quad \text{for every } n \geq 1. \tag{19}$$

Without loss of generality, we can assume that (18) holds, since

$$\mathcal{O}_{q_0^-} = (\mathcal{O}_{q_0})^-$$

is dense in N, which means that the roles of q_0 and q_0^- are symmetric.

Since $\varepsilon \mapsto A_{q_0}(M,\varepsilon)$ is monotone with respect to the inclusion, then q_0 belongs to the set

$$W = \{q \in N \mid q^- \in A_q(M,\varepsilon) \text{ for every } \varepsilon > 0\}.$$

The UCC property for Dubins' problem on M is equivalent to the equality $N = W$ (see Remark 2.2). It is enough to show that $\mathcal{O}_{q_0} \subset W$, due to (5) and to the density of $\mathcal{O}_{q_0}$ in N.

This is trivial for what concerns the negative orbit of q_0. Fix now $\varepsilon, t > 0$. We have to show that there exists an ε-admissible trajectory connecting $e^{tf}(q_0)$ and $(e^{tf}(q_0))^-$. Notice that $A_{e^{tf}(q_0)}(M,\varepsilon)$ contains a neighborhood V of $q' = e^{(t+1)f}(q_0)$. For every $\varepsilon' > 0$, let $q(\cdot,\varepsilon') : [0, T_{\varepsilon'}] \to N$ be an ε'-admissible trajectory such that $q(0,\varepsilon') = q_0$ and $q(T_{\varepsilon'},\varepsilon') = q_0^-$. Then, as ε' goes to zero, we have that $T_{\varepsilon'}$ goes to infinity, while, for every $L > 0$, the restriction of $q(\cdot,\varepsilon')$ on $[0, L]$ (respectively, $[T_{\varepsilon'} - L, T_{\varepsilon'}]$) converges uniformly to $[0, L] \ni s \mapsto e^{sf}(q_0)$ (respectively, $[T_{\varepsilon'} - L, T_{\varepsilon'}] \ni s \mapsto e^{(s-T_{\varepsilon'})f}(q_0^-)$). In particular, for ε' small enough, $q(t+1,\varepsilon')$ and $q(T_{\varepsilon'} - t - 1, \varepsilon')^-$ belong to V (see Figure 3). Provided that $\varepsilon' \leq \varepsilon$, an ε-admissible trajectory steering $e^{tf}(q_0)$ to $(e^{tf}(q_0))^-$ can be obtained by gluing three ε-admissible subtrajectories, the first going from $e^{tf}(q_0)$ to $q(t+1,\varepsilon')$, the second coinciding with the restriction of $q(\cdot,\varepsilon')$ on $[t+1,T_{\varepsilon'}-t-1]$, and the last connecting $q(T_{\varepsilon'}-t-1,\varepsilon')$ and $(e^{tf}(q_0))^-$. □

Recall that the *injectivity radius at a point p of a Riemannian manifold M* is defined as the least upper bound $i_p(M)$ of all $r>0$ such that the map $(t,q)\mapsto \exp(t,q)$, restricted to the punctured disk $(0,r)\times U_pM$, is injective. Before stating sufficient conditions for UCC on nonpositively curved Riemannian surfaces, we prove the following technical result.

LEMMA 5.8. *Let M be a Riemannian surface, $q=(p,v)\in N$, and $\varepsilon,R,\delta>0$. Assume that $R<i_p(M)$, that $K\le 0$ on $B_M(p,R)$, and that $\int_{B_M(p,R)}(-K)dA\le\delta$. There exist $R(\varepsilon),\delta(\varepsilon)>0$, depending only on ε, such that if $R\ge R(\varepsilon)$ and $\delta\le\delta(\varepsilon)$, then there exists an ε-admissible trajectory going from q to q^- and contained in $UB_M(p,R)$.*

Proof. Notice that it is enough to prove the existence of $\delta(\varepsilon)$ and $R(\varepsilon)$ in the case $\varepsilon=1$. Indeed, given $\varepsilon>0$, denote by M^ε the Riemannian surface $(M,\varepsilon m)$. Let, moreover, K^ε and dA^ε be, respectively, the curvature and the surface form on M^ε. Then $K^\varepsilon=\varepsilon K$, $dA=\varepsilon dA^\varepsilon$, and, for every $p\in M$ and $r>0$, $B_M(p,\varepsilon r)=B_{M^\varepsilon}(p,r)$. Thus,

$$\int_{B_M(p,\varepsilon r)}(-K)dA=\int_{B_{M^\varepsilon}(p,r)}(-K^\varepsilon)dA^\varepsilon.$$

The reduction to the case $\varepsilon=1$ is proved (with $\varepsilon R(\varepsilon)=R(1)$ and $\delta(\varepsilon)=\varepsilon\delta(1)$), since an ε-admissible trajectory in M is a 1-admissible trajectory in M^ε and $i_p(M)=\varepsilon i_p(M^\varepsilon)$.

Let $q=(p,v)$ and R be, respectively, a point of N and a positive constant such that $R<i_p(M)$ and $K\le 0$ on $B_M(p,R)$. Denote by Q the square $[-R/2,R/2]\times[-R/2,R/2]\subset\mathbf{R}^2$, endowed with the metric ϕ_q^*m. Recall that Dubins' problem on Q is described by (13)–(15).

Let $u:[0,T]\to[-1,1]$ be a 1-admissible control, and denote by $(x(\cdot),y(\cdot),\theta(\cdot))$ the coordinates of the corresponding trajectory starting from a given $(x_0,y_0,\theta_0)\in UQ$.

Assume that $(x(t),y(t),\theta(t))\in UQ$ and $\dot x(t)>0$ for every $t\in(0,T)$. We can define a map $\tau:[x_0,x(T)]\to[0,T]$ by means of the relation $x(\tau(\xi))=\xi$. Notice that τ is continuous, as well as the function $\eta:[x_0,x(T)]\to\mathbf{R}$ which maps ξ to $\eta(\xi)=y(\tau(\xi))$. Let Ω be the open region of Q defined by

$$\Omega=\cup_{\xi\in(x_0,x(T))}I(\eta(\xi)),$$

where, for every $l\in\mathbf{R}$, $I(l)$ denotes the open interval with 0 and l as boundary points.

Let (X,Y,Θ) be the solution of

$$\begin{cases}\dot X=\cos\Theta,\\ \dot Y=\sin\Theta,\\ \dot\Theta=u\end{cases}$$

with initial condition $(X(0),Y(0),\Theta(0))=(x_0,y_0,\theta_0)$. For every $t\in[0,T]$,

$$\begin{aligned}|\theta(t)-\Theta(t)|=\left|\int_0^t\cos(\theta(s))\frac{B_y(x(s),y(s))}{B(x(s),y(s))}ds\right| &= \left|\int_{x_0}^{x(t)}B_y(\xi,\eta(\xi))d\xi\right|\\ \le\int_{x_0}^{x(t)}d\xi\left(\int_{I(\eta(\xi))}B_{yy}(\xi,v)dv\right) &= \int_{x_0}^{x(t)}\int_{I(\eta(\xi))}-K(\xi,v)B(\xi,v)dv\,d\xi,\end{aligned}$$

where the last equality follows from (8). Since the surface element dA of Q is equal to $B(\xi, v)dv\, d\xi$, we have

$$|\theta(t) - \Theta(t)| \leq \int_Q (-K)dA. \tag{20}$$

Integrating the equations satisfied by $y(\cdot)$ and $Y(\cdot)$, we get, for every $t \in [0, T]$,

$$|y(t) - Y(t)| \leq \int_0^t |\theta(s) - \Theta(s)| ds \leq t \int_Q (-K)dA. \tag{21}$$

Similarly, we obtain

$$\begin{aligned}
|x(t) - X(t)| &= \left| \int_0^t \left(\frac{\cos(\theta(t))}{B(x(t), y(t))} - \cos(\Theta(t)) \right) dt \right| \\
&\leq \left| \int_0^t (\cos(\theta(t)) - \cos(\Theta(t)))\, dt \right| + \left| \int_{x_0}^{x(t)} (1 - B(\xi, \eta(\xi)))\, d\xi \right| \\
&\leq t \int_Q (-K)dA + \int_{x_0}^{x(t)} \left(\int_{I(\eta(\xi))} B_y(\xi, v) dv \right) d\xi \\
&= t \int_Q (-K)dA + \int_{x_0}^{x(t)} \left(\int_{I(\eta(\xi))} \left(\int_{I(v)} (-K(\xi, l) B(\xi, l)) dl \right) dv \right) d\xi \\
&= t \int_Q (-K)dA + \int_{x_0}^{x(t)} \left(\int_{I(\eta(\xi))} |\eta(\xi) - l| (-K(\xi, l) B(\xi, l)) dl \right) d\xi \\
&\leq \left(t + (x(t) - x_0) \frac{R}{2} \right) \int_Q (-K)dA \\
&\leq t \left(1 + \frac{R}{2} \right) \int_Q (-K)dA. \qquad (22)
\end{aligned}$$

Let

$$\delta = \int_Q (-K)dA. \tag{23}$$

Our aim is to show that under suitable assumptions on R and δ, there exists a 1-admissible trajectory contained in UQ which steers $(0, 0, 0)$ to $(0, 0, -\pi)$. As a model we consider the teardrop in the plane of size 1, whose projection is contained in a square of center $(0, 0)$ and side length 6.

Let $\alpha \in (0, \pi/2)$ and consider the trajectory $(x^+(\cdot, \alpha), y^+(\cdot, \alpha), \theta^+(\cdot, \alpha))$ in UQ with initial condition $(x^+(0, \alpha), y^+(0, \alpha), \theta^+(0, \alpha)) = (0, 0, 0)$ corresponding to the control

$$u^+(s, \alpha) = \begin{cases} 1 & \text{if} \quad 0 \leq s \leq \alpha, \\ -1 & \text{if} \quad \qquad s > \alpha. \end{cases}$$

Let $t^+(\alpha)$ be the first time t at which $\theta^+(t, \alpha) = -\pi/2$. The fact that a finite $t^+(\alpha)$ with such property exists, for δ and $1/R$ small enough, depends on the following considerations. First, notice that the trajectory is defined as long as its projection on the plane (x, y) stays in Q; thus, for a time $T_{\max}$ larger than $R/2$. If $\dot{x}^+(s, \alpha) > 0$ for

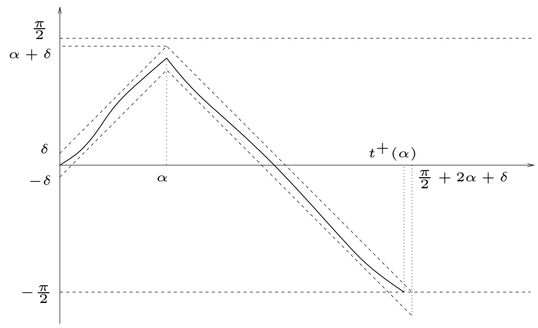

FIG. 4. *The graph of* $\theta^+(\cdot,\alpha)$.

every $s\in[0,t]$, then $|\theta^+(t,\alpha)-\int_0^t u^+(s,\alpha)ds|\leq\delta$, as it follows from (20). Assume that

$$\delta<\pi/2-\alpha. \tag{24}$$

In particular, $\dot{x}^+(t,\alpha)>0$ until $\theta^+(\cdot,\alpha)$ assumes the value $-\pi/2$ or up to time $T_{\max}$. If, moreover,

$$R\geq 2\pi+2\alpha, \tag{25}$$

then $T_{\max}\geq R/2>\pi/2+2\alpha+\delta$ and, since $\int_0^{\pi/2+2\alpha}u^+(s,\alpha)ds=-\pi/2$, $t^+(\alpha)$ is well defined and satisfies

$$\left|t^+(\alpha)-\frac{\pi}{2}-2\alpha\right|\leq\delta. \tag{26}$$

(See Figure 4.) Let $u^-(\cdot,\alpha)=-u^+(\cdot,\alpha)$, define $(x^-(\cdot,\alpha),y^-(\cdot,\alpha),\theta^-(\cdot,\alpha))$ as the trajectory in UQ corresponding to the control $u^-(\cdot,\alpha)$ with initial condition

$$(x^-(0,\alpha),y^-(0,\alpha),\theta^-(0,\alpha))=(0,0,0).$$

As above, if δ and R satisfy (24) and (25), then $t^-(\alpha)$, the first time t for which $\theta^-(t,\alpha)=\pi/2$, is well defined and the estimate

$$\left|t^-(\alpha)-\frac{\pi}{2}-2\alpha\right|\leq\delta \tag{27}$$

holds true.

Fix $R=3\pi$ and notice that, as a consequence, (25) is always satisfied. Let U be a neighborhood of $(\pi/3,\pi/3)$ compactly contained in $(0,\pi/2)\times(0,\pi/2)$. Thus there exists $\delta_0>0$ such that if $\delta\leq\delta_0$, then

$$\mathcal{F}(\alpha_0,\alpha_1)=(x^+(t^+(\alpha_0),\alpha_0),y^+(t^+(\alpha_0),\alpha_0))-(x^-(t^-(\alpha_1),\alpha_1),y^-(t^-(\alpha_1),\alpha_1))$$

is well defined for every (α_0,α_1) in U, and

$$t^+(\alpha_0),t^-(\alpha_1)\leq\frac{3}{2}\pi.$$

Notice that δ_0 depends only on the choice of U. The quantity $|\mathcal{F}(\alpha_0,\alpha_1)|$ measures how far the two trajectories corresponding to the controls $u^+(\cdot,\alpha_0)$ and $u^-(\cdot,\alpha_1)$ are

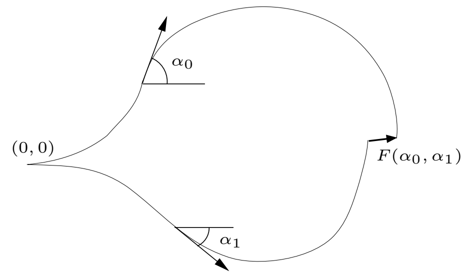

FIG. 5.

from matching their ends (see Figure 5). The lemma is proved if we show that the minimum of $|\mathcal{F}|$ on U is equal to zero.

For every $\alpha \in (0,\pi/2)$, let $(X^{\pm}(\cdot,\alpha), Y^{\pm}(\cdot,\alpha), \Theta^{\pm}(\cdot,\alpha))$ be the trajectories in $U\mathbf{R}^2 \cong \mathbf{R}^2 \times \mathcal{S}^1$ corresponding to the controls $u^{\pm}(\cdot,\alpha)$ and to the initial conditions $(X^{\pm}(0,\alpha), Y^{\pm}(0,\alpha), \Theta^{\pm}(0,\alpha)) = (0,0,0)$. For every (α_0,α_1) in U, let

$$\begin{aligned}\mathcal{G}(\alpha_0,\alpha_1) &= \left(X^{+}\left(\frac{\pi}{2}+2\alpha_0,\alpha_0\right), Y^{+}\left(\frac{\pi}{2}+2\alpha_0,\alpha_0\right)\right)\\ &\quad - \left(X^{-}\left(\frac{\pi}{2}+2\alpha_1,\alpha_1\right), Y^{-}\left(\frac{\pi}{2}+2\alpha_1,\alpha_1\right)\right)\\ &= 2(\sin\alpha_0 - \sin\alpha_1, -\cos\alpha_0 - \cos\alpha_1),\end{aligned}$$

which is the flat counterpart of $\mathcal{F}$. Note that, by construction, $\mathcal{G}(\pi/3,\pi/3) = (0,0)$. Moreover,

$$\det D\mathcal{G}(\pi/3,\pi/3) = \det\begin{pmatrix}\cos\frac{\pi}{3} & -\cos\frac{\pi}{3}\\ \sin\frac{\pi}{3} & \sin\frac{\pi}{3}\end{pmatrix} = \frac{\sqrt{3}}{2},$$

and so if $\mathcal{S}^1 \ni s \mapsto (\alpha_0(s),\alpha_1(s)) \in U$ is a simple closed loop encircling $(\pi/3,\pi/3)$ and being close enough to $(\pi/3,\pi/3)$, then $s \mapsto \mathcal{G}(\alpha_0(s),\alpha_1(s))$ encircles $(0,0)$. Fix a loop $(\alpha_0(\cdot),\alpha_1(\cdot))$ of such kind and let

$$r = \min_{s\in\mathcal{S}^1} d_{\mathbf{R}^2}((0,0),\mathcal{G}(\alpha_0(s),\alpha_1(s))) > 0.$$

We stress that r, by construction, is independent of the specific point p and of the manifold M. In fact, it depends only on the dynamics of the flat Dubins problem.

According to (21) and (22), we get, for every $(\alpha_0,\alpha_1)\in U$ and for $j=0,1$,

$$\begin{aligned}|x^{\pm}(t^{\pm}(\alpha_j),\alpha_j) - X^{\pm}(t^{\pm}(\alpha_j),\alpha_j)| &\le t^{\pm}(\alpha_j)\left(1+\frac{R}{2}\right)\delta \quad\le\quad \frac{3}{2}\pi\left(1+\frac{3}{2}\pi\right)\delta,\\ |y^{\pm}(t^{\pm}(\alpha_j),\alpha_j) - Y^{\pm}(t^{\pm}(\alpha_j),\alpha_j)| &\le t^{\pm}(\alpha_j)\delta \quad\le\quad \frac{3}{2}\pi\delta.\end{aligned}$$

On the other hand, for for every $(\alpha_0,\alpha_1)\in U$ and $j=0,1$,

$$\begin{aligned}|X^{\pm}(t^{\pm}(\alpha_j),\alpha_j) - X^{\pm}(\pi/2+2\alpha_j,\alpha_j)| &\le \delta,\\ |Y^{\pm}(t^{\pm}(\alpha_j),\alpha_j) - Y^{\pm}(\pi/2+2\alpha_j,\alpha_j)| &\le \delta,\end{aligned}$$

as it follows from (26) and (27). Therefore, by triangle inequality,

$$|\mathcal{F}(\alpha_0,\alpha_1)-\mathcal{G}(\alpha_0,\alpha_1)|\leq c_0\delta$$

for every $(\alpha_0,\alpha_1)\in U$, where c_0 is a universal positive constant. If

$$\delta\leq\min\left\{\delta_0,\frac{r}{c_0}\right\}, \tag{28}$$

then it follows from standard degree theory considerations that there exists (α_0,α_1) in U such that $\mathcal{F}(\alpha_0,\alpha_1)=(0,0)$. The lemma is proved, with $R(1)=3\pi$ and $\delta(1)$ equal to the right-hand side of (28). □

From now on, let, for every complete Riemannian surface M, for every $p\in M$ and $r>0$,

$$\Phi(p,r)=\int_{B_M(p,r)}KdA.$$

PROPOSITION 5.9. *Let M be of the first kind. Then the Dubins problem on M is* UCC.

Proof. Fix $q\in N$ and $\varepsilon>0$. Motivated by Remark 2.2, we are going to prove the so-called weak controllability from q, i.e., that there exists $\hat{q}\in A(q,\varepsilon)$ such that $\hat{q}^-\in A(\hat{q},\varepsilon)$.

Let $\tilde{q}\in UX$ be such that $\Pi_*(\tilde{q})=q$ and define $\tilde{q}_1=e^f(\tilde{q})$. Recall that, due to (5), $\tilde{q}_1$ belongs to the interior of $A_{\tilde{q}}(X,\varepsilon)$. Therefore, there exists $\rho>0$ such that for every $\theta\in[-\rho,\rho]$, the ray $[0,\infty)\ni s\mapsto\exp(s,e^{\theta g}(\tilde{q}_1))$ is projected by Π in the interior of $\pi(A_q)$. Denote by S the sector of X spanned by all rays $\exp(\cdot,e^{\theta g}(\tilde{q}_1))$, $\theta\in[-\rho,\rho]$.

Consider the case in which there exists $p\in S$ such that $\Phi(p,R(\varepsilon))\geq-\delta(\varepsilon)$, where $R(\cdot)$ and $\delta(\cdot)$ are the functions whose existence was established by Lemma 5.8. Notice that $i_p(X)=\infty$. Then, for every $\hat{q}\in U_pX$, we would have $\hat{q}^-\in A_{\hat{q}}(X,\varepsilon)$. Since $U_pX\cap A_{\tilde{q}}(X,\varepsilon)\neq\emptyset$, then the weak controllability from q would follow. Thus, we can assume that S is heavy.

Recall that M contains at least one closed geodesic G (see Remark 3.3). The set $\Pi^{-1}(G)$ is the union of infinitely many complete geodesics. Let $\widetilde{G}$ be one of these lifts of G. Fix one endpoint of $\widetilde{G}$ and denote it by z.

Lemma 5.3 implies that, without loss of generality, $z\in S(\infty)$ and $\widetilde{G}$ is not the boundary of a flat half-plane. Since $\widetilde{G}$ is the axis of a hyperbolic isometry contained in Γ, it follows from (6) that there exist components of $\Pi^{-1}(G)$ which are entirely contained in S. Thus, we can assume that $\widetilde{G}\subset S$ (see Figure 6).

Choose a parameterization $c:\mathbf{R}\to X$ of $\widetilde{G}$. A standard application of Topogonov theorem to the triangle $(\pi(\tilde{q}_1),c(t),c(0))$ shows that the angle between $\widetilde{G}$ and the segment from $\pi(\tilde{q}_1)$ to $c(t)$ goes to zero as t goes to plus or minus infinity (see [4, p. 34]).

Fix $p\in G$ and let $\sigma>0$ be such that for every $q'\in U_pM$, $B_N(q',\sigma)\subset A^2_{e^{-f}(q')}(M,\varepsilon)$. There exists a ray from $\pi(\tilde{q})$, contained in S, which intersects $\widetilde{G}$ at a point which projects to p, with an angle smaller than σ, and at a distance from $\pi(\tilde{q})$ larger than one.

Therefore, there exists an admissible curve for $(\mathcal{D}^M_\varepsilon)$ which starts from the point-with-direction q and arrives tangentially on G. Moreover, the same procedure can be applied in order to arrive tangentially on G with the opposite orientation. Gluing

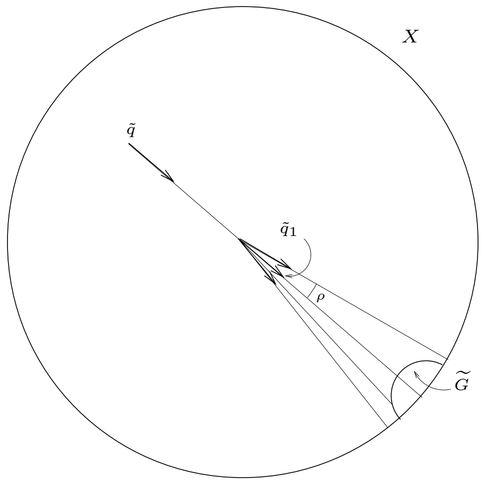

FIG. 6.

pieces of admissible trajectories, one obtains an admissible strategy which connects q with q^-. ∎

In the above proof we additionally gave an argument for the following proposition.

PROPOSITION 5.10. *Let M be a connected complete nonpositively curved Riemannian surface and denote by X its universal covering. If for every $r > 0$ and every sector S of X, $\sup_{p\in S} \Phi(p, r) = 0$, then the Dubins problem on M is* UCC.

Remark 5.11. If $\lim_{p\to\infty,\, p\in M} K(p) = 0$, then $\lim_{p\to\infty,\, p\in X} \Phi(p, r) = 0$ for every $r > 0$, and therefore the Dubins problem on M is UCC. This does not come unexpected. Indeed, it is a special case of [6, Theorem 5.14], where unrestricted complete controllability was proved for all complete, connected Riemannian surfaces whose curvature tends to zero at infinity.

COROLLARY 5.12. *If $K \le 0$ and $\int_M KdA$ is finite, then the Dubins problem on M is* UCC.

Proof. Assume that the UCC property does not hold. In particular, M is of the second kind and, according to Proposition 5.10, there exist $r > 0$ and a sector S in X such that

$$\sup_{p\in S} \Phi(p, r) < 0, \tag{29}$$

where X denotes the universal covering of M. Then S is heavy and Lemma 5.4 guarantees the existence of a half-plane $H \subset S$ embedded by Π in M. Finally, $\int_M KdA = \int_H KdA = -\infty$. ∎

Remark 5.13. Let M be a complete, connected unbounded Riemannian surface and assume that $K \le 0$ outside a compact subset C of M. Then, for every $p \in M\setminus C$, for every $q \in U_pM$, the square $[-d(p,C)/2, d(p,C)/2] \times [-d(p,C)/2, d(p,C)/2]$ can be endowed with the metric $\phi_q^* m$. Therefore, for every $r > 0$, on a complement of a compact subset of M, which depends on r, it is possible to define

$$\widetilde{\Phi}(p, r) = \int_{\tilde{B}(p,r)} KdA,$$

where $\tilde{B}(p, r)$ is the ball of center $(0, 0)$ and radius r contained in a sufficiently large square of the type $[-\rho, \rho] \times [-\rho, \rho]$ endowed with $\phi_q^* m$, for some $q \in U_pM$. The definition makes sense, since the value of $\int_{\tilde{B}(p,r)} KdA$ does not depend on the choice of q. Clearly, if $i_p(M) \geq r$, then $\widetilde{\Phi}(p,r) = \Phi(p,r)$.

If $\lim_{p\to\infty} \widetilde{\Phi}(p,r) = 0$, then the Dubins problem on M is UCC. Indeed, it is a general fact that on an unbounded surface M every attainable set is unbounded (see Lemma 5.5). Then, for every $\varepsilon > 0$ and every $q \in N$, there exists $\hat{q} \in A_q(M,\varepsilon)$ such that $\widetilde{\Phi}(\pi(\hat{q}), R(\varepsilon))$ is well defined and it is larger than $-\delta(\varepsilon)$. Applying Lemma 5.8 to $(-R(\varepsilon), R(\varepsilon)) \times (-R(\varepsilon), R(\varepsilon))$, endowed with $\phi_{\hat{q}}^* m$, we obtain the desired weak controllability from q.

5.3. Noncontrollability results.

LEMMA 5.14. *Let M be a complete Riemannian surface. If a half-plane H in M exists, such that $\sup_H K < 0$, then the Dubins problem on M is not* UCC.

Proof. Let $a = |\sup_H K| > 0$. Fix a system of geodesic coordinates (x,y) on H such that H is parameterized by $\{(x,y)|\ x \in \mathbf{R}, y \geq 0\}$. Let us compare $F(x,y)$, defined as in (12), with

$$G(y) = \sqrt{a}\tanh\left(\sqrt{a}y\right),$$

the solution of (16) corresponding to $K \equiv -a$. If $(x,y) \in H$ and $F(x,y) = G(y)$, then $F_y(x,y) - G_y(y) = -K(x,y) - a \geq 0$. Therefore

$$F(x,y) \geq \sqrt{a}\tanh\left(\sqrt{a}y\right). \tag{30}$$

Let $q = (0,1,\pi/2) \in UH$. We claim that for ε small enough, the attainable set $A_q(M,\varepsilon)$ is contained in the interior of UH. Indeed, let $\varepsilon \in (0, \sqrt{a}\tanh(\sqrt{a}))$ and consider any admissible control $u : [0,\infty) \to [-\varepsilon,\varepsilon]$. Denote by $(x(\cdot), y(\cdot), \theta(\cdot))$ its corresponding trajectory such that $(x(0), y(0), \theta(0)) = q$. Define $\theta_0 \in (0,\pi/2)$ through the relation

$$\sqrt{a}\tanh\left(\sqrt{a}\right)\cos\theta_0 = \varepsilon \tag{31}$$

and let

$$T = \sup\{\tau > 0 \,|\, \pi - \theta_0 \leq \theta(\tau) \leq \theta_0\}.$$

Notice that $y(t) \geq 1$ for every $t \in [0,T)$. It follows from (30) that $F(x(t), y(t)) \geq \sqrt{a}\tanh(\sqrt{a})$ for every $t \in [0,T)$. Therefore, if T were finite, then

$$F(x(T), y(T))\cos\theta_0 > \varepsilon,$$
$$F(x(T), y(T))\cos(\pi - \theta_0) < -\varepsilon.$$

This would not be compatible with (15), the equation satisfied by $\theta(\cdot)$, and the definition of T. $\square$

COROLLARY 5.15. *If K is uniformly negative and M is of the second kind, then the Dubins problem is not* UCC. *More precisely, if $0 < \varepsilon < |\sup_M K|$, then (D_ε^M) is not completely controllable.*

Proof. The first part of the statement follows from Lemma 5.4, which ensures that Π embeds a half-plane of X in M. The quantitative assertion is easily derived from the proof proposed for Lemma 5.14, where the point $q_1 = (0,1,\pi/2)$ can be replaced by $(0,y,\pi/2)$, with y arbitrarily large. $\square$

LEMMA 5.16. *Let M be a complete Riemannian surface. Let $H \subset M$ be a half-plane such that K is nonpositive and bounded on H. Assume that there exists $r > 0$ such that $\sup_{p\in H}\Phi(p,r) < 0$. Then the Dubins problem on M is not* UCC.

Proof. Parameterize H over $\{(x,y)|\ x\in\mathbf{R}, y\geq 0\}$ with a system of geodesic coordinates. By contradiction, let us assume that the Dubins problem on M is UCC. Therefore, for every $\varepsilon>0$, there exists a curve leaving from $(0,0)$ in the direction $(0,0,\pi/2)$ and coming back to ∂H in finite time, having geodesic curvature bounded by ε. In particular, there exists an ε-admissible trajectory $q=(x,y,\theta):[0,T]\to N$ such that $q(0)=(0,0,\pi/2)$ and $q(T)=(x(T),y(T),k\pi)$, with $x(T)\in\mathbf{R}$, $y(T)>0$, and $k\in\{0,1\}$.

Without loss of generality, $\theta(t)\in(0,\pi)$ for every $t\in[0,T)$. Hence, $t\mapsto y(t)$ is nonnegative and strictly increasing on $[0,T]$. Notice that T tends to infinity as ε goes to zero, since otherwise the Ascoli theorem would immediately lead to a contradiction.

The following claim will be repeatedly applied.

CLAIM 5.17. *For every $\rho,\tau>0$, there exists $\varepsilon(\rho,\tau)>0$ with the following property: let $\gamma:[0,\tau]\to H$ be a curve with geodesic curvature bounded by $\varepsilon(\rho,\tau)$, parameterized by arc-length and such that $\phi_{\dot\gamma(0)}([0,\tau]\times[-\rho,\rho])$ is contained in H; then, for every $t\in[0,\tau]$, $\mathrm{dist}(\gamma(t),\exp(t,\dot\gamma(0)))<\rho$.*

The claim can be obtained by a standard perturbation argument, as ε approaches zero, applied to the system (13)–(15). The use of geodesic coordinates is justified by the hypothesis that $\phi_{\dot\gamma(0)}([0,\tau]\times[-\rho,\rho])\subset H$. Notice that ε can be chosen only depending on τ and ρ, because K is uniformly bounded from below.

Up to a global rescaling of M, as performed in the proof of Lemma 5.8, we can assume that $r=1/4$. Assume from now on that $\varepsilon<\varepsilon(1/4,1)$ and $T>2$.

Let T_1 be the largest number between $T-2$ and the maximal time $t\in[0,T)$ such that $\theta(t)=\pi/2$. Let, moreover,

$$\Omega=\cup_{t\in(T_1,T)}\{x(t)\}\times(0,y(t)),$$

and D be the disk of center $p=\exp(1/2,\mathcal{R}q(T-1))$ and radius $1/4$. Assume that $\theta(T)=0$ (symmetric arguments hold when $\theta(T)=\pi$). The Gauss–Bonnet theorem, applied to Ω, implies that

$$\theta(T_1)+\int_\Omega(-K)dA=-\int_{T_1}^T u(t)dt.$$

Since both $\theta(T_1)$ and $\int_\Omega(-K)dA$ are nonnegative, then

$$\max\left\{\theta(T_1),\int_\Omega(-K)dA\right\}\leq\left|\int_{T_1}^T u(t)dt\right|\leq 2\varepsilon. \tag{32}$$

A contradiction is reached if we show that for ε small enough, D is contained in Ω. It follows from (32) and the definition of T_1 that for $\varepsilon<\pi/4$, $T_1=T-2$. Claim 5.17 implies that $y(1)\geq 3/4$. Thus, the y-component of p is larger than $y(T-1)-1/2>y(1)-1/2\geq 1/4$. Therefore, D does not intersect the line $\{(x,0)|\ x\in\mathbf{R}\}$. If D contained a point of the type $\pi(q(\tau))$, for $\tau\in[T-2,T]$, then the distance from p to the geodesic $s\mapsto\exp(s,q(T-1))$ would be smaller than

$$d(p,\pi(q(\tau)))+d(\pi(q(\tau)),\exp(\tau-T+1,q(T-1)))<\frac{1}{2}.$$

(See Figure 7.) Therefore, drawing the minimizing segment from p to the considered geodesic, we would construct a triangle (having p and $\pi(q(T-1))$ as two of its vertices) with two orthogonal angles. This is impossible, due to the Gauss–Bonnet theorem.

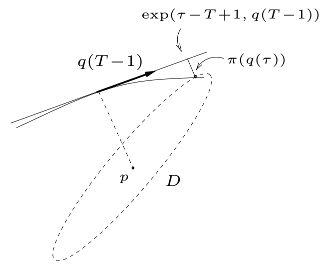

FIG. 7.

Assume now that $y \in [0, y(T)]$ exists such that $(x(T), y)$ belongs to D. Then the distance from $\exp(-1, q(T))$ to the geodesic $s \mapsto (x(T), s)$ is strictly smaller than one. Again, this leads to a contradiction.

The same argument shows that D does not intersect $\Sigma = \{\exp(s, \mathcal{R}q(T-2)) | \, s \in J\}$, where J is the maximal interval containing zero such that $\exp(s, \mathcal{R}q(T-2)) \in H$ for every $s \in J$. Since $\theta(T-2)$ is contained in $(0, \pi/2)$, and Σ cannot intersect $\{(x(T-2, y)| \, y \geq 0\}$ twice, then the stated inclusion of D in Ω follows. ∎

From Lemma 5.4, one easily obtains the next corollary.

COROLLARY 5.18. *Let M be a complete, connected Riemannian surface. Assume that K is nonpositive and bounded on M and that M is of the second kind. Let X be the universal covering of M. Assume that* $\sup_{p \in S} \Phi(p, r) < 0$ *for some sector S of X and some $r > 0$. Then the Dubins problem on M is not* UCC.

5.4. Riemannian surfaces with nonpositive curvature outside a compact set. In this section, let us focus on Riemannian surfaces with sign-varying curvature, allowing K to be positive on a bounded set. Recall that a subset U of a finitely connected complete Riemannian surface M is called a *Riemannian half-cylinder* if it is diffeomorphic to $\mathcal{S}^1 \times [0, \infty)$. With a Riemannian half-cylinder U, we can associate its curvature at infinity $K_\infty(U) = -\int_U K dA - k(\partial U)$, where $k(\partial U)$ denotes the integral of the geodesic curvature of ∂U. The Cohn-Vossen theorem [8, 27] guarantees that $K_\infty(U) \geq 0$. When $K_\infty(U)$ is strictly positive, U is called a *strict Riemannian half-cylinder*. Notice that if U' is a Riemannian half-cylinder homotopic to U, then $K_\infty(U) = K_\infty(U')$. In particular, U' is strict if and only if U is.

We can prove the following.

PROPOSITION 5.19. *Let M be a finitely connected complete Riemannian surface such that K is bounded and nonpositive outside a compact set. Assume, moreover, that every Riemannian half-cylinder in M is strict. Then the Dubins problem on M is* UCC *if and only if, for every half-plane H contained in M and every $r > 0$,* $\sup_{p \in H} \Phi(p, r) = 0$. *In particular, the Dubins problem on M is* (i) UCC *if* $\int_M K dA > -\infty$*;* (ii) *not* UCC *if there exists a strict Riemannian half-cylinder $U \subset M$ such that* $\sup_{p \in U} \Phi(p, r) < 0$ *for some $r > 0$.*

We do not provide a complete argument, since Proposition 5.19 basically follows from what precedes and from the results of Shioya [27]. Let us just sketch the main points of the proof.

Shioya proved that if U is a strict Riemannian half-cylinder, then, through every point of U far enough from ∂U, passes a complete geodesic $c : \mathbf{R} \to M$, contained in U and which has a finite number $n(U)$ of self-intersections (only depending on U). In addition, $n(U) = 0$ if $\int_U K dA = -\infty$, which means that U contains a half-plane of M. When $n(U) \geq 1$, the self-intersections of c are *regular* in the sense defined in

Chapter 1 of [27]. Such notion of regularity guarantees that up to reorientation of c, the restriction of $\phi_{\dot c(0)}$ to $\mathbf{R}\times[0,+\infty)$ takes values in U and each of its preimages is finite, with cardinality bounded by $n(U)+1$. Therefore, $\phi_{\dot c(0)}(\mathbf{R}\times[0,+\infty))$ lifts to a nonpositively curved half-plane.

If there exist $r>0$ and a half-plane H contained in M such that $\sup_{p\in H}\Phi(p,r)<0$, then the Dubins problem on M is not UCC (Proposition 5.16).

Fix $q\in N$ and $\varepsilon>0$ and assume that, for every half-plane H contained in M and every $r>0$, $\sup_{p\in H}\Phi(p,r)=0$. Recall that $A_q(M,\varepsilon)$ is unbounded. If $\int_U K dA > -\infty$ for some Riemannian half-cylinder U, then, for every $r>0$, $\lim_{p\in U,p\to\infty}\widetilde{\Phi}(p,r)=0$. (For the definition of $\widetilde{\Phi}$, see Remark 5.13.) If $A_q(M,\varepsilon)$ contains a point $\hat q$ such that $\widetilde{\Phi}(\pi(\hat q),R(\varepsilon))=0$, then the weak controllability from q is proved. Otherwise, there must exist a Riemannian half-cylinder U such that $\int_U K dA=-\infty$ and $\pi(A_q(M,\varepsilon))\cap U$ is unbounded. Thus, there exist a parameterized complete geodesic $c(\cdot)$ of the type described above and $\hat q\in\pi^{-1}(c(0))\cap A(q,\varepsilon)$ pointing inside the half-plane $\phi_{\dot c(0)}(\mathbf{R}\times[0,+\infty))$. From the usual reasoning based on (5), $\phi_{\dot c(0)}(\mathbf{R}\times[0,+\infty))\cap\pi(A_q(M,\varepsilon))$ contains a sector S. If $\lim_{p\in S,\,p\to\infty}\Phi(p,R(\varepsilon))=0$, then the weak controllability from q follows. Otherwise, $\int_S K dA=-\infty$. From a visibility property analogous to Lemma 5.2, it follows that S contains a half-plane of M. The weak controllability from q follows.

Remark 5.20. The argument above shows that when $\sup_{p\in H}\Phi(p,r)=0$ for every half-plane H contained in M and every $r>0$, UCC holds even in the case in which K is unbounded. Analogously, Lemma 5.14 implies that, if K is uniformly negative (possibly unbounded) on a strict Riemannian half-cylinder, then the Dubins problem on M is not UCC.

6. Remarks on optimal control. Let M be any complete oriented Riemannian surface, with no assumption on the sign of K. Denote by $\langle\lambda,\cdot\rangle$ the action of a covector $\lambda\in T_q^*N$ on T_qN.

Fix $q_1=(p_1,v_1)$, $q_2=(p_2,v_2)\in N$ and assume that $q_2\in A_{q_1}(M,\varepsilon)$. Then, there exists an ε-admissible curve $q:[0,T]\to N$ such that $q(0)=q_1$, $q(T)=q_2$, and for which T is minimal. According to the Pontryagin maximum principle, the time-optimality of $q(\cdot)$ implies that there exist an absolutely continuous curve $\lambda:[0,T]\to T^*N$ and a constant $c\ge 0$ such that for almost every $t\in[0,T]$, $\lambda(t)\in T^*_{q(t)}N\setminus\{0\}$ and the function

$$H_w(\lambda)=\langle\lambda,f(q)\rangle+w\,\langle\lambda,g(q)\rangle\,,\quad \lambda\in T_q^\star N,\quad q\in N,\quad w\in[-\varepsilon,\varepsilon],$$

verifies

$$(33)\quad H_{u(t)}(\lambda(t))=\max_{w\in[-\varepsilon,\varepsilon]}H_w(\lambda(t)) \;=\; \langle\lambda,f(q)\rangle+\varepsilon|\langle\lambda(t),g(q(t))\rangle| \;\equiv\; c,$$

$$(34)\quad \dot\lambda(t)=\overrightarrow{H_{u(t)}}(\lambda(t)),$$

where, for every $w\in[-\varepsilon,\varepsilon]$, $\overrightarrow{H_w}$ denotes the Hamiltonian vector field associated with $H_w:T^*N\to\mathbf{R}$.

A simple computation shows that

$$[f\pm g,[f,g]]=-K\,g\pm f.$$

In particular, $(f,g,[f,g])$, $(g,[f,g],[f+g,[f,g]])$, and $(g,[f,g],[f-g,[f,g]])$ are three moving frames in N. Therefore, the results by Schättler [24, 25] (see also [2]) imply that $q(\cdot)$ is piecewise smooth. A subinterval I of $[0,T]$ is called a *bang arc* if, up to a modification on a set of measure zero, u is constant and equal to ε or $-\varepsilon$ along I, while it is called a *singular arc* if $u|_I$ is smooth (again, up to a modification on a set of measure zero) but $|u|$ is not constantly equal to ε on I. Schättler's results give restrictions on the structure of time-optimal controls, stating that for every $q \in N$, there exists a neighborhood U of q such that every time-optimal trajectory of (D_ε) contained in U is the concatenation of three bang arcs or of a bang, a singular, and a bang arc (where some arc possibly has zero length).

In order to show that every time-optimal trajectory follows locally Dubins' pattern, let us prove that every singular arc corresponds to a geodesic of M, i.e., that $u = 0$ almost everywhere along singular arcs.

The covector lift $\lambda(\cdot)$ of a time-optimal trajectory $q : [0,t] \to N$ can be represented through the three real-valued functions:

$$\begin{aligned}\varphi_1(t) &= \langle \lambda(t), f(q(t)) \rangle\,,\\ \varphi_2(t) &= \langle \lambda(t), g(q(t)) \rangle\,,\\ \varphi_3(t) &= \langle \lambda(t), [f,g](q(t)) \rangle\,.\end{aligned}$$

Note that, according to (33),

$$u(t) = \operatorname{sign}(\varphi_2(t))\,\varepsilon \tag{35}$$

for almost every t such that $\varphi_2(t) \neq 0$ and that

$$|\varphi_1(t)| + |\varphi_2(t)| + |\varphi_3(t)| \neq 0 \tag{36}$$

for every t, since $\lambda(t) \neq 0$ and f, g, and $[f,g]$ are everywhere linearly independent.

Equation (34) implies that φ_1, φ_2, φ_3 satisfy the system of differential equations

$$\dot\varphi_1(t) = -u(t)\varphi_3(t), \tag{37}$$

$$\dot\varphi_2(t) = \varphi_3(t), \tag{38}$$

$$\dot\varphi_3(t) = u(t)\varphi_1(t) - K(q(t))\varphi_2(t). \tag{39}$$

Let $I \subset [0,T]$ be a singular arc. Modify, if necessary, u on a set of measure zero in order to render it smooth on I. By definition of singular arc, $\varphi_2|_{I'} \equiv 0$ on some subinterval I' of I. From (38), we have that $\varphi_3 = 0$ on I'. Plugging the information in (37) and (39), we get that φ_1 is constant and that $\varphi_1(t)u(t) = 0$ for every $t \in I'$. Due to (36), $\varphi_1 \neq 0$, and thus $u = 0$ along I'. Since u is smooth on I, it follows that $\varphi_2 = u = 0$ on I.

We proved that every singular arc of a time-optimal trajectory of (D_ε) (independently of the sign of K) corresponds to a geodesic segment and that time-optimal trajectories follow locally Dubins' pattern. The following proposition provides further restrictions on the structure of time-optimal trajectories under the hypothesis that K is uniformly negative.

PROPOSITION 6.1. *If K is uniformly negative and ε is small enough, then every time-optimal trajectory $q : [0,T] \to N$ of (D_ε) is the concatenation of a bang, a singular, and a bang arc (some of which possibly having zero length). In particular, $q(\cdot)$ follows Dubins' pattern.*

Proof. Assume that

$$K \leq -\varepsilon^2. \tag{40}$$

From (33) and (35), we get that $u\,\varphi_1 = \mathrm{sign}(\varphi_2)\,\varepsilon\,c - \varepsilon^2\varphi_2$ and so, for almost every $t \in [0,T]$,

$$\ddot{\varphi}_2(t) = -(K(q(t)) + \varepsilon^2)\varphi_2(t) + \mathrm{sign}(\varphi_2(t))\,\varepsilon\,c. \tag{41}$$

Therefore, $\ddot{\varphi}_2\varphi_2 \geq 0$ almost everywhere. If τ is such that $\varphi_2(\tau) = 0$ and $\varphi_2 > 0$ on a left neighborhood of τ, then $\varphi_2 > 0$ on $(\tau, T]$. Symmetric arguments show that the set $\{t \in [0,T] \mid \varphi_2(t) \neq 0\}$ has at most two connected components and that $q(\cdot)$ is the concatenation of a bang, a singular, and a bang arc. $\square$

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REFERENCES

[1] A. A. Agrachev and Y. L. Sachkov *Control Theory from the Geometric Viewpoint*, Encyclopaedia Math. Sci., 87, Springer-Verlag, Berlin, 2004.

[2] A. A. Agrachev and M. Sigalotti, *On the local structure of optimal trajectories in* $\mathbf{R}^3$, SIAM J. Control Optim., 42 (2003), pp. 513–531.

[3] W. Ballmann, *Axial isometries of manifolds of non-positive curvature*, Math. Ann., 259 (1982), pp. 131–144.

[4] W. Ballmann, M. Gromov, and V. Schroeder, *Manifolds of Nonpositive Curvature*, Progr. Math., 61, Birkhäuser, Boston, 1985.

[5] U. Boscain and Y. Chitour, *Time optimal synthesis for left-invariant control systems on* $SO(3)$, SIAM J. Control Optim., to appear.

[6] Y. Chitour, *Applied and Theoretical Aspects of the Controllability of Nonholonomic Control System*, Ph.D. Thesis, Rutgers University, New Brunswick, NJ, 1996.

[7] Y. Chitour and M. Sigalotti, *Dubins' problem on surfaces.* I. *Nonnegative curvature*, J. Geom. Anal., 15 (2005), pp. 565–587.

[8] S. Cohn-Vossen, *Kürzeste Wege und Totalkrümmung auf Flächen*, Compos. Math., 2 (1935), pp. 69–133.

[9] L. E. Dubins, *On curves of minimal length with a constraint on average curvature and with prescribed initial and terminal positions and tangents*, Amer. J. Math., 79 (1957), pp. 497–516.

[10] P. Eberlein and B. O'Neill, *Visibility manifolds*, Pacific J. Math., 46 (1973), pp. 45–110.

[11] A. Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv., 32 (1957), pp. 13–72.

[12] V. Jurdjevic, *Geometric Control Theory*, Cambridge Stud. Adv. Math. 52, Cambridge University Press, Cambridge, UK, 1997.

[13] W. Klingenberg, *A Course in Differential Geometry*, Graduate Texts in Math. 51, Springer-Verlag, New York, Heidelberg, 1978.

[14] A. Krener, *A generalization of Chow's theorem and the bang-bang theorem to non-linear control problems*, SIAM J. Control, 12 (1974), pp. 43–51.

[15] C. Lobry, *Contrôlabilité des systèmes non linéaires*, SIAM J. Control, 8 (1970), pp. 573–605.

[16] L. A. Lyusternik and A. I. Fet, *Variational problems on closed manifolds*, Dokl. Akad. Nauk SSSR (N.S.), 81 (1951), pp. 17–18 (in Russian).

[17] D. Mittenhuber, *Dubins' problem in the hyperbolic plane using the open disc model*, in Geometric Control and Non-Holonomic Mechanics (Mexico City, 1996), CMS Conf. Proc. 25, Amer. Math. Soc., Providence, RI, 1998, pp. 115–152.

[18] D. Mittenhuber *Dubins' problem is intrinsically three-dimensional*, ESAIM Control Optim. Calc. Var., 3 (1998), pp. 1–22.

[19] F. Monroy–Perez, *Three-dimensional non-Euclidean Dubins' problem*, in Geometric Control and Non-Holonomic Mechanics (Mexico City, 1996), CMS Conf. Proc. 25, Amer. Math. Soc., Providence, RI, 1998, pp. 153–181.

[20] R. N. Murray, Z. X. Li, and S. S. Sastry, *A Mathematical Introduction to Robotic Manipulation*, CRC Press, Boca Raton, FL, 1994.

[21] J. C. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Graduate Texts in Math. 149, Springer-Verlag, New York, 1994.

[22] J. A. Reeds and L. A. Shepp, *Optimal paths for a car that goes both forwards and backwards*, Pacific J. Math., 145 (1990), pp. 367–393.

[23] T. Sakai, *Riemannian Geometry*, Transl. Math. Monogr. 149, Amer. Math. Soc., Providence, RI, 1996.
[24] H. Schättler, *On the local structure of time-optimal bang-bang trajectories in* $\mathbf{R}^3$, SIAM J. Control Optim., 26 (1988), pp. 186–204.
[25] H. Schättler *The local structure of time-optimal trajectories in dimension three under generic conditions*, SIAM J. Control Optim., 26 (1988), pp. 899–918.
[26] U. Serres, *On the curvature of two-dimensional optimal control systems and Zermelo's navigation problem*, J. Math. Sci. to appear.
[27] T. Shioya, *Behavior of distant maximal geodesics in finitely connected complete* 2*-dimensional Riemannian manifolds*, Mem. Amer. Math. Soc., 108, Providence, RI, 1994.
[28] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Brandeis University, Waltham, MA, 1970.
[29] H. J. Sussmann, *Shortest* 3*-dimensional paths with a prescribed curvature bound*, in Proc. IEEE CDC (New Orleans, 1995), IEEE Publications, New York, 1995, pp. 3306–3312.
[30] H. J. Sussmann and G. Tang, *Shortest Paths for the Reeds-Shepp Car: A Worked Out Example of the Use of Geometric Techniques in Nonlinear Optimal Control*, Rutgers Center for Systems and Control Technical Report 91-10, Rutgers University, New Brunswick, 1991.
[31] G. Thorbergsson, *Closed geodesics on non-compact Riemannian manifolds*, Math. Z., 159 (1978), pp. 249–258.