1

Linear Systems Subject to Input Saturation and Time Delay: Global Asymptotic Stabilization

Karim Yakoubi Yacine Chitour

***Abstract*— This paper deals with the problem of global asymptotic stabilization of linear systems by bounded static feedbacks subject to time delay. Under standard necessary conditions, we provide two constructions of controllers of nested saturation type which extends to the general case partial results of [2], [3], for arbitrary small bound on the control and large (constant) delay. To validate the approach, a third-order integrator and oscillator with multiplicity two example is presented.**

***Index Terms*— saturation, bounded controls, Lyapunov functions, time-delay systems, uncertainty, stabilization.**

I. Introduction

In this paper, we address the global asymptotic stabilization issue for linear systems subject to input saturation, i.e. of the type,

$$(S) : \dot{x}(t) = Ax(t) + Bu(t-h), \tag{1}$$

where (i) $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, with n the dimension of the system and m the number of inputs; (ii) the control u verifies $\|u\| \leq r$, where $r \in (0,1]$ only depends on (S); (iii) there is an arbitrary constant delay $h \geq 0$ in the input.

We use $(S)_h^r$, $r \in (0,1]$, $h > 0$, to denote the control system (S) with input bound r and input time delay h. We omit the index r if it is equal to one and, similarly for the index h if it is equal to zero.

Our problem is that of globally asymptotically stabilizing $(S)_h^r$ to the origin by mean of a static feedback. We then seek u as

$$u(t-h) = -r\sigma\left(F_h^r(x(t-h))\right), \tag{2}$$

where the non-linearity σ is of "saturation" type (definitions are given in section (2)) and the function $F_h^r : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is at least locally Lipschitz (to obtain at least locally solutions).

In the zero-delay case, the stabilization of linear systems with saturating actuators has been widely investigated in the last years: see for example, [10], [11] and [4] and references herein where the authors globally asymptotically stabilized the system (S) by static feedback with nested saturation ([10], [11]) and by maximal ellipsoid saturation ([4]). It is well-known that such a global asymptotic stabilization is possible if and only if (S) satisfies (C) : all eigenvalues of A are in the closed left-half plane and (A, B) is stabilizable.

It is trivial to see that condition (C) is also necessary in the case of non zero delay and it seems natural to expect condition (C) to be also sufficient. In that regard, partial results have been recently obtained by Mazenc, Mondié and Niculescu. More precisely, they extended in [2] the nested saturation construction when A is nilpotent and proved global asymptotic stabilization (GAS) of $(S)_h^r$, for any input rate $r \in (0,1]$ and arbitrary delay $h > 0$. We refer to such a property as delay-independent GAS. In the case of a two-dimensional oscillator, they showed a similar result in [3]. Our main results is to complete that line of work, namely to show that condition (C) is sufficient for delay-independent GAS.

We will actually provide a solution for the delay-independent GAS problem, which directly uses the nested saturated feedbacks of [10] and can be seen as a generalization of [2], [3]. However, the argument is an extension to the non-zero delay case of that of [10]. Recall that, at the heart of the argument of [10], lies a result on finite-gain L^∞-stability for one and two dimensional neutrally stable linear systems subject to input saturation. Such an argument was first introduced in [1], where was addressed the issue of finite-gain L^p-stability of neutrally stable linear systems subject to input saturation. In a related work ([13]), we extend to the non zero delay case the results of [1]. The goal consists of determining a suitable "storage function" V and establishing for V an appropriate dissipation inequality of the form $\frac{dV(x_u(t))}{dt} \leq -\|x_u(t)\|^2 + \lambda_p\|u(t)\|^2$, for some constant $\lambda_p > 0$ possibly depending on the input bound r and the delay h. Recall that the "storage function" in ([1]), denoted here by V_0, is non-smooth. In the present situation, the "storage function" V is the sum of a term similar to V_0 and a Lyapunov-Krasovskii functional (as used in [2], [3], in order to take care of the delay). One has then to pay attention to the several constants involved in the computations in order to get delay-independent GAS.

Organization of the paper. In Section 2, we provide the main notations used in the paper and state the delay-independent stabilization results. In Section 3, the solution based on feedbacks of saturation type is described together. A third-order integrator and oscillator with multiplicity two example is proposed in section 4.

II. Notations

For $x \in \mathbb{R}^n$, $\|x\|$ and x^T denote respectively the Euclidean norm of x and the transpose of x. Similarly, for any $n \times m$ matrix K, K^T and $\|K\|$ denote respectively the transpose of K and the induced $2-$norm of K. Moreover, $\lambda_-(K)$ and $\lambda_+(K)$ denote the minimal and the maximal singular values of the matrix K. If $f(.)$ and $g(.)$ are two real-valued functions, we mean by $f(\tau) \asymp_0 g(\tau)$, that there are positive constants ξ_1 and ξ_2, independent of τ, with $\xi_1 \leq \xi_2$ such that for τ is in some neighborhood of 0,

$$\xi_1 g(\tau) \leq f(\tau) \leq \xi_2 g(\tau).$$

K. Yacoubi and Y. Chitour are with the Laboratoire des signaux et systèmes, Université Paris-Sud, CNRS, Supélec, 91192 Gif-sur-Yvette, France.

For $h > 0$, let $x_t(\theta) := x(t+\theta)$, $-h \le \theta \le 0$. Initial conditions for time-delay systems are elements of $C_h := C([-h,0],\mathbb{R}^n)$, the Banach space of continuous $\mathbb{R}^n$-valued functions defined on $[-h,0]$, equipped with the supremum norm, $\|x_t\|_h = \sup_{-h\le\theta\le 0}\|x(t+\theta)\|$. Then, for $x_t \in L^\infty([-h,\infty),\mathbb{R}^n)$, we have $\|x_t\|_{L^\infty} = \text{ ess } \sup_{-h\le s<\infty}\|x(t+s)\|$.

Definition 1:. (*Saturation function*) We call $\sigma:\mathbb{R}\longrightarrow\mathbb{R}$ a saturation function ("S-function" for short) if there exist two real numbers $0<a\le K_\sigma$ such that for all $t,t'\in\mathbb{R}$,

(i) $|\sigma(t)-\sigma(t')|\le K_\sigma \inf(1,|t-t'|)$,

(ii) $|\sigma(t)-at|\le K_\sigma t\sigma(t)$,

Hypothesis (ii) implies that σ is differentiable at $t=0$ and $\sigma'(0)=a$. A constant K_σ defined as above is called an S-bound for σ. When the context is clear, we simply use K to denote an S-bound. Note that (i) is equivalent to the fact that σ is bounded and globally Lipschitz, while (ii) is equivalent to $t\sigma(t)>0$ for $t\neq 0$, $\liminf_{|t|\to\infty}|\sigma(t)|>0$ and $\limsup_{t\to 0}(\sigma(t)-t)/t^2$ is finite.

For an $m-$tuple $k=(k_1,\ldots,k_m)$ of nonnegative integers, define $|k| = k_1+\ldots+k_m$. We say that σ is an $\mathbb{R}^{|k|}-$ valued S-function if $\sigma = (\sigma_1,\ldots,\sigma_{|k|}) = (\sigma_1^1,\ldots,\sigma_{k_1}^1,\ldots,\sigma_1^m,\ldots,\sigma_{k_m}^m) = \left((\sigma_i^1)_{1\le i\le k_1},(\sigma_i^2)_{1\le i\le k_2},\ldots,(\sigma_i^m)_{1\le i\le k_m}\right)$, where, for $1\le j\le m, (\sigma_i^j)_{1\le i\le k_j}$ is an $\mathbb{R}^{k_j}$-valued S-function ($i.e$: $(\sigma_i^j)_{1\le i\le k_j} = (\sigma_1^j,\ldots,\sigma_{k_j}^j)$ where each component $\sigma_i^j, 1\le i\le k_j$ is an S-function and $(\sigma_i^j)_{1\le i\le k_j}(x) = \left(\sigma_1^j(x_1),\ldots,\sigma_{k_j}^j(x_{k_j})\right)$, for $x=(x_1,\ldots,x_{k_j})^T\in\mathbb{R}^{k_j}$.)

Definition 2:. Consider the functional differential equation of retarded type

$$(\Sigma)_h : \begin{cases} \dot{x}(t) = f(x_t), t\ge t_0; \\ \\ x_{t_0}(\theta) = \Psi(\theta), \forall\ \theta\in[-h,0]. \end{cases}$$

It is assumed that $\Psi\in C_h$ and the map f is continuous and Lipschitz in Ψ and $f(0)=0$. We say that the trivial solution $x(t)\equiv 0$ of $(\Sigma)_h$ is globally asymptotically stable (GAS for short) if the following conditions holds: (i) for every $\varepsilon>0$, there exists a $\delta>0$ such that, for any $\Psi\in C_h$, with $||\Psi||_h\le\delta$, there exists $t_0\ge 0$, such that the solution x_Ψ of $(\Sigma)_h$ satisfies $||(x_\Psi)_t||_h\le\varepsilon$, for all $t\ge t_0$; (ii) for all $\Psi\in C_h$, the trajectory of $(\Sigma)_h$ with initial condition Ψ converges to zero as $t\to\infty$.

III. Feedbacks of nested saturation type

A. Statements of the results

We next determine two explicit expressions of globally asymptotically stabilizing feedbacks for general time-delay linear systems, both of nested saturation type, according to the results of the stabilization of delay free-system. The above problem was first studied for delay-free continuous-time systems. It was shown in [10] that, under condition (C), there exists explicit expressions of globally asymptotically stabilizing feedbacks. Then, it is natural to investigate whether this technique can be extended to the case where there is a delay in the input. In this section, we will take for simplicity the initial state to be zero. We start by giving some definitions, first introduced in [10] and adapted here to the delay case.

Definition 3:. (cf. [10]) Given an $n-$dimensional system $(\Sigma)_h : \dot{x}(t) = f(x(t-h))$, we say that $(\Sigma)_h$ has the **CICS** (converging-input converging-state) property if, whenever $e:[0.\infty)\to\mathbb{R}^n$ is any bounded measurable function which converges to zero as $t\to\infty$, every solution $t\to x(t)$ of $\dot{x}(t) = f(x(t-h))+e(t)$ converges to zero as $t\to\infty$.(Such a concept is needed in order to state Theorem 1 and an intermediary result (Lemma 2 given below), which is useful for the induction step in the proof of Theorem 1).

For a system $\dot{x}(t) = f(x(t),u(t-h)), x\in\mathbb{R}^n, u\in\mathbb{R}^m$, we say that a feedback $u(t-h) := \bar{u}(x(t-h))$ is stabilizing if 0 is a GAS equilibrium of the closed-loop system $\dot{x}(t) = f\left(x(t),\bar{u}(x(t-h))\right)$. If, in addition, this closed-loop system has the CICS property then we will say that $\bar{u}$ is CICS-stabilizing.

Definition 4:. (cf. [10]) For a square matrix A, let $N(A)$ be the sum $s(A)+z(A)$, where $s(A)$ is the number of conjugate pairs of nonzero purely imaginary eigenvalues of A (counting multiplicity) and $z(A)$ is the multiplicity of zero as an eigenvalue of A.

In the next theorem, we summarize our results.

Theorem 1:. Let $(S)_h^r$ be a linear system $\dot{x}(t) = Ax(t)+Bu(t-h)$ with state space $\mathbb{R}^n$ and input space $\mathbb{R}^m$. Assume that $(S)_h^r$ is stabilizable and A has no unstable eigenvalues. Let $N=N(A)$. Let $\sigma=(\sigma_1,\ldots,\sigma_N)$ be an arbitrary sequence of S-functions. Then, for all $h\ge 0$, there exist a number $r^*(h)\in(0,1]$, an $m-$tuple $k=(k_1,\ldots,k_m)$ of non negative integers such that $|k|=N$ and for each $1\le j\le m$, linear functions $f_{h,i}^j, g_{h,i}^j:\mathbb{R}^n\longrightarrow\mathbb{R}, 1\le i\le k_j$, such that for all $r\in(0,r^*(h)]$, there are CICS-stabilizing feedbacks $(*)$ $u_j(t-h) = -r\sigma_{k_j}^j\{f_{h,k_j}^j(x(t-h)) + \alpha_{k_j-1}^j\sigma_{k_j-1}^j[f_{h,k_j-1}^j(x(t-h)) + \ldots + \alpha_1^j\sigma_1^j(f_{h,1}^j(x(t-h)))\ldots]\}$ where $\alpha_i^j\ge 0$, for all $i\in[1,k_j-1]$, and $(**)$ $u_j(t-h) = -r[\beta_{k_j}^j\sigma_{k_j}^j\left(g_{h,k_j}^j(x(t-h))\right) + \beta_{k_j-1}^j\sigma_{k_j-1}^j\left(g_{h,k_j-1}^j(x(t-h))\right) + \ldots + \beta_1^j\sigma_1^j\left(g_{h,1}^j(x(t-h))\right)]$, where $\beta_1^j,\ldots,\beta_{k_j}^j$ are nonnegative constants such that $\beta_1^j+\ldots+\beta_{k_j}^j\le 1$.

B. Proof of Theorem 1

The proof of Theorem 1 is based on two lemmas, exactly as in the argument of [10]. More precisely, Lemma 1 below is Lemma 3.1 of [10] and Lemma 2 below, which is actually the main technical point, is the nonzero delay version of Lemma 3.2 of [10]. In order to facilitate the analysis of the stabilizability properties by bounded feedback of $(S)_h^r$, a linear transformation is carried out.

Lemma 1:. (cf. [10]) Let $(S_1)_h^r : \dot{x}(t) = Ax(t)+bu(t-h)$ be an n-dimensional linear single-input system. Suppose that (A,b) is a controllable pair and all eigenvalues of A are critical. Then, (i) if 0 is an eigenvalue of A, then there exists a linear coordinate transformation $y=Tx$ which transforms $(S_1)_h^r$ into $\dot{\bar{y}}(t) = A_1\bar{y}(t)+(y_n(t)+u(t-h))\,b_1$, $\dot{y}_n(t) =$

$u(t-h)$, where (A_1, b_1) is controllable, y_n is a scalar variable, and $\bar{y} = (y_1, \ldots, y_{n-1})^T$; (ii) if A has an eigenvalue of the form $i\omega$, with $\omega > 0$, then there is a linear change of coordinates $Tx = (y_1, \ldots, y_n)^T = (\bar{y}^T, y_{n-1}, y_n)^T$ of $\mathbb{R}^n$ that puts $(S_1)_h^r$ in the form: $\dot{\bar{y}}(t) = A_1\bar{y}(t) + (y_n(t) + u(t-h))\, b_1$, $\dot{y}_{n-1}(t) = \omega y_n(t)$, $\dot{y}_n(t) = -\omega y_{n-1}(t) + u(t-h)$, with (A_1, b_1) being controllable and y_{n-1}, y_n scalar variables.

Lemma 2:. Let $\rho > 0$. Then, there exist a constant $v_0 > 0$ and, for every $h \geq 0$ there is an $r^*(h) \in (0, 1]$ and an 2×1 matrix F_h such that, for any two bounded measurable functions $\alpha(t), \beta(t)$ converging both to zero as $t \longrightarrow \infty$ and for all $r \in (0, r^*(h)]$, if $x = (x_1, x_2)^T : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$ is any solution of the control system $(S_2)_h^r$ given by

$$\begin{cases} \dot{x}_1(t) = \rho x_2(t) + r\alpha(t), \\ \\ \dot{x}_2(t) = -\rho x_1(t) - r\sigma\left(F_h^T x(t-h) + u(t-h)\right) \\ \\ +rv(t-h) + r\beta(t), \end{cases} \tag{3}$$

$x_0 = ((x_1)_0, (x_2)_0)^T = \bar{0}$ on $[-h, 0]$, with $\bar{0}$ the zero function in C_h, and $u, v \in L^\infty([-h, \infty), \mathbb{R})$ with $\|v\|_{L^\infty} \leq v_0$, it holds (i) $\exists$ $M_\infty > 0$ independent of r, such that $\limsup_{t\longrightarrow\infty} \|x(t)\| \leq M_\infty(\|u\|_{L^\infty} + \|v\|_{L^\infty} + \|f\|_{L^\infty})$. where $f = (\alpha, \beta)^T$; (ii) For $u = v = f = 0$, $(0, 0)$ is GAS.

Remark 1:. We will in fact actually obtain the following stronger ISS-like property (see [8] and references there):

$$\lim \sup_{t\longrightarrow\infty} \|x(t)\| \leq \theta_\infty(\|\Psi\|_h) + M_\infty(\|u\|_{L^\infty} + \|v\|_{L^\infty} + \|f\|_{L^\infty}),$$

where Ψ_h is the initial condition for x and θ_∞ is a class-$\mathcal{K}$ function (i.e. $\theta_\infty : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, strictly increasing and satisfies $\theta_\infty(0) = 0$.)

Proof of lemma 2. Let $h > 0$. Consider the feedback law $F_h = e^{-\rho A_0 h} b$, where $A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then $\|F_h\| = 1$ and the systems $(S_2)_h^r$ becomes $\dot{x}(t) = \rho A_0 x(t) - rb\left(\sigma(b^T e^{\rho A_0 h} x(t-h) + u(t-h)) + v(t-h)\right) + rf(t)$. Set $A_{\rho,r} := \rho A_0 - rbb^T$. Let y be the solution of

$$\dot{y}(t) = A_{\rho,r} y(t) - rv(t-h)b + rf(t), \tag{4}$$

$y(0) = \bar{0}$ on $[-h, 0]$, with $v \in L^\infty$, and $\|v\|_{L^\infty} \leq v_0$, where v_0 is a positive constant to be determined later. (By an obvious abuse of notation, L^∞ denotes here $L^\infty([-h, \infty), \mathbb{R}^n)$.) Since $A_{\rho,r}$ is Hurwitz, system (4) is L^∞ stable. If γ_∞ denotes its $L^\infty-$gain, then $\|y\|_\infty \leq \gamma_\infty(\|v\|_{L^\infty} + \|f\|_{L^\infty})$. Let x be the solution of $(S_2)_h^r$ corresponding to u, v and let $z = x - y$. Then z satisfies $\dot{z}(t) = \rho A_0 z(t) - rb\left[\sigma\left(F_h^T z(t-h) + F_h^T y(t-h) + u(t-h)\right) - b^T y(t)\right]$, $z(0) = \bar{0}$ on $[-h, 0]$. Let $\tilde{u}(t-h) = F_h^T y(t-h) + u(t-h)$ and $\tilde{v}(t) = b^T y(t)$. Let v_0 verify $\gamma_\infty(v_0 + \|f\|_{L^\infty}) \leq \Gamma$, with $\Gamma = \lim_{|\xi|\to\infty} \inf \sigma(\xi) > 0$, we get $\|\tilde{v}\|_{L^\infty} \leq \|y\|_{L^\infty} \leq \gamma_\infty(\|v\|_{L^\infty} + \|f\|_{L^\infty}) \leq \Gamma$. The dynamics of z becomes $\dot{z}(t) = \rho A_0 z(t) - rb\left[\sigma\left(F_h^T z(t-h) + \tilde{u}(t-h)\right) - \tilde{v}(t)\right]$, $z(0) = \bar{0}$ on $[-h, 0]$. Then, $z(t) = e^{\rho A_0 h} z(t-h) -$

$$r\int_{t-h}^t e^{\rho A_0(t-\xi)} b\left[\sigma\left(F_h^T z(\xi-h) + \tilde{u}(\xi-h)\right) - \tilde{v}(\xi)\right] d\xi.$$

Then, $F_h^T z(t-h) + \tilde{u}(t-h) = b^T z(t) + \tilde{d}(t)$, where $g(v) = b^T e^{\rho A_0 v} b$ and $\tilde{d}(t) = \tilde{u}(t-h) + r\int_{t-h}^t g(t-\xi)\left[\sigma\left(F_h^T z(\xi-h) + \tilde{u}(\xi-h)\right) - \tilde{v}(\xi)\right] d\xi$. It follows that $\dot{z}(t) = \rho A_0 z(t) - rb\left[\sigma\left(b^T z(t) + \tilde{d}(t)\right) - \tilde{v}(t)\right]$, $z(0) = \bar{0}$ on $[-h, 0]$.

Let $\tilde{z}(t) = b^T z(t) + \tilde{d}(t)$. We consider the Lyapunov function V_2 used in [1] for $p = 2$ (or in [10]) and given by

$$V_2(z) := \lambda_{\rho,r} V_0(z) + V_1(z), \tag{5}$$

where $\lambda_{\rho,r} > 0$ will be chosen later and $V_0(z) := \frac{\|z\|^3}{3}$, $V_1(z) := z^T P_{\rho,r} z$, with $P_{\rho,r}$, the positive definite symmetric matrix, satisfying $P_{\rho,r} A_{\rho,r} + A_{\rho,r}^T P_{\rho,r} = -2I$, where the Hurwitz matrix $A_{\rho,r}$ defined by $\rho A_0 - rbb^T$. Computations yield $\lambda_\pm(P_{\rho,r}) = (1 - \pm\sqrt{1 - 4/(4/r + r/\rho^2)})^{-1}$, with

$$P_{\rho,r} = \begin{pmatrix} 2/r + r/\rho^2 & 1/\rho \\ 1/\rho & 2/r \end{pmatrix}. \tag{6}$$

Moreover, one has $\lambda_\pm(P_{\rho,r}) \asymp_0 1/r$. From [1], $\dot{V}_0(z(t)) = -r\|z\| z^T b\left(\sigma(b^T z + \tilde{d}) - \tilde{v}\right) = -r\|z\|\tilde{z}^T (\sigma(\tilde{z}) - \tilde{v}) + r\|z\|\tilde{d}^T (\sigma(\tilde{z}) - \tilde{v}) \leq -r\|z\|\tilde{z}^T (\sigma(\tilde{z}) - \tilde{v}) + r\|z\|\|\tilde{d}\| (K + \Gamma)$. As in [1], Eq.(18) p.1199, we separate each of the cases $\|\tilde{z}\| \leq M_1$ and $\|\tilde{z}\| > M_1$ where, by definition of Γ, there is some $M_1 \geq 1$ so that $\inf_{|\xi|\geq M_1} |\sigma(\xi)| \geq \frac{1}{2}\Gamma$. We get $\dot{V}_0(z(t)) \leq -r\|z(t)\|\tilde{z}^T(t)\sigma(\tilde{z}(t)) + r\|z(t)\|\left((K + \Gamma)\|\tilde{d}(t)\| + M_1\|\tilde{v}(t)\|\right)$. Moreover $\dot{V}_1(z(t)) = 2z^T(t) P_{\rho,r}\dot{z}(t) = z^T(t)\left(P_{\rho,r} A_{\rho,r} + A_{\rho,r}^T P_{\rho,r}\right) z(t) + 2rz^T(t) P_{\rho,r} b[\tilde{z}(t)\sigma(\tilde{z}(t)) - \tilde{z}(t) + \tilde{d}(t) + \tilde{v}(t)] \leq -\|z(t)\|^2 + 2rK\|P_{\rho,r} b\|\|z(t)\|(\tilde{z}^T(t)\sigma(\tilde{z}(t)) + \frac{1}{K}(\|\tilde{d}(t)\| + \|\tilde{v}(t)\|))$. We choose $\lambda_{\rho,r}$ as $\lambda_{\rho,r} 2K\|P_{\rho,r} b\| = 2K/r\sqrt{4 + r/\rho^2} \asymp_0 1/r$. We deduce

$$\dot{V}_2 \leq -\|z(t)\|^2 + \kappa_{\rho,r}\|z(t)\|\left(\|\tilde{d}(t)\| + \|\tilde{v}(t)\|\right), \tag{7}$$

where $\kappa_{\rho,r} = r\lambda_{\rho,r} \max\{1 + K + \Gamma, 1 + M_1\} \asymp_0 1$. Let

$$H_\mu(t) = \mu\int_{t-2h}^t ds \int_s^t \|z(l)\|^2 dl = \mu\int_0^{2h} ds \int_{t-s}^t \|z(l)\|^2 dl,$$

with $\mu > 0$ chosen later. Its derivative satisfies $\dot{H}_\mu(t) = 2h\mu\|z(t)\|^2 - \mu\int_{t-2h}^t \|z(s)\|^2 ds$. Finally, consider the following function $V(z(t), t) := V_2(z(t)) + H_\mu(t)$. Then $\dot{V}(z(t)) \leq -(1 - 2h\mu)\|z(t)\|^2 + \kappa_{\rho,r}\|z(t)\|(\|\tilde{d}(t)\| + \|\tilde{v}(t)\|)$ $\mu\int_{t-2h}^t \|z(s)\|^2 ds$. From the definition of $\tilde{d}(t)$ and Cauchy-Schwartz inequality, $\|\tilde{d}(t)\| + \|\tilde{v}(t)\| \leq \|\tilde{u}(t-h)\| + \|\tilde{v}(t)\| + r\max\{1, K\}\int_{t-h}^t (\|\tilde{u}(\xi-h)\| + \|\tilde{v}(\xi)\|) d\xi$ $+(2h)^{\frac{1}{2}}\left(\int_{t-2h}^t \|z(s)\|^2 ds\right)^{1/2}$. In addition, $\dot{V}(z(t)) \leq -(1 - 2h\mu)\|z(t)\|^2 + \kappa_{\rho,r}\|z(t)\|(\|\tilde{u}(t-h)\| + \|\tilde{v}(t)\|)$ $+r\max\{1, K\}\kappa_{\rho,r}\|z(t)\|\int_{t-h}^t (\|\tilde{u}(\xi-h)\| + \|\tilde{v}(\xi)\|) d\xi$ $+$ $\kappa_{\rho,r}(2h)^{1/2}\|z(t)\|\left(\int_{t-2h}^t \|z(s)\|^2 ds\right)^{\frac{1}{2}} - \mu\int_{t-2h}^t \|z(s)\|^2 ds$ $\leq -(1 - 3h\mu)\|z(t)\|^2 + \kappa_{\rho,r}\|z(t)\|(\|\tilde{u}(t-h)\| + \|\tilde{v}(t)\|) +$ $r\max\{1, K\}\kappa_{\rho,r}\|z(t)\|\left(\int_{t-h}^t (\|\tilde{u}(\xi-h)\| + \|\tilde{v}(\xi)\|) d\xi\right)$ $-\left(\mu - 1/2\mu^{-1} r^2\kappa_{\rho,r}^2\right)\int_{t-2h}^t \|z(s)\|^2 ds$. Choosing $mu^*(h)$ and $r^*(h) \in (0, 1]$ as $\mu^*(h) = \min\{1/6h, \kappa_{\rho,r}/\sqrt{2}\}$ $r^*(h) = \min\{(3\sqrt{2}h\kappa_{\rho,r})^{-1}, 1\}$, we get for $\mu \leq \mu^*(h)$ and $r \leq r^*(h)$, $\dot{V}(z(t)) \leq -\frac{1}{2}\|z(t)\|^2 + \kappa_{\rho,r}\|z(t)\|(\|\tilde{u}(t-h)\| + \|\tilde{v}(t)\|)$

$+r\max\{1,K\}\kappa_{\rho,r}\|z(t)\|\left(\int_{t-h}^{t}\left(\|\tilde{u}(\xi-h)\|+\|\tilde{v}(\xi)\|\right)d\xi\right)=$ $-1/2\|z(t)\|^2 \quad +\kappa_{\rho,r}\|z(t)\|\Big\{\|\tilde{u}(t-h)\|+\|\tilde{v}(t)\|+ r\max\{1,K\}\Big(\|\tilde{m}(t)\|+\|\tilde{n}(t)\|\Big)\Big\}$, where $\tilde{m}(t)=\int_{t-h}^{t}\|\tilde{u}(\xi-h)\|d\xi$ and $\tilde{n}(t)=\int_{t-h}^{t}\|\tilde{v}(\xi)\|d\xi$. We have

$$\dot{V}(z(t))\leq-\frac{1}{2}\|z(t)\|^2+\delta_{\rho,r}\|z(t)\|\Big(\|\tilde{u}\|_{L^\infty}+\|\tilde{v}\|_{L^\infty}\Big),\quad (8)$$

where $\delta_{\rho,r}:=\kappa_{\rho,r}(1+rh\max\{1,K\})\asymp_0 1$, since $\|\tilde{m}\|_{L^\infty}\leq h\|\tilde{u}\|_{L^\infty}$ and $\|\tilde{n}\|_{L^\infty}\leq h\|\tilde{v}\|_{L^\infty}$.
Thus, $\dot{V}$ is negative outside the ball centered at the origin, of radius $2\delta_{\rho,r}(\|\tilde{u}\|_{L^\infty}+\|\tilde{v}\|_{L^\infty})=\delta_{\rho,r}R$, where $R=2(\|\tilde{u}\|_{L^\infty}+\|\tilde{v}\|_{L^\infty})$. It follows that $V(z(t))\leq\sup_{|\xi|\leq\delta_{\rho,r}R}V(\xi)\leq\frac{\lambda_{\rho,r}}{3}(\delta_{\rho,r})^3R^3+\lambda_+(P_{\rho,r})(\delta_{\rho,r})^2R^2+2\mu(\delta_{\rho,r})^2R^2h^2$. First assume that $R\leq 1$. Then, we have $\lambda_{\min}(P_{\rho,r})\|z(t)\|^2\leq V(z(t))\leq\left(\frac{\lambda_{\rho,r}}{3}(\delta_{\rho,r})^3+\lambda_+(P_{\rho,r})(\delta_{\rho,r})^2+2\mu(\delta_{\rho,r})^2h^2\right)R^2$. This implies that $\limsup_{t\to\infty}\|z(t)\|\leq K^1_{\rho,r}R$, where

$$K^1_{\rho,r}:=\left(\frac{\lambda_{\rho,r}(\delta_{\rho,r})^3+3\lambda_+(P_{\rho,r})(\delta_{\rho,r})^2+6\mu(\delta_{\rho,r})^2h^2}{3\lambda_-(P_{\rho,r})}\right)^{\frac{1}{2}}.$$

If $R\geq 1$, $\left(\frac{\lambda_{\rho,r}}{3}(\delta_{\rho,r})^3+\lambda_+(P_{\rho,r})(\delta_{\rho,r})^2+2\mu(\delta_{\rho,r})^2h^2\right)R^3$, and we get $\limsup_{t\to\infty}\|z(t)\|\leq K^2_{\rho,r}R$, where

$$K^2_{\rho,r}:=\left(\frac{\lambda_{\rho,r}(\delta_{\rho,r})^3+3\lambda_+(P_{\rho,r})(\delta_{\rho,r})^2+6\mu(\delta_{\rho,r})^2h^2}{\lambda_{\rho,r}}\right)^{\frac{1}{3}}.$$

Let $K_{\rho,r}=\max(K^1_{\rho,r},K^2_{\rho,r})$. Then, we get $K_{\rho,r}\asymp_0 1$. So, by choosing $rh\leq 1$, the finite-gain $K_{\rho,r}$ is delay-independent. More precisely, there exists a positive constant C_0 independent of r and h such that $\limsup_{t\to\infty}\|z(t)\|\leq C_0\left(\|\tilde{u}\|_{L^\infty}+\|\tilde{v}\|_{L^\infty}\right)$. It implies that $\limsup_{t\to\infty}\|z(t)\|\geq\limsup_{t\to\infty}\|x(t)\|-\|y\|_{L^\infty}\geq\limsup_{t\to\infty}\|x(t)\|-\gamma_\infty\left(\|v\|_{L^\infty}+\|f\|_{L^\infty}\right)$. We conclude that $\limsup_{t\to\infty}\|x(t)\|\leq C_0\left(\|\tilde{u}\|_{L^\infty}+\|\tilde{v}\|_{L^\infty}\right)+\gamma_\infty\left(\|v\|_{L^\infty}+\|f\|_{L^\infty}\right)\leq C_0\Big\{2\gamma_\infty(\|v\|_{L^\infty}+\|f\|_{L^\infty})+\|u\|_\infty\Big\}+\gamma_\infty\left(\|v\|_{L^\infty}+\|f\|_{L^\infty}\right)\leq M_\infty\Big(\|u\|_{L^\infty}+\|v\|_{L^\infty}+\|f\|_{L^\infty}\Big)$, where $M_\infty=\max\Big\{C_0,\gamma_\infty(1+2C_0)\Big\}\asymp_0 1$. □

Corollary 1:. For $n=1,2$, let J be equal to 0 if $n=1$ and equal to $\rho A_0,\rho>0$ if $n=2$. Let $b=1$ if $n=1$ and $b=(0,1)^T$ if $n=2$. Then, there exist a constant $v_0>0$ and for every $\varepsilon>0$ and $h>0$ there is an $r^*(h)\in(0,1]$, and an $n\times 1$ matrix $\tilde{F}_h$, such that for any functions $v\in L^\infty([-h,\infty),\mathbb{R})$, with $\|v\|_{L^\infty}\leq v_0$, and $\tilde{e}:\mathbb{R}_{\geq 0}\longrightarrow\mathbb{R}^n$, converges to zero as $t\longrightarrow\infty$, if $\chi:\mathbb{R}_{\geq 0}\longrightarrow\mathbb{R}^n$, is any solution of the system $\dot{\tilde{y}}=J\tilde{y}-rb\left(\sigma\left(\tilde{F}_h^T\tilde{y}(t-h)-\xi v(t-h)\right)-\eta v(t-h)\right)+r\tilde{e}(t)$, where $\xi+\eta=1$ and $\xi\eta=0$, it follows that for, $0<r\leq r^*(h)$, $\limsup_{t\longrightarrow\infty}\|\chi(t)\|<\varepsilon$.

Proof of Corollary 1. For $n=1$, the contents of the above result essentially result from Lemma 6 of [2] ($\tilde{F}_h^T$ is simply equal to one) and, for $n=2$, the conclusion follows from Lemma 2.

Proof of Theorem 1. Without loss of generality, making a change of coordinates if necessary, we may assume that the system $(S)^r_h$ has the following partitioned form:

$$(S)^r_h:\begin{cases}\dot{x}_1(t)=A_1x_1(t)+B_1u(t-h), & x_1\in\mathbb{R}^{n_1},\\ \dot{x}_2(t)=A_2x_2(t)+B_2u(t-h), & x_2\in\mathbb{R}^{n_2},\end{cases}$$

where $n_1+n_2=n$, all the eigenvalues of A_1 are critical and A_2 is an Hurwitz matrix. Here, we take

$$A=\begin{pmatrix}A_1 & 0\\ 0 & A_2\end{pmatrix},B=\begin{pmatrix}B_1\\ B_2\end{pmatrix}$$

The controllability assumption on (A,B) implies that the pair (A_1,B_1) is also controllable. Since A_2 is Hurwitz, it will be sufficient to show that if we find an CICS-stabilizing feedback of type (*) or type (**) for the system $\dot{x}_1(t)=A_1x_1(t)+B_1u(t-h)$ then the same feedback will stabilize $(S)^r_h$ as well because the second equation, $\dot{x}_2(t)=A_2x_2(t)+B_2u(t-h)$, can be seen as an asymptotically stable linear system forced by a function that converges to zero. Thus, in order to stabilize $(S)^r_h$, it is enough to stabilize the subsystem $\dot{x}_1(t)=A_1x_1(t)+B_1u(t-h)$.

Without loss of generality, in our proof of Theorem we will suppose that $(S)^r_h$ is already in this form, that is, we assume that all the eigenvalues of A have zero real part and the pair (A,B) is controllable.

i) The single-input case: We prove the theorem by induction on the dimension of the system. For dimension zero, the conclusion holds true. Consider now a single-input $n-$dimensional system, $n\geq 1$, and suppose that Theorem 1 has been established for all single-input systems of dimensions less than or equal to $n-1$. Write $N=N(A)$, and pick any $\varepsilon>0$. If 0 is an eigenvalue of A, we apply the first part of Lemma 1 and rewrite our system in the form $\dot{\bar{y}}(t)=A_1\bar{y}(t)+(y_n(t)+u(t-h))\,b_1$, $\dot{y_n}(t)=u(t-h)$, where $\bar{y}=(y_1,\ldots,y_{n-1})^T$. In the case when zero is not an eigenvalue of A, we apply the second part of Lemma 1. Then we take $\dot{\bar{y}}(t)=A_1\bar{y}(t)+(y_n(t)+u(t-h))\,b_1$, $\dot{y}_{n-1}(t)=-\omega y_n(t)$, $\dot{y}_n(t)=\omega y_{n-1}(t)+u(t-h)$, where $\bar{y}=(y_1,y_2,\ldots,y_{n-2})^T$. So, in either case, we can rewrite our system in the form $\dot{\bar{y}}(t)=A_1\bar{y}(t)+(y_n(t)+u(t-h))\,b_1$, $\dot{\tilde{y}}(t)=J\tilde{y}(t)+u(t-h)b$, where J is a skew-symmetric matrix, (J,b) is a controllable pair, with $\tilde{y}=y_n,b=1$ in the first case and $\tilde{y}=(y_{n-1},y_n)^T,b=(0,1)^T$ in the second case. To consider the problem of converging-input converging-state (CICS) property, we must study solutions of the following system: $\dot{\bar{y}}(t)=A_1\bar{y}(t)+(y_n(t)+u(t-h))\,b_1+\bar{e}(t)$, $\dot{\tilde{y}}(t)=J\tilde{y}(t)+u(t-h)b+\tilde{e}(t)$, where $\tilde{e},\bar{e}$ are any measurable functions on $[0,\infty)$, bounded by $\tilde{e}_0$ and $\bar{e}_0$, which converge to zero as $t\longrightarrow\infty$ and have the same dimensions as $\tilde{y}$, $\bar{y}$ respectively. We will design a feedback of the form $u(t-h)=-r\Big[\sigma_N\left(\tilde{F}_h^T\tilde{y}(t-h)-\xi v(t-h)\right)-\eta v(t-h)\Big]$, $\tilde{F}_h=e^{-Jh}b$, where ξ and η are constants such that $\xi\eta=0,\xi+\eta=1$ and v is to be chosen later. To simplify the exposition, we assume that $\sigma'_N(0)=1$ (achieved by a time rescaling). From Corollary 1, there exists $r_1^*(h)\in(0,1]$ and $0<v_0<\frac{\varepsilon}{2}$ such that, if $\|v\|_{L^\infty}\leq v_0$, then trajectories of the system satisfy $\|\tilde{y}\|_{L^\infty}<\frac{\varepsilon}{2}$ for $r\in(0,r_1^*(h)]$. Along such trajectory, one has $\sigma_N\left(\tilde{F}_h^T\tilde{y}(t-h)-\xi v(t-h)\right) = \tilde{F}_h^T\tilde{y}(t-h) -\xi v(t-h) + O\left(\|\tilde{F}_h^T\tilde{y}(t-h)-\xi v(t-h)\|^2\right)$. So, we have $\dot{\bar{y}}(t) = A_1\bar{y}(t) + rv(t-h)b_1+ \left(y_n(t)-r\tilde{F}_h^T\tilde{y}(t-h)+rO(\|\tilde{F}_h^T\tilde{y}(t-h)-\xi v(t-h)\|^2)\right)b_1 +\bar{e}(t) = A_1\bar{y}(t) + rv(t-h)b_1 + \bar{\bar{e}}(t)$, where $\bar{\bar{e}}(t) := \bar{e}(t)+ (y_n(t)-r\tilde{F}_h^T\tilde{y}(t-h)+rO(\|\tilde{F}_h^T\tilde{y}(t-h)-\xi v(t-h)\|^2))b_1$. Then for every $\varepsilon > 0$ there exists $r_2^*(h) \in (0, r_1^*(h)]$, such that $\limsup_{t\to\infty}\bar{\bar{e}}(t) < \varepsilon, \forall r \in (0, r_2^*(h)]$. Note that (A_1, b_1) is controllable and all eigenvalues of A_1 have non-positive real part. Applying the inductive hypothesis to the single-input system of dimension less than or equal to $n-1$, $\dot{\bar{y}}(t) = A_1\bar{y}(t)+rv(t-h)b_1+\bar{\bar{e}}(t)$, we know that there is a feedback $v(t-h) = \bar{u}(\bar{y}(t-h))$ having $(*)$ or $(**)$ form (cases $\xi = 1, \eta = 0$, and $\xi = 0, \eta = 1$, respectively), such that all the trajectories of $\dot{\bar{y}}(t) = A_1\bar{y}(t) + rb_1\bar{u}(\bar{y}(t-h))$ go to zero as $t \to \infty$, for every vector function $\bar{\bar{e}}$ that converges to zero (see [10]). The proof for the single-input case is completed. □

ii) The general case: Next, we deal with the general case of $m \geq 1$ inputs and prove Theorem 1 by induction on m. First, we know from the proof above that the theorem is true if $m = 1$. Assume that Theorem 1 has been established for all $k-$input systems, for all $k \leq m-1, m \geq 2$, and consider an $m-$input system $(S)_h^r : \dot{x}(t) = Ax(t)+Bu(t-h)+e(t)$ where $e(t)$ is any decaying vector function. Assume without loss of generality that the first column b_1 of B is nonzero and consider the Kalman controllability decomposition of the system $(S^1)_h^r : \dot{x}(t) = Ax(t)+b_1u(t-h)$ (see [7], Lemma 3.3.3). We conclude that, after a change of coordinates $y = T_{yx}^{-1}x$, $(S^1)_h^r$ has the form $\dot{y_1}(t) = A_1y_1(t) + A_2y_2(t) + \bar{b}_1u_1(t-h)$, $\dot{y_2}(t) = A_3y_2(t)$, Where $(A_1, \bar{b}_1)$ is a controllable pair. In these coordinates, $(S)_h^r$ has the form $\dot{y}_1(t) = A_1y_1(t) + A_2y_2(t)+\bar{b}_1u_1(t-h)+\bar{B}_1\bar{u}(t-h)+e_1(t)$, $\dot{y}_2(t) = A_3y_2(t)+ \bar{B}_2\bar{u}(t-h) + e_2(t)$, where $\bar{u} = (u_2, \ldots, u_m)^T$ and $\bar{B}_1, \bar{B}_2$ are matrices of appropriate dimensions. So it suffices to show the conclusion for the previous system. Let n_1, n_2 denote the dimensions of y_1, y_2 respectively. Recall that $N = N(A)$. Let $\sigma = (\sigma_1, \cdots, \sigma_N)$ be any finite sequence of saturation functions. Then, for the single-input controllable system $\dot{y}_1(t) = A_1y_1(t)+\bar{b}_1u_1(t-h)$, there is a feedback $u_1(t-h) = v_1(y_1(t-h))$, such that (i) v_1 is $N_1 = N(A_1)(*)-$form (or $N_1 = N(A_1)(**)-$form); (ii) the resulting closed-loop system has the CICS property. By controllability, we conclude that the $(m-1)-$input subsystem $\dot{y}_2 = A_3y_2 + \bar{B}_2\bar{u}(t-h)$ is controllable as well. By the inductive hypothesis, this subsystem can be stabilized by a feedback $\bar{u}(t-h) = \bar{v}_2(y_2(t-h)) = [v_2(y_2(t-h)), \ldots, v_m(y_2(t-h))]^T$, such that (i) there exists an $(m-1)-$tuple $\bar{k} = (N_2, \ldots, N_m)$ of nonnegative integers and $|\bar{k}| = N - N_1$, such that $\bar{v}_2$ is $|\bar{k}|(*)-$form (or $|\bar{k}|(**)-$form); (ii) the resulting closed-loop system has the CICS property. Since $A_2y_2(t) + \bar{B}_1\left[v_2(y_2(t-h)), \ldots, v_m(y_2(t-h))\right]^T + e_1(t)$ still converges to zero, we conclude that y_1, with $u_1, \bar{u}$ as above, converges to zero. Therefore, the above feedback renders $\dot{x}(t) = Ax(t) + Bu(t-h)$ GAS and the resulting closed-loop system has the CICS property. □

IV. Example

Let the following system where triple integrator and oscillator with multiplicity two are considered:

$$\dot{x}(t) = Ax(t) + bu(t-h), \quad x \in \mathbb{R}^7, u \in \mathbb{R}, \tag{9}$$

where $b = (0\ 0\ 1\ 0\ 0\ 0\ 1)^T$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \tag{10}$$

and u is required to satisfy the constraint $|u| \leq r$, $r \in (0, 1]$. To get a feedback of the form $u(.) = -r\sigma_5\Big\{f_{h,5}(x) + \sigma_4\Big[f_{h,4}(x) + \sigma_3\Big(f_{h,3}(x) + \sigma_2\Big(f_{h,2}(x) + \sigma_1\Big(f_{h,1}(x)\Big)\Big)\Big)\Big]\Big\}$, we need to find a linear transformation that puts (9) in the adapted form. In our case, it reduces to, $\dot{y}(t) = \tilde{A}y(t)+\tilde{b}u(t-h)$, with $\tilde{b} = (0\ 1\ 0\ 1\ 1\ 1\ 1)^T$ and

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

A simple computation then shows that

$$R(A, b) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 \end{pmatrix} \tag{12}$$

$$R(\tilde{A}, \tilde{b}) = \begin{pmatrix} 0 & 1 & 4 & 5 & -1 & -7 & -1 \\ 1 & 4 & 5 & -1 & -7 & -1 & 9 \\ 0 & 1 & 3 & 2 & -2 & -2 & 2 \\ 1 & 3 & 2 & -2 & -2 & 2 & 2 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

If we let $T = R(A, b)R(\tilde{A}, \tilde{b})^{-1}$, then the coordinate change $y = T^{-1}x$ transforms (9) into (11) (see [10]). The matrix T^{-1} is easily computed, and the transformation $y = T^{-1}x$ turns out to be $y_1 = x_1 + 4x_2 + 5x_3 + 2x_4 - 2x_5 - 3x_6 - 5x_7$, $y_2 = x_2+4x_3+2x_4+2x_5+3x_6-3x_7$, $y_3 = x_1+3x_2+2x_3-2x_6-2x_7$, $y_4 = x_2+3x_3+2x_6-2x_7$, $y_5 = x_1+2x_2+x_3$, $y_6 = x_2+x_3$, $y_7 = x_3$. We now need to find r and h so that $u(t-h) = -rsat\Big\{y_7(t-h) + sat\Big[y_6(t-h) + sat\Big(y_5(t-h) + sat\Big(-\sin(h)y_3(t-h)+ +\cos(h)y_4(t-h) + sat\Big(-\sin(h)y_1(t-h) + \cos(h)y_2(t-h)\Big)\Big)\Big)\Big]\Big\}$.

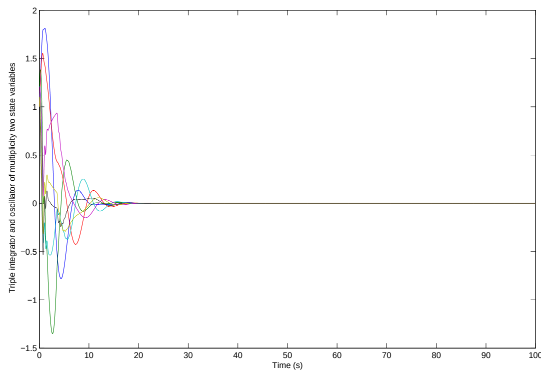

Fig. 1. Closed-loop states ($h = 0.1$ and $r = 2.5$).

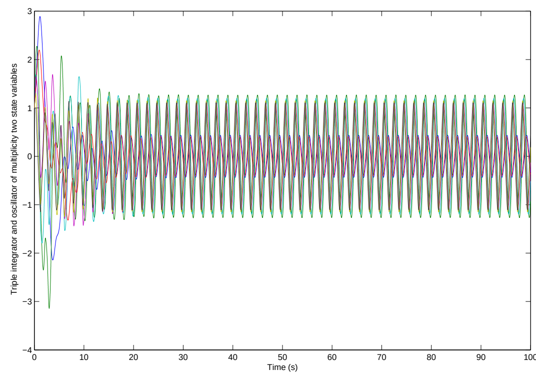

Fig. 2. Closed-loop states ($h = 0.3$ and $r = 2.5$).

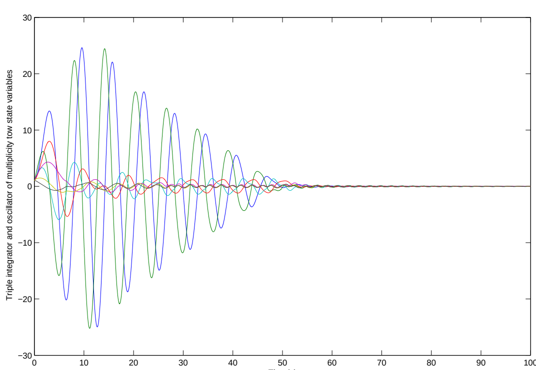

Fig. 3. Closed-loop states ($h = 0.3$ and $r = 0.55$).

In fig. 1, one can observe the behavior of the triple integrator and oscillator with multiplicity two when the delay is 0.1 and when the control law is $u = -2.5sat\Big\{y_7 + sat\Big[y_6 + sat\Big(y_5 + sat\Big(-\sin(0.1)y_3 + \cos(0.1)y_4 + sat\Big(-\sin(0.1)y_1 + \cos(0.1)y_2\Big)\Big)\Big)\Big]\Big\}$. Figs. 2 shows that the same system is closed loop with the same feedback is not asymptotically stable when the delay is not 0.1 but the larger delay 0.3. We observe on fig. 3 that in this case (when $h = 0.3$) the control law given above globally uniformly asymptotically stabilizes the system with the amplitude $r = 0.55$.

References

[1] W. Liu, Y.Chitour and E.D. Sontag, *On finite gain stabilizability of linear systems subject to input saturation*, Siam J. Control Optim., Vol 34, no 4, pp. 1190-1219, 1996.

[2] F. Mazenc, S. Mondie and S. Niculescu, *Global Asymptotic Stabilization for Chains of Integrators with a Delay in the Input*, IEEE Trans. Automat. Contr., Vol. 48, No. 1, 2003, pp. 57-63.

[3] F. Mazenc, S. Mondie and S. Niculescu, *Global Stabilization of oscillators with bounded Delayed Input*, Systems and Control Letters, 53 (2004) 415-422.

[4] A. Megretsky, *A gain scheduled for systems with saturation which makes the closed loop system L^2- bounded*, IFAC congress 1996.

[5] S.I. Niculescu, *Delay effects on stability': a control perspective*, Lecture Notes in Control and Inform. Springer, London (2001).

[6] A. Saberi, P. Hou, A. Stoorvogel, *On simultaneous global external and internal stabilization of critically unstable linear systems with saturating actuators*, IEEE Transactions on Automatic Control , Vol. 45, No. 6, pp. 1042-1052, June 2000.

[7] E.D. Sontag, *Mathematical control theory: Deterministic Finite Dimensional systems*, Springer-Verlag, New York, (1990).

[8] E.D. Sontag, *Comments on integral invariants of ISS*, Systems and Control Letters 43, pp. 93-100, 1998.

[9] E.D. Sontag and H.J. Sussmann, *Nonlinear output feedback design for linear systems with saturating controls*, Proc. IEEE Conf. Decision and Control, Honolulu, Dec. 1990, IEEE Publications, 1990, pp. 3414-3416.

[10] H.J. Sussmann, E.D. Sontag and Y. Yang, *A general result on the stabilization of linear systems using bounded controls*, IEEE Trans. Automat. Contr. Vol. 39 pp. 2411-2425, 1994.

[11] A. Teel, *Global stabilization and restricted tracking for multiple integrators with bounded controls*, Syst. Control lett. vol. 18, pp. 165-171, 1992.

[12] Y. Yang, *Global stabilization of nonlinear systems*, PhD Thesis, Rutgers University, (1993).

[13] K. Yakoubi and Y. Chitour, *Linear systems subject to input saturation and time delay: Finite-gain L^p-stabilization*, SIAM J. Control Optim., Vol 45, no. 3, 1084–1115, 2006.