ON FINITE GAIN STABILIZABILITY OF LINEAR SYSTEMS SUBJECT TO INPUT SATURATION*

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Abstract. This paper deals with (global) finite-gain input/output stabilization of linear systems with saturated controls. For neutrally stable systems, it is shown that the linear feedback law suggested by the passivity approach indeed provides stability, with respect to every L^p-norm. Explicit bounds on closed-loop gains are obtained, and they are related to the norms for the respective systems without saturation.

These results do not extend to the class of systems for which the state matrix has eigenvalues on the imaginary axis with nonsimple (size > 1) Jordan blocks, contradicting what may be expected from the fact that such systems are globally asymptotically stabilizable in the state-space sense; this is shown in particular for the double integrator.

Keywords: input saturations, linear systems, finite gain stability, Lyapunov functions, dissipative systems

1. Introduction. In this work we are interested in those nonlinear systems which are obtained when cascading a linear system with a memory-free input nonlinearity:

$$(\Sigma) \qquad \dot{x} = Ax + B\sigma(u), \quad y = Cx\,.$$

The nonlinearity σ is of a "saturation" type (definitions are given later). Figure 1 shows the type of system being considered, where the linear part has transfer function $W(s)$ and the function σ shown is the standard semilinear saturation (results will apply to more general σ's).

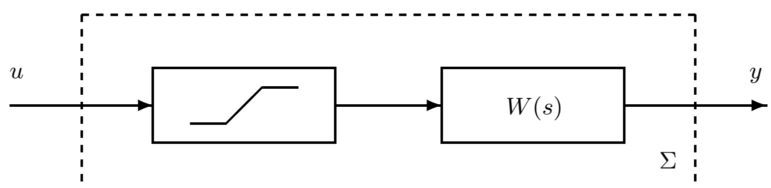

FIG. 1. *Input-Saturated Linear System*

Linear systems with actuator saturation constitute one of the most important classes of nonlinear systems encountered in practice. Surprisingly, until recently few general theoretical results were available regarding global feedback design problems for them. One such general result was given in [14], which showed that global state-space stabilization for such systems is possible under the assumptions that all the eigenvalues of A are in the closed left-hand plane, plus stabilizability and detectability of (A, B, C). (These conditions are best possible, since they are also necessary. The controller consists of an observer followed by a smooth static nonlinearity.) For more recent work, see [20], which showed —based upon techniques introduced in [16] for a particular case— how to simplify the controller that had been proposed in [14]. See also [8] for closely related work showing that such systems can be semiglobally (that is, on compact sets) stabilized by means of linear feedback.

In this paper, we are interested in studying not merely closed-loop *state-space* stability, but also stability with respect to measurement and actuator noise. This is the notion of stability that

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is often found in input/output studies. The problem is to find a controller C so that the operator $(u_1, u_2) \mapsto (y_1, y_2)$ defined by the standard systems interconnection

$$\begin{aligned} y_1 &= P(u_1 + y_2) \\ y_2 &= C(u_2 + y_1) \end{aligned}$$

is well-posed and finite-gain stable, where P denotes the input/output behavior of the original plant Σ. See Figure 2.

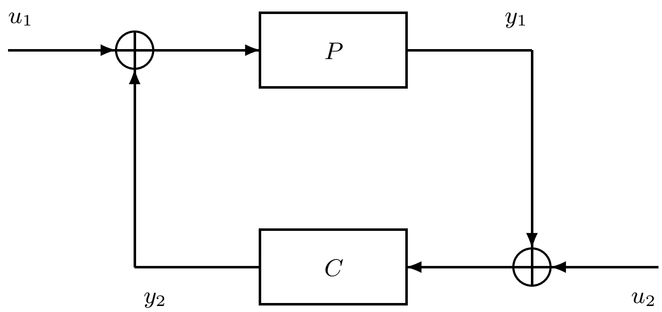

FIG. 2. *Standard Closed-Loop*

(In our main results, we will take for simplicity the initial state to be zero. However, nonzero initial states can be studied as well, and some remarks in that regard are presented in a latter section of the paper.) Once that such i/o stability is achieved, geometric operator-theoretic techniques can be applied; see for instance [3] and the references there. For other work on computing norms for nonlinear systems in state space form, see for instance [18] and the references given there.

We focus in this paper in a case which would be trivial if one would only be interested in state stability, namely that in which the original matrix A is neutrally stable, that is, all eigenvalues have nonpositive real part and there are no nontrivial Jordan blocks for eigenvalues in the imaginary axis. (The whole point of [14] and [20] was of course to deal with such possible nontrivial blocks, e.g. multiple integrators.) In this case, a standard passivity approach suggests the appropriate stabilization procedure. For instance, assume that σ is the identity (so the original system is linear), $A + A' \leq 0$, and $C = B'$. Then the system is passive, with storage function $V(x) = \|x\|^2/2$, since integrating the inequality $dV(x(t))/dt \leq y(t)'u(t)$ gives $\int_0^t y(s)'u(s)ds \geq V(x(t)) - V(x(0))$. Thus the negative feedback interconnection with the identity (strictly passive system), that is, $u = -y$, results in finite gain stability. For this calculation, and more on passivity, see for instance [7] and the references given there. (For the use of the same formulas for just *state-space* stabilization, but applying to linear systems with saturations, see [5] and [9]; see also the discussion on the "Jurdevic-Quinn" method in [13].)

In this paper, we essentially generalize the passivity technique to systems with saturations. We first establish finite gain stability in the various p-norms, using linear state feedback stabilizers. Then we show how outputs can be incorporated into the framework. Our work is very much in the spirit of the well-known "absolute stability" area, but we have not been able to find a way to deduce our results from that classical literature.

These results do not extend to the class of systems for which the state matrix has eigenvalues on the imaginary axis with nonsimple (size>1) Jordan blocks, contradicting what may be expected from the fact that such systems are globally asymptotically stabilizable in the state-space sense; this is shown in particular for the double integrator.

One remark on terminology. In the operator approach to nonlinear systems, see e.g. [19], a "system" is typically defined as a partially defined operator between normed spaces, and "stability" means that the domain of this operator is the entire space. In that context, finite gain stability is the requirement that the operator be everywhere defined and bounded; the norm of the operator is by definition the gain of the system. In this paper, we use simply the term L^p-stability to mean this stronger finite gain condition.

The reader is referred to the companion paper [2] for complementary results to those in this paper, dealing with Lipschitz continuity ("incremental gain stability") and continuity of the operators in question. The two papers are technically independent.

Organization of Paper. Section 2 provides definitions and statements of the main results, as well as some related comments. Proofs of the main results are given in Section 3. Section 4 estimates gains in terms of the corresponding gains for systems without saturation, in particular for p=2 (H_∞-norms). Results regarding nonzero initial states and global asymptotic stability of the origin are collected in Section 5. Section 6 shows how to enlarge the class of input nonlinearities even more, so as to include in non-saturations as well. The paper closes with Section 7, which contains the double integrator counterexample.

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2. Statements of Main Results. We introduce now the class of saturation functions to be considered, and state the main results on finite-gain stability. Some remarks are also provided. Proofs are deferred to a later section.

2.1. Saturation Functions. We next formally define what we mean by a "saturation." Essentially, we ask only that this be a function which has the same sign as its argument, stays away from zero at infinity, is bounded and is not horizontal near zero.

DEFINITION 1. We call $\sigma : \mathbb{R} \to \mathbb{R}$ a *saturation* function if it satisfies the following two conditions:
(*i*) σ is locally Lipschitz and bounded;
(*ii*) $t\sigma(t) > 0$ if $t \neq 0$, $\liminf_{t\to 0} \frac{\sigma(t)}{t} > 0$ and $\liminf_{|t|\to\infty} |\sigma(t)| > 0$.

For convenience we will simply call a saturation function σ an *S-function*. We say that σ is an $\mathbb{R}^n$*-valued S-function* if $\sigma = (\sigma_1, \ldots, \sigma_n)'$ where each component σ_i is an S-function and

$$\sigma(x) \stackrel{\text{def}}{=} (\sigma_1(x_1), \ldots, \sigma_n(x_n))'$$

for $x = (x_1, \ldots, x_n)' \in \mathbb{R}^n$. Here we use $(\cdots)'$ to denote the transpose of the vector $(\cdots)$.

REMARK 1. It follows directly from Definition 1 that most reasonable saturation-type functions are indeed S-functions in our sense. Included are $\arctan(t)$, $\tanh(t)$, and the standard saturation function $\sigma_0(t) = \operatorname{sign}(t)\min\{|t|, 1\}$, i.e.,

$$\sigma_0(t) = \begin{cases} 1 & \text{if } t > 1\,, \\ t & \text{if } |t| \le 1\,, \\ -1 & \text{if } t < -1\,. \end{cases}$$

REMARK 2. It is easy to see that if σ satisfies a bound $|\sigma(t)| \le M|t|$ for t near 0 (in particular if $\sigma(0) = 0$ and (i) in Definition 1 holds), then Condition (ii) in Definition 1 is equivalent to the following condition:

(c) There exist positive numbers a, b, K and a measurable function $\tau : \mathbb{R} \to [a, b]$ such that for all $t \in \mathbb{R}$ we have $|\sigma(t) - \tau(t)t| \le Kt\sigma(t)$.

It is clear that (c) implies (ii). To see the converse, let $\delta > 0$ be such that $|\sigma(t)| \leq M|t|$ for $|t| \leq \delta$. Then just let

$$\tau(t) = \begin{cases} 1 & \text{if } t = 0\,, \\ \frac{\sigma(t)}{t} & \text{if } t \in [-\delta, \delta]/\{0\}\,, \\ \frac{\sigma(\delta)}{\delta} & \text{if } t > \delta\,, \\ -\frac{\sigma(-\delta)}{\delta} & \text{if } t < -\delta\,, \end{cases}$$

It is easily verified that there exist positive constants a, b, K such that (c) holds for this τ.

DEFINITION 2. We say that a constant $K > 0$ is an *S-bound* for σ if there exist $a, b > 0$ and a measurable function $\tau : \mathbb{R} \to [a, b]$ such that, for all $t \in \mathbb{R}$:
(i) $b \leq K$,
(ii) $|\sigma(t)| \leq K$,
(iii) $|\sigma(t)| \leq K|t|$,
(iv) $|\sigma(t) - \tau(t)t| \leq Kt\sigma(t)$.
The above discussion shows that such (finite) S-bounds always exist.

A constant $K > 0$ is called an *S-bound* for an $\mathbb{R}^m$-valued S-function σ if K is an S-bound for each component of σ.

2.2. L^p-Stability. Consider the initialized control system given by

$$\begin{aligned} \dot{x} &= f(x, u), \\ x(0) &= 0\,, \end{aligned} \tag{1}$$

where the state x and the control u take respectively values in $\mathbb{R}^n$ and $\mathbb{R}^m$. We assume that the function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz with respect to (x, u). Terminology for systems will be as in any standard reference, such as [13].

Throughout this paper, if ξ is a point in $\mathbb{R}^n$, we use $\|\xi\| = (\sum_{i=1}^n \xi_i^2)^{1/2}$ to denote the usual Euclidean norm. For each matrix S, $\|S\|$ denotes the induced operator norm, and $\|S\|_F$ the Frobenius norm, i.e, $\|S\|_F$=$\mathrm{Tr}(SS')^{1/2}$, where $\mathrm{Tr}(\cdot)$ denotes trace. Recall that $\|S\| \leq \|S\|_F$.

For each $1 \leq p \leq \infty$ and each integrable (essentially bounded, for $p = \infty$) vector-valued function $x \in L^p([0, \infty), \mathbb{R}^n)$, we let $\|x\|_{L^p}$ denote the usual L^p-norms:

$$\|x\|_{L^p} = \left(\int_0^\infty \|x(t)\|^p dt\right)^{1/p}$$

if $p < \infty$ and

$$\|x\|_{L^\infty} = \text{ ess } \sup_{0 \leq t < \infty} \|x(t)\|\,.$$

DEFINITION 3. Let $1 \leq p \leq \infty$ and $0 \leq M \leq \infty$. We say that (1) has L^p-gain less than or equal to M if for any $u \in L^p([0, \infty), \mathbb{R}^m)$, the solution x of (Σ) corresponding to u is in $L^p([0, \infty), \mathbb{R}^n)$ and satisfies

$$\|x\|_{L^p} \leq M\|u\|_{L^p}\,.$$

The infimum of such numbers M will be called the L^p-gain of (Σ). We say that system (Σ) is L^p-stable if its L^p-gain is finite.

By a *neutrally stable* $n \times n$ matrix A we mean one for which all solutions of $\dot{x} = Ax$ are bounded; equivalently, A has no eigenvalues with positive real part and each Jordan block corresponding to a purely imaginary eigenvalue has size one. Another well-known characterization of such matrices is that they are the ones for which there exists a symmetric positive definite matrix Q such that $A'Q+QA\leq 0$.

We now state our main result:

THEOREM 1. *Let A,B be $n\times n, n\times m$ matrices respectively. Let σ be an $\mathbb{R}^m$-valued S-function. Assume that A is neutrally stable. Then there exists an $m\times n$ matrix F such that the system*

$$\begin{aligned} \dot{x} &= Ax+B\sigma(Fx+u)\,, \\ x(0) &= 0\,, \end{aligned} \tag{2}$$

is L^p-stable for all $1\leq p\leq\infty$.

Theorem 1 is an immediate consequence of the more general technical result contained in Theorem 2 below. In order to state that theorem in great generality, we recall first a standard notion. Let (Σ) $\dot{x}=Ax+Bu$ be a linear system, where x and u take values in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. For each measurable and locally essentially bounded $u:[0,\infty)\to\mathbb{R}^m$ and each $x_0\in\mathbb{R}^n$, let $x_u(t,x_0)$ be the solution of (Σ) corresponding to u with $x_u(0,x_0)=x_0$. Following the terminology of [6], the *stabilizable subspace* $S(A,B)$ of (A,B) is the subspace of $\mathbb{R}^n$ which consists of all those initial states $x_0\in\mathbb{R}^n$ for which there is some u so that $x_u(t,x_0)\to 0$ as $t\to\infty$. In other words, $S(A,B)$ is the subspace of $\mathbb{R}^n$ made up of all the states that can be asymptotically controlled to 0 (so this includes in particular the reachable subspace). Observe that the pair (A,B) is stabilizable (asymptotically null controllable) iff $S(A,B)=\mathbb{R}^n$.

THEOREM 2. *Let A and B be $n\times n$ and $n\times m$ matrices respectively. Let $S(A,B)$ be the stabilizable subspace of (A,B). Let σ be an $\mathbb{R}^m$-valued S-function and let $\theta:\mathbb{R}^k\to S(A,B)\subseteq\mathbb{R}^n$ be a locally Lipschitz function such that $\|\theta(\xi)\|\leq\min\{L,L\|\xi\|\}$ for all $\xi\in\mathbb{R}^k$, where $L>0$ is a constant and $k>0$ is some integer. Assume that A is neutrally stable. Then there exist an $m\times n$ matrix F and an $\varepsilon>0$ such that the system*

$$\begin{aligned} \dot{x} &= Ax+B\sigma(Fx+u)+\varepsilon\theta(v)\,, \\ x(0) &= 0\,, \end{aligned} \tag{3}$$

is L^p-stable for each $1\leq p\leq\infty$, i.e. there exists for each p a finite constant $M_p>0$ such that for any $u\in L^p([0,\infty),\mathbb{R}^m), v\in L^p([0,\infty),\mathbb{R}^k)$,

$$\|x\|_{L^p}\leq M_p(\|u\|_{L^p}+\|v\|_{L^p})\,.$$

The proof is deferred to Section 3.

Theorem 2 implies Theorem 1 (just take $\theta\equiv 0$) as well as a result dealing with small "nonmatching" state perturbations.

REMARK 3. It is possible to make the result even more general, by weakening the Lipschitz assumption on θ. Moreover, even the Lipschitz property of σ is not needed. The main problem in dropping this last assumption is that uniqueness of solutions of the closed-loop system is then not guaranteed, so that there is no well-defined input-to-state operator. Nonetheless, one could rephrase all statements by asserting that all possible solutions satisfy the stated bounds. This is consistent with the way stability is defined in some texts on input/output stability, where well-possedness (existence and uniqueness of solutions) is stated as a property independent of stability itself.

2.3. Output Stabilization. Consider the initialized linear input/output system

$$\begin{aligned} (\Sigma_{ao})\ \dot{x} &= Ax+B\sigma(u)\,, \\ x(0) &= 0\,, \\ y &= Ex\,, \end{aligned}$$

where A, B, E are respectively $n \times n$, $n \times m$, $r \times n$ matrices. Assume that system (Σ_{ao}) is *asymptotically observable* (that is, it is *detectable*). Our main result for input/output systems is as follows:

THEOREM 3. *Assume that system (Σ_{ao}) is asymptotically observable, A is neutrally stable and the $\mathbb{R}^m$-valued S-function σ is globally Lipschitz. Then there exist an $m \times n$ matrix F and an $n \times r$ matrix L such that the following property holds. Let $1 \leq p \leq \infty$. Pick any $u_1 \in L^p([0,\infty),\mathbb{R}^m)$ and $u_2 \in L^p([0,\infty),\mathbb{R}^r)$, and consider the solution $x = (x_1, x_2)$ of*

$$\begin{aligned} \dot{x_1} &= Ax_1 + B\sigma(y_2 + u_1)\ , \quad y_1 = Ex_1\ , \\ \dot{x_2} &= (A + LE)x_2 + B\sigma(Fx_2) - L(y_1 + u_2)\ , \quad y_2 = Fx_2\ , \end{aligned}$$

with $x(0) = 0$. Consider the total output function $y = (y_1, y_2) = (Ex_1, Fx_2)$. Then y is in $L^p([0,\infty),\mathbb{R}^{r+m})$ and

$$\|y\|_{L^p} \leq M_p(\|u_1\|_{L^p} + \|u_2\|_{L^p})$$

for some constant $M_p > 0$.

2.4. Not Every Feedback Stabilizes. One may ask if *any* F which would stabilize when the saturation is not present also provides finite gain for (2). Not surprisingly, the answer is negative. In order to give an example, we need first a simple technical remark.

LEMMA 2.1. *Consider the system $\dot{x} = Ax + B\sigma(Fx + u)$, where the matrix A is assumed to have all eigenvalues in the imaginary axis and where each component of σ is a continuous function whose range contains a neighborhood of the origin (this holds, for instance, if it is an S-function). Furthermore, assume that the pair (A, B) is controllable. Then, given any state $x_0 \in \mathbb{R}^n$, there is some measurable essentially bounded control u steering the origin to x_0 in finite time.*

Proof. Since all eigenvalues of A have zero real part and the pair (A, B) is controllable, for each $\varepsilon > 0$ there is some control v_0 for the system $\dot{x} = Ax + Bu$ so that $|v_0(t)| < \varepsilon$ for all t and v_0 drives in finite time the origin to x_0 (see e.g. [12]). Since the range of σ contains a neighborhood of the origin, and using a measurable selection (Fillipov's Theorem), it is also true that there is a measurable control v which achieves the same transfer, for the system $\dot{x} = Ax + B\sigma(u)$. Now let, along the corresponding trajectory, $u(t) = v(t) - Fx(t)$. It follows that this achieves the desired transfer for $\dot{x} = Ax + B\sigma(Fx + u)$. ▯

The next two examples show that even if A is neutrally stable, Theorem 1 may not be true if F only satisfies the condition that $A + BF$ is Hurwitz.

EXAMPLE 1. Let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad F = -\,(1/2\ \ ,\ 1)\ ,$$

and any σ so that $\sigma(1/2) = 1$. Then both the origin and $(-1, 0)'$ are equilibrium points of the system

$$\dot{x} = Ax + B\sigma(Fx).$$

By Lemma 2.1, there is some input u_0 which steers the origin to $(-1, 0)'$ in some finite time T_0. Consider the input u_1 equal to u_0 for $0 \leq t \leq T_0$ and to 0 for $t > T_0$. Then if x is the trajectory of (2) corresponding to u_1, we have that $x(t) = (-1, 0)'$ for all $t \geq T_0$. Clearly, for any $1 \leq p < \infty$, $u_1 \in L^p([0,\infty),\mathbb{R})$ and $x \notin L^p([0,\infty),\mathbb{R}^2)$. Therefore, system (3) is not L^p-stable for any $1 \leq p < \infty$. If we use multiple inputs, a different example that includes $p = \infty$ is as follows.

EXAMPLE 2. Assume that $m = n = 2$. Let

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} -3 & 7 \\ -1 & 2 \end{pmatrix}.$$

Then $A+BF=F$ is Hurwitz. Let $\sigma=(\sigma_0,\sigma_0)'$, where σ_0 is the standard saturation function. Then the system

$$\begin{aligned} \dot{x} &= \sigma(Fx+u)\,, \\ x(0) &= (0,0)' \end{aligned} \tag{4}$$

is not L^p-stable for any $1\le p\le\infty$. To see this, take a control v on some interval $[0,T]$ that steers $(0,0)'$ to $(1,1)'$. Let $u=v$ on $[0,T]$ and $u=(0,0)'$ on (T,∞). Let $x=(x_1,x_2)'$ be the solution of (4) corresponding to u. Then on $[T,\infty)$, we have $x_1(t)=x_2(t)=t-T+1$. Thus (4) is not L^p-stable for any $1\le p\le\infty$. (In fact, the trajectory is not even bounded for a bounded input.)

3. Proofs of the Main Results. For notational convenience (to avoid having too many negative signs in the formulas) we will prove the main theorem for systems written in the form

$$\begin{aligned} \dot{x} &= Ax-B\sigma(Fx+u)+\varepsilon\theta(v)\,, \\ x(0) &= 0\,. \end{aligned} \tag{5}$$

A trivial remark is needed before we start.

Remark 4. Assume that $\sigma_1:\mathbb{R}^{k_1}\to\mathbb{R}^m$ and $\sigma_2:\mathbb{R}^{k_2}\to\mathbb{R}^n$ each satisfies a growth estimate of the type $\|\sigma_1(u)\|\le C\|u\|$, $\|\sigma_2(v)\|\le C\|v\|$ for $u\in\mathbb{R}^{k_1}, v\in\mathbb{R}^{k_2}$. It follows from classical linear systems theory that if the system $\dot{x}=Ax$ is globally asymptotically stable—that is, A is a Hurwitz matrix—then the controlled system $\dot{x}=f(x,u,v)=Ax+B\sigma_1(u)+\sigma_2(v)$ is automatically also L^p-stable for all $1\le p\le\infty$. We will be interested in the case in which A is merely stable, but this remark will be used at various points.

We now prove Theorem 2. First note that we can assume that (A,B) is controllable.

3.1. Reduction to the Controllable Case. Suppose Theorem 2 is already known to be true for controllable (A,B); we show how the general case follows. It is an elementary linear systems exercise to show that the stabilizable subspace $S(A,B)$, for any two A,B, is invariant under A; this follows for instance from its characterization as a sum of the reachable subspace and the space of stable modes. Thus the restriction of A to $S(A,B)$ is well-defined, and it is again neutrally stable. Now since θ takes values in $S(A,B)$, the trajectories of (5) lie in $S(A,B)$. So we may assume that (A,B) is stabilizable, i.e. $S(A,B)=\mathbb{R}^n$, since otherwise we can restrict ourselves to $S(A,B)$. Then, up to a change of coordinates, we may assume that

$$A=\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix},\quad B=\begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

where (A_1,B_1) is controllable and A_1 is neutrally stable. Assume that A_1 is an $r\times r$ matrix and B_1 is an $r\times m$ matrix.

Let $\tilde{\theta}:\mathbb{R}^r\to\mathbb{R}^r$ be given by $\tilde{\theta}(\xi)=(\tilde{\theta}_0(\xi_1),\dots,\tilde{\theta}_0(\xi_r))'$ for $\xi\in\mathbb{R}^r$, where $\tilde{\theta}_0$ is the standard saturation function, i.e. $\tilde{\theta}_0(t)=\operatorname{sign}(t)\,\min\{1,|t|\}$.

By our assumption that the result is known in the controllable case, there exists an $m\times r$ matrix F_1 and $\varepsilon_1>0$ so that the system

$$\begin{aligned} \dot{x}_1 &= A_1x_1-B_1\sigma(F_1x_1+u)+\varepsilon_1\tilde{\theta}(w)\,, \\ x_1(0) &= 0 \end{aligned} \tag{6}$$

is L^p-stable for all $1\le p\le\infty$. Let Γ_p be the L^p-gain of this system, so $\|x_1\|_{L^p}\le\Gamma_p(\|u\|_{L^p}+\|w\|_{L^p})$ for all $u\in L^p([0,\infty),\mathbb{R}^m)$ and $w\in L^p([0,\infty),\mathbb{R}^r)$.

Since (A,B) is stabilizable, we can find an $m\times n$ matrix E such that $A+BE$ is Hurwitz. Then the system

$$\begin{aligned} \dot y &= (A+BE)y+v\,, \\ y(0) &= 0\,. \end{aligned} \tag{7}$$

is L^p-stable for any $1\le p\le\infty$. Let γ_p be the L^p-gain of (7), so $\|y\|_{L^p}\le\gamma_p\|v\|_{L^p}$.

Take an $\varepsilon>0$ such that $\varepsilon L\gamma_\infty\|BE\|\le\varepsilon_1$. Let $F=(F_1,0)$. We show that for this choice of F and ε, system (5) is L^p-stable for any $1\le p\le\infty$. For this purpose, let $u\in L^p([0,\infty),\mathbb{R}^m), v\in L^p([0,\infty),\mathbb{R}^k)$. Let x be the solution of (5) corresponding to u,v. Let y be the solution of

$$\begin{aligned} \dot y &= (A+BE)y+\varepsilon\theta(v)\,, \\ y(0) &= 0\,. \end{aligned} \tag{8}$$

Then we have $\|y\|_{L^\infty}\le\varepsilon L\gamma_\infty$ and $\|y\|_{L^p}\le\varepsilon L\gamma_p\|v\|_{L^p}$ (note that $\|\theta(\xi)\|\le\min\{L,L\|\xi\|\}$ for all $\xi\in\mathbb{R}^k$). Let $z=x-y$. Then z satisfies

$$\begin{aligned} \dot z &= Az-B\sigma(Fz+Fy+u)-BEy\,, \\ z(0) &= 0\,. \end{aligned}$$

Write $z=(z_1,z_2)'$. Then we have $z_2\equiv 0$ and z_1 satisfies

$$\begin{aligned} \dot z_1 &= A_1z_1-B_1\sigma(F_1z_1+Fy+u)-B_1Ey\,, \\ z_1(0) &= 0\,. \end{aligned}$$

Since $\|B_1Ey\|_{L^\infty}\le\|B_1E\|\,\|y\|_{L^\infty}\le\varepsilon L\gamma_\infty\|B_1E\|\le\varepsilon_1$, we have

$$-\frac{B_1Ey}{\varepsilon_1}=\tilde\theta\left(-\frac{B_1Ey}{\varepsilon_1}\right)\,,$$

So z_1 satisfies

$$\begin{aligned} \dot z_1 &= A_1z_1-B_1\sigma(F_1z_1+Fy+u)+\varepsilon_1\tilde\theta(-B_1Ey/\varepsilon_1)\,, \\ z_1(0) &= 0\,. \end{aligned}$$

By the L^p-stability of (6) we get that

$$\begin{aligned} \|z\|_{L^p}=\|z_1\|_{L^p} &\le \Gamma_p\left(\|Fy+u\|_{L^p}+\left\|\frac{B_1Ey}{\varepsilon_1}\right\|_{L^p}\right) \\ &\le \Gamma_p\left(\|F\|\,\|y\|_{L^p}+\|u\|_{L^p}+\frac{\|B_1E\|\|y\|_{L^p}}{\varepsilon_1}\right) \\ &\le \Gamma_p\left(\|u\|_{L^p}+\left(\frac{\|B_1E\|}{\varepsilon_1}+\|F\|\right)\varepsilon L\gamma_p\|v\|_{L^p}\right)\,. \end{aligned}$$

This shows that (5) is L^p-stable, which concludes the proof that we may assume that (A,B) is controllable.

3.2. Proof of Theorem 2 Assuming Controllability. From elementary linear algebra, we know that any neutrally stable matrix A is similar to a matrix

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \tag{9}$$

where A_1 is an $r \times r$ Hurwitz matrix and A_2 is an $(n-r) \times (n-r)$ skew-symmetric matrix. So, up to a change of coordinates, we may assume that A is already in the form (9). In these coordinates, we write

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

where B_2 is an $(n-r) \times m$ matrix, and we write vectors as $x = (x_1, x_2)'$ and also $\theta = (\theta_1, \theta_2)'$. Consider the feedback law $F = (0, B_2')$. Then system (5), with this choice of F, can be written as

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 - B_1 \sigma(B_2' x_2 + u) + \varepsilon \theta_1(v)\,, \\ \dot{x}_2 &= A_2 x_2 - B_2 \sigma(B_2' x_2 + u) + \varepsilon \theta_2(v)\,, \\ x_1(0) &= 0, \ \ x_2(0) = 0\,. \end{aligned} \tag{10}$$

Since A_1 is Hurwitz, it will be sufficient to show that there exists an $\varepsilon > 0$ such that the x_2-subsystem is L^p-stable (we may think of x_2 as an additional input to the first subsystem, and apply Remark 4).

The controllability assumption on (A, B) implies that the pair (A_2, B_2) is also controllable. Since A_2 is skew-symmetric, the matrix $\tilde{A} := A_2 - B_2 B_2'$ is Hurwitz. (Just observe that the Lyapunov equation $\tilde{A}' I_{n-r} + I_{n-r} \tilde{A} = -2 B_2 B_2'$ holds, and the pair $(\tilde{A}, B_2)$ is controllable; see [13], Exercise 4.6.7.) Therefore, the theorem is a consequence of the following lemma. This is where the main parts of our argument lie (except for a small technical point, whose proof is deferred to subsection 3.5).

Lemma 3.1. *Let σ, θ be as in Theorem 2. Let A be a skew-symmetric matrix. Assume that $\tilde{A} := A - BB'$ is Hurwitz. Then there exists an $\varepsilon > 0$ such that the system*

$$\begin{aligned} \dot{x} &= Ax - B\sigma(B'x + u) + \varepsilon \theta(v)\,, \\ x(0) &= 0\,, \end{aligned} \tag{11}$$

is L^p-stable for all $1 \le p \le \infty$.

Proof. Assume that $\sigma = (\sigma_1, \ldots, \sigma_m)'$. Let $0 < a \le b < \infty, K > 0$ be constants and $\tau_i : \mathbb{R} \to [a, b], i = 1, \ldots, m$, be measurable functions so that the components σ_i of σ satisfy (i), (ii), (iii) and (iv) in Definition 2 with the respective τ_i's. We may assume that K is large enough such that $K \ge L$. Let

$$\Gamma \stackrel{\text{def}}{=} \min_{i=1,\ldots,m} \liminf_{|\xi| \to \infty} |\sigma_i(\xi)|.$$

Then $\Gamma > 0$. Let $\varepsilon > 0$ satisfy

$$\varepsilon < \frac{\Gamma}{K \gamma_\infty \sqrt{m} \|B\|}\,, \tag{12}$$

where γ_∞ is the L^∞-gain of the initialized linear control system

$$\begin{aligned} \dot{y} &= (A - BB')y + u\,, \\ y(0) &= 0. \end{aligned} \tag{13}$$

By (12) there exists a $\delta \in (0, 1/2]$ such that

$$\varepsilon \le \frac{(1 - 2\delta)\Gamma}{K \gamma_\infty \sqrt{m} \|B\|}\,.$$

Let $u \in L^p([0, \infty), \mathbb{R}^m), v \in L^p([0, \infty), \mathbb{R}^k)$. Let y be the solution of

$$\begin{aligned} \dot{y} &= (A - BB')y + \varepsilon \theta(v)\,, \\ y(0) &= 0. \end{aligned} \tag{14}$$

Let x be the solution of (11) corresponding to u, v and let $z = x - y$. Then z satisfies

$$\begin{aligned} \dot{z} &= Az - B\sigma(B'z + u + B'y) + BB^T y\,, \\ z(0) &= 0. \end{aligned} \tag{15}$$

Let $\tilde{u} = u + B'y$ and $\tilde{v} = B'y$. Then we get

$$\|\tilde{v}\|_{L^\infty} \leq \|B\|\;\|y\|_{L^\infty} \leq \varepsilon\|B\|\gamma_\infty\|\theta\|_{L^\infty} \leq \frac{(1-2\delta)\Gamma}{\sqrt{m}}. \tag{16}$$

Now (15) can be written as

$$\begin{aligned} \dot{z} &= Az - B\left(\sigma(B'z + \tilde{u}) - \tilde{v}\right)\,, \\ z(0) &= 0. \end{aligned} \tag{17}$$

(We have brought the problem to one of a "matched uncertainty" type, in robust control terms, if we think of $\tilde{v}$ as representing a source of uncertainty.)

Let $\tilde{z}(t) = B'z(t) + \tilde{u}(t)$. For each $1 \leq p < \infty$, consider the function $V_{0,p} : \mathbb{R}^n \to \mathbb{R}$ given by

$$V_{0,p}(x) \;=\; \frac{\|x\|^{p+1}}{p+1}\,.$$

Along the trajectories of (17), we have

$$\dot{V}_{0,p}(z(t)) = -\|z(t)\|^{p-1}z'(t)B\left(\sigma\left(B'z(t) + \tilde{u}(t)\right) - \tilde{v}(t)\right)$$

$$= -\|z(t)\|^{p-1}\tilde{z}'(t)\left[\sigma\left(\tilde{z}(t)\right) - \tilde{v}(t)\right] + \|z(t)\|^{p-1}\tilde{u}'(t)\left[\sigma\left(\tilde{z}(t)\right) - \tilde{v}(t)\right]\,.$$

Since K is an S-bound for σ and considering (16), we have the following decay estimate:

$$\begin{aligned} \dot{V}_{0,p}(z(t)) \;&\leq\; -\|z(t)\|^{p-1}\tilde{z}'(t)\left(\sigma\left(\tilde{z}(t)\right) - \tilde{v}(t)\right) \\ &\quad + (K + \frac{(1-2\delta)\Gamma}{\sqrt{m}})\|z(t)\|^{p-1}\|\tilde{u}(t)\|. \end{aligned} \tag{18}$$

We next need to bound the first term in the right hand side of (18). For that purpose, we will partition $[0, \infty)$ into two subsets. By the definition of Γ, there is some $M_1 \geq 1$ so that

$$\min_{i=1,\cdots,m}\inf_{|\xi|\geq M_1}|\sigma_i(\xi)| \geq (1-\delta)\Gamma.$$

The first subset consists of those t for which $\|\tilde{z}'(t)\| \leq M_1\sqrt{m}$. For such t, trivially:

$$\tilde{z}'(t)\left(\sigma\left(\tilde{z}(t)\right) - \tilde{v}(t)\right) \geq \tilde{z}'(t)\sigma\left(\tilde{z}(t)\right) - M_1\sqrt{m}\|\tilde{v}(t)\|\,. \tag{19}$$

Next we consider those t for which $\|\tilde{z}'(t)\| > M_1\sqrt{m}$. First we note some general facts about any vector $\xi \in \mathbb{R}^m$ for which

$$\|\xi\| \;>\; M_1\sqrt{m}\,. \tag{20}$$

If we pick i_0 so that $|\xi_{i_0}| = \max_{i=1,\ldots,m}\{|\xi_i|\}$, then $|\xi_{i_0}| > M_1$, and therefore, by the choice of M_1, $\left|\sigma_{i_0}(\xi_{i_0})\right| \geq (1-\delta)\Gamma$. We conclude that if ξ satisfies (20) then

$$\xi'\sigma(\xi) \geq \xi_{i_0}\sigma_{i_0}(\xi_{i_0}) \geq \frac{\|\xi\|}{\sqrt{m}}\,(1-\delta)\Gamma\,,$$

or equivalently

$$\|\xi\| \ \le \ \frac{\sqrt{m}\,\xi'\sigma(\xi)}{(1-\delta)\Gamma}\,.$$

From this and (16), if $\|\tilde{z}(t)\| > M_1\sqrt{m}$, we have

$$\begin{aligned} \tilde{z}'(t)\left(\sigma\left(\tilde{z}(t)\right)-\tilde{v}(t)\right) &\ge \tilde{z}'(t)\sigma\left(\tilde{z}(t)\right) - \|\tilde{z}'(t)\|\ \|\tilde{v}(t)\| \\ &\ge \tilde{z}'(t)\sigma\left(\tilde{z}(t)\right) - \frac{\sqrt{m}\|\tilde{v}\|_{L^\infty}}{(1-\delta)\Gamma}\tilde{z}'(t)\sigma\left(\tilde{z}(t)\right) \\ &\ge \left(1-\frac{1-2\delta}{1-\delta}\right)\tilde{z}'(t)\sigma\left(\tilde{z}(t)\right) \\ &= \frac{\delta}{1-\delta}\tilde{z}'(t)\sigma\left(\tilde{z}(t)\right). \end{aligned} \tag{21}$$

Note also that $\frac{\delta}{1-\delta} \le 1$ for $0 < \delta \le 1/2$. Combining (19) and (21) we have a common estimate valid for all $t \ge 0$:

$$\tilde{z}'(t)\left(\sigma\left(\tilde{z}(t)\right)-\tilde{v}(t)\right) \ge \frac{\delta}{1-\delta}\tilde{z}'(t)\sigma\left(\tilde{z}(t)\right) - M_1\sqrt{m}\|\tilde{v}(t)\|\,.$$

Using this and (18) we get

$$\begin{aligned} \dot{V}_{0,p}(z(t)) &\le -\frac{\delta}{1-\delta}\|z(t)\|^{p-1}\tilde{z}'(t)\sigma\left(\tilde{z}(t)\right) \\ &+ \|z(t)\|^{p-1}\left((K+\frac{\Gamma}{\sqrt{m}})\|\tilde{u}(t)\| + M_1\sqrt{m}\|\tilde{v}(t)\|\right). \end{aligned} \tag{22}$$

Let $\tau = \operatorname{diag}(\tau_1,\ldots,\tau_m)$ with $\tau(\xi) = \operatorname{diag}(\tau_1(\xi_1),\ldots,\tau_m(\xi_m))$ for $\xi \in \mathbb{R}^m$. Then $aI \le \tau(\xi) \le bI$ for all $\xi \in \mathbb{R}^m$. We have for any $\xi \in \mathbb{R}^m$,

$$\begin{aligned} \|\tau(\xi)\xi - \sigma(\xi)\| &= \left(\sum_{i=1}^m |\tau_i(\xi_i)\xi_i - \sigma_i(\xi_i)|^2\right)^{1/2} \\ &\le K\left(\sum_{i=1}^m \xi_i^2(\sigma_i(\xi_i))^2\right)^{1/2} \le K\xi'\sigma(\xi). \end{aligned} \tag{23}$$

Now we rewrite (17) in the form

$$\begin{aligned} \dot{z} &= \bar{A}(t)z + B\left[\tau\left(\tilde{z}(t)\right)\tilde{z}(t) - \sigma\left(\tilde{z}(t)\right) - \tau\left(\tilde{z}(t)\right)\tilde{u}(t) + \tilde{v}(t)\right], \\ z(0) &= 0 \end{aligned} \tag{24}$$

where $\bar{A}(t) = A - B\tau\left(\tilde{z}(t)\right)B'$. Then $\bar{A}$ satisfies the conditions of Corollary 1 below. Therefore, for each $1 < p < \infty$, there exist a differentiable function $V_{1,p}$ and positive real numbers a_p, b_p and c_p such that

(P1) $a_p\|x\|^p \le V_{1,p}(x) \le b_p\|x\|^p$;
(P2) $\|DV_{1,p}(x)\| \le c_p\|x\|^{p-1}$;
(P3) $DV_{1,p}(x)\bar{A}(t)x \le -\|x\|^p$;

for all $x \in \mathbb{R}^n$ and $t \ge 0$. (Note that the constants a_p, b_p, c_p depend only on A, B, a, b.) Moreover $V_{1,p}$ can be chosen so that

(P4) $\limsup_{p\to 1+} c_p = c_1 < \infty$, and the limit $V_{1,1}(x) = \lim_{p\to 1+} V_{1,p}(x)$ exists for all $x \in \mathbb{R}^n$.

Using (23) and (24), we get, for $1 < p < \infty$,

$$\begin{aligned} \frac{dV_{1,p}\left(z(t)\right)}{dt} &\le -\|z(t)\|^p + c_p\|B\|\|z(t)\|^{p-1}\left(\|\tilde{v}(t)\| + b\|\tilde{u}(t)\|\right) \\ &\quad + c_p\|B\|\|z(t)\|^{p-1}\left\{\|\tau\left(\tilde{z}(t)\right)\tilde{z}(t) - \sigma\left(\tilde{z}(t)\right)\|\right\} \\ &\le -\|z(t)\|^p + c_p\|B\|\|z(t)\|^{p-1}\left(\|\tilde{v}(t)\| + b\|\tilde{u}(t)\|\right) \\ &\quad + c_pK\|B\|\|z(t)\|^{p-1}\tilde{z}'(t)\sigma\left(\tilde{z}(t)\right). \end{aligned} \tag{25}$$

For $1 \le p < \infty$, let

$$\lambda_p = \frac{K\|B\|c_p(1-\delta)}{\delta}. \tag{26}$$

(Observe that this constant does not depend on the particular u and v being considered, but only on the system and on p.) Finally, consider, for each $1 \le p < \infty$, the following function:

$$V_p = \lambda_p V_{0,p} + V_{1,p}\,, \tag{27}$$

where λ_p is given in (26). Using (22), (25) and the fact that $b \le K$, for $1 < p < \infty$, we have along trajectories of (17),

$$\frac{dV_p\,(z(t))}{dt} \le -\|z(t)\|^p + \kappa_p\|z(t)\|^{p-1}(\|\tilde{u}(t)\| + \|\tilde{v}(t)\|)\,, \tag{28}$$

where

$$\kappa_p = \lambda_p \max\{1 + K + \frac{\Gamma}{\sqrt{m}}, \frac{1}{K} + \sqrt{m}M_1\}\,.$$

For any $t \ge 0$, integrating (28) from 0 to t, we have:

$$V_p\,(z(t)) + \int_0^t \|z(s)\|^p ds \le \kappa_p \int_0^t \|z(s)\|^{p-1}(\|\tilde{u}(s)\| + \|\tilde{v}(s)\|)ds\,.$$

When $p = 1$, this inequality is also true as an easy consequence of the Lebesgue Dominated Convergence Theorem (applied to a sequence $\{p^j\}_{j=1}^{\infty}$ decreasing to 1). Thus the inequality is true for all $1 \le p < \infty$.

Applying Hölder's inequality to $\int_0^t \|z(s)\|^{p-1}(\|\tilde{u}(s)\| + \|\tilde{v}(s)\|)ds$, we conclude that for all $1 \le p < \infty$ and $t \ge 0$,

$$V_p\,(z(t)) + \|z\|^p_{L^p[0,t]} \le \kappa_p\|z\|^{p-1}_{L^p[0,t]}(\|\tilde{u}\|_{L^p} + \|\tilde{v}\|_{L^p})\,. \tag{29}$$

Since $V_p \ge 0$, we get that $z \in L^p([0,\infty), \mathbb{R}^n)$ and

$$\|z\|_{L^p} \le \kappa_p(\|\tilde{u}\|_{L^p} + \|\tilde{v}\|_{L^p})\,. \tag{30}$$

Now since $z = x - y$, $\tilde{u} = u + B'y, \tilde{v} = B'y$, we have

$$\begin{aligned}
\|\tilde{v}\|_{L^p} &\le \|B\|\,\|y\|_{L^p} \le \varepsilon K\gamma_p\|B\|\|v\|_{L^p} \\
\|\tilde{u}\|_{L^p} &\le \|u\|_{L^p} + \varepsilon K\gamma_p\|B\|\|v\|_{L^p}\,, \\
\|z\|_{L^p} &\ge \|x\|_{L^p} - \|y\|_{L^p} \ge \|x\|_{L^p} - \varepsilon K\gamma_p\|v\|_{L^p}\,,
\end{aligned}$$

where γ_p is the L^p-gain of (13). Combining this with (30) we have

$$\|x\|_{L^p} \le \kappa_p\|u\|_{L^p} + \varepsilon K\gamma_p(1 + 2\kappa_p\|B\|)\|v\|_{L^p}\,.$$

This finishes the proof of the Lemma, and hence our main theorem, for the case when $1 \le p < \infty$.

We now prove the lemma for $p = \infty$. For this, we need to show that system (11) has the *uniform bounded input bounded state* property, i.e., there exists a finite constant M such that $\|x\|_{L^\infty} \le M(\|u\|_{L^\infty} + \|v\|_{L^\infty})$ for all $u \in L^\infty([0,\infty), \mathbb{R}^m)$ and $v \in L^\infty([0,\infty), \mathbb{R}^k)$. Letting $p = 2$, from (28) we have

$$\frac{dV_2\,(z(t))}{dt} \le -\|z(t)\|\,(\|z(t)\| - \kappa_2(\|\tilde{u}\|_{L^\infty} + \|\tilde{v}\|_{L^\infty}))\,. \tag{31}$$

Let $\beta = \|\tilde{u}\|_{L^\infty} + \|\tilde{v}\|_{L^\infty}$. Thus, $\dot{V}_2$ is negative outside the ball of radius $\kappa_2\beta$ centered at the origin. It follows that

$$V_2\left(z(t)\right) \le \sup_{\|\xi\|\le\kappa_2\beta} V_2(\xi) \le \frac{\lambda_2\kappa_2^3}{3}\beta^3 + b_2\kappa_2^2\beta^2 \,.$$

First assume that $\beta \le 1$. Then we have

$$a_2\|z(t)\|^2 \le V_2\left(z(t)\right) \le \left(\frac{\lambda_2\kappa_2^3}{3} + b_2\kappa_2^2\right)\beta^2 \,,$$

which implies that

$$\|z\|_{L^\infty} \le \left\{\frac{\lambda_2\kappa_2^3 + 3b_2\kappa_2^2}{3a_2}\right\}^{\frac{1}{2}} \beta \,.$$

If $\beta > 1$, we have

$$\frac{\lambda_2\|z(t)\|^3}{3} \le V_2\left(z(t)\right) \le \left(\frac{\lambda_2\kappa_2^3}{3} + b_2\kappa_2^2\right)\beta^3 \,.$$

We then get that

$$\|z\|_{L^\infty} \le \left\{\frac{\lambda_2\kappa_2^3 + 3b_2\kappa_2^2}{\lambda_2}\right\}^{\frac{1}{3}} \beta \,.$$

Let

$$\bar{G}_\infty = \max\left\{\left\{\frac{\lambda_2\kappa_2^3 + 3b_2\kappa_2^2}{3a_2}\right\}^{\frac{1}{2}}, \left\{\frac{\lambda_2\kappa_2^3 + 3b_2\kappa_2^2}{\lambda_2}\right\}^{\frac{1}{3}}\right\} .$$

We have $\|z\|_{L^\infty} \le \bar{G}_\infty\beta$. Now

$$\beta = \|\tilde{u}\|_{L^\infty} + \|\tilde{v}\|_{L^\infty} \le \|u\|_{L^\infty} + 2\varepsilon K\gamma_\infty\|B\|\,\|v\|_{L^\infty} \,.$$

and

$$\|z\|_{L^\infty} \ge \|x\|_{L^\infty} - \varepsilon K\gamma_\infty\|v\|_{L^\infty} \,.$$

We conclude that

$$\|x\|_\infty \le \bar{G}_\infty\|u\|_{L^\infty} + \varepsilon K\gamma_\infty(1 + 2\bar{G}_\infty\|B\|)\|v\|_{L^\infty} \,.$$

Now the proof of Lemma 3.1 is complete. □

3.3. Proof of the Output Feedback Theorem. We now provide a proof of Theorem 3. We will show a somewhat stronger statement, namely, that the state trajectory x also satisfies an estimate as required. The proof will be the usual Luenberger-observer construction, but a bit of care has to be taken because of the nonlinearities.

Asymptotic observability means that there is some $n \times r$ matrix L such that $A + LE$ is Hurwitz. Let F be as in Theorem 2. Let $e = x_1 - x_2$. Then $(x_1, e)'$ satisfies

$$\begin{aligned} \dot{x_1} &= Ax_1 + B\sigma(Fx_1 - Fe + u_1)\,, \\ \dot{e} &= (A + LE)e + B\left(\sigma(Fx_1 - Fe + u_1) - \sigma(Fx_1 - Fe)\right) + Lu_2\,. \end{aligned}$$

Let $\tilde{v} = \sigma(Fx_1 - Fe + u_1) - \sigma(Fx_1 - Fe)$. Since $\|\tilde{v}(t)\| \leq K\|u_1(t)\|$ (here K is a Lipschitz constant for σ) and $A+LE$ is Hurwitz, we know that e is in $L^p([0,\infty),\mathbb{R}^n)$ and $\|e\|_{L^p} \leq \widehat{M}(\|u_1\|_{L^p} + \|u_2\|_{L^p})$ for some constant $\widehat{M} > 0$. Then the conclusion follows from Theorem 2 applied to the x_1-subsystem.

Note that the conclusion of this theorem can be restated in terms of the finite gain stability of a standard systems interconnection

$$y_1 = P(u_1 + y_2),$$
$$y_2 = C(u_2 + y_1),$$

where P denotes the input/output behavior of the original system Σ and C is the input/output behavior of the controller with state space x_2 and output y_2.

3.4. Operator Stability Among Different Norms. We can actually prove a stronger result than that stated in Theorem 2, namely that the input to state operator $(u,v) \to x$ from $L^p([0,\infty),\mathbb{R}^m) \times L^p([0,\infty),\mathbb{R}^k)$ to $L^p([0,\infty),\mathbb{R}^n)$ is a bounded operator from $L^p([0,\infty),\mathbb{R}^m) \times L^p([0,\infty),\mathbb{R}^k)$ to $L^q([0,\infty),\mathbb{R}^n)$, for any $q \geq p$.

REMARK 5. From (29), (30) we get that, for $u \in L^p([0,\infty),\mathbb{R}^m), v \in L^p([0,\infty),\mathbb{R}^k)$ and $t \geq 0$,

$$a_p\|z(t)\|^p \leq V_p\left(z(t)\right) \leq \kappa_p\|z\|_{L^p}^{p-1}\|\left(\|\tilde{u}\|_{L^p} + \|\tilde{v}\|_{L^p}\right) \leq \kappa_p^p(\|\tilde{u}\|_{L^p} + \|\tilde{v}\|_{L^p})^p\,,$$

and then, $\|z\|_{L^\infty} \leq C_1(\|\tilde{u}\|_{L^p} + \|\tilde{v}\|_{L^p})$ with $C_1 = \kappa_p a_p^{-1/p}$. Therefore we obtain for $q \geq p$,

$$\|z\|_{L^q}^q \leq \|z\|_{L^\infty}^{q-p}\|z\|_{L^p}^p \leq C_1^{q-p}\kappa_p^p(\|\tilde{u}\|_{L^p} + \|\tilde{v}\|_{L^p})^q\,. \tag{32}$$

From this one can easily deduce that for any $q \geq p$ the solution x of (11) satisfies

$$\|x\|_{L^q} \leq M_{p,q}(\|u\|_{L^p} + \|v\|_{L^p})$$

for some constants $M_{p,q} > 0$. The same results then hold for the original system in Theorem 2, as is clear from the reduction to (11). That is, for any $u \in L^p([0,\infty),\mathbb{R}^m), v \in L^p([0,\infty),\mathbb{R}^k)$, the solution x of (5) satisfies a similar inequality.

3.5. A Remark on Robustness of a Linear Feedback. It is worth pointing out that the same method used to prove Lemma 3.1 allows establishing the next proposition, which is a result regarding time-varying multiplicative uncertainties on a linear feedback law $u = -B'x$. For that, we need the following lemma:

LEMMA 3.2. *Fix two positive real numbers c,d. Let A be an $n \times n$ skew symmetric matrix, B an $n \times m$ matrix, and assume that the pair (A,B) is controllable (or equivalently that $A - BB'$ is Hurwitz). Then there is a symmetric positive definite matrix P so that*

$$P(A - BDB') + (A' - BD'B')P \;\leq\; -I \tag{33}$$

for all $m \times m$ matrices D so that $D + D' \geq cI$ and $\|D\| \leq d$.

Proof. Since (A,B) is controllable, the same is true for (A,rB) for any $r > 0$; thus $A - rBB'$ is Hurwitz for any $r > 0$. Pick $P_1 > 0$ so that $P_1(A - cBB') + (A' - cBB')P_1 = -2I$. We will choose P of the form $P_1 + \beta I$ for a suitable β. Note that

$$2x'P_1(A - BDB')x = -2\|x\|^2 + 2x'P_1B(cI - D)B'x,$$

where the last term has norm bounded above by $C\|x\|\|B'x\|$ for some constant C which depends on c and d. On the other hand,

$$2\beta x'(A - BDB')x = -2\beta x'BDB'x \leq -c\beta\|B'x\|^2\,.$$

So $2x'P(A-BDB')x \le -2\|x\|^2 + C\|x\|\|B'x\| - \beta c\|B'x\|^2$ and picking β large enough guarantees that this quadratic form is always less than $-\|x\|^2$. ◻

Corollary 1. *Let A and B be as in Lemma 3.2. Let $c, d > 0$ and $\bar{A}(t) = A - BD(t)B'$, where $D(t)$ is any measurable $m \times m$ matrix such that $D(t) + D'(t) \ge cI$ for almost all t in $[0,\infty)$ and $\sup\{\|D(t)\| : t \in [0,\infty)\} \le d$. Then for each $1 < p < \infty$, there exist a differentiable function V_p and positive real numbers a_p, b_p and c_p such that*

(P0) V_p, a_p, b_p, c_p depend only on A, B, c, d;

and for all $x \in \mathbb{R}^n$, $t \in [0,\infty)$,

(P1) $a_p\|x\|^p \le V_p(x) \le b_p\|x\|^p$;

(P2) $\|DV_p(x)\| \le c_p\|x\|^{p-1}$;

(P3) $DV_p(x)\bar{A}(t)x \le -\|x\|^p$.

Moreover, we may choose V_p so that

(P4) $\limsup_{p\to 1+} c_p = c_1 < \infty$, and the limit $V_1(x) := \lim_{p\to 1+} V_p(x)$ exists for all $x \in \mathbb{R}^n$.

Proof. Proof: Just take $V_p(x) = \alpha_p(x'Px)^{p/2}$, where $\alpha_p > 0$ is a proper constant and P is chosen as in Lemma 3.2. ◻

As a direct application of Corollary 1, we get

Corollary 2. *Let A be an $n \times n$ skew-symmetric matrix and B an $n \times m$ matrix. Assume that $A - BB'$ is Hurwitz. Let $D(t)$ be a measurable $m \times m$ matrix with bounded entries. Assume also that there exists a constant $a > 0$ such that $D(t) + D'(t) \ge aI$ for almost all t in $[0,\infty)$. Then the following initialized system*

$$(\tilde{\Sigma}) \qquad \begin{aligned} \dot{x} &= \bar{A}(t)x + u\,, \\ x(0) &= 0, \end{aligned}$$

where $u \in L^p([0,\infty), \mathbb{R}^n)$ and $\bar{A}(t) := A - BD(t)B'$, is L^p-stable for $1 \le p \le \infty$, and the L^p-gain depends only on p, a, A, B, and $M = \sup\{\|D(t)\| : t \in [0,\infty)\}$.

Proof. Let V_p be a function satisfying Conditions (P0)—(P3) in Corollary 1 with respect to $\bar{A}$. Along the trajectories of $(\tilde{\Sigma})$, we have

$$\dot{V}_p(x(t)) \le -\|x(t)\|^p + c_p\|x(t)\|^{p-1}\|u(t)\|\,,$$

for some $c_p > 0$. The conclusion follows after applying Hölder's inequality. ◻

4. Comparison with Linear Gains. From the proof of Lemma 3.1, we can also obtain explicit bounds for the L^p-gain for (11). For simplicity, we deal only with the case when $\theta \equiv 0$ and we will assume that each component σ_i of σ satisfies a stronger estimate:

$$\forall t \in \mathbb{R}, \;\; |\sigma_i(t) - a_i t| \le Kt\sigma_i(t),$$

where $a_i > 0$ are some constants. Of course this implies that $(d\sigma_i(t)/dt)\,|_{t=0} = a_i$. Specifically, we will compare these bounds with the L^p-gain of the system that is obtained by linearizing (11):

$$\begin{aligned} \dot{x} &= \tilde{A}x - BDu\,, \\ x(0) &= 0\,, \end{aligned} \tag{34}$$

where $\tilde{A} = A - BDB'$ with $D = \mathrm{diag}(a_1, \cdots, a_m)$. (Note that $\tilde{A}$ is Hurwitz.) For the cases $p = 1, 2$ we have the following:

Corollary 3. *Let A, B be as in Lemma 3.1 and σ as above. Let G_1 and G_2 be respectively the L^1 and L^2-gains of the system*

$$\begin{aligned} \dot{x} &= Ax - B\sigma(B'x + u)\,, \\ x(0) &= 0\,. \end{aligned} \tag{35}$$

Let γ_1, γ_2 be respectively the L^1 and L^2-gains of (34) and let $d = \min\{a_1, \ldots, a_m\}$. Then we have

1. $G_1 \le (\frac{K^2}{d}+1)\gamma_1$,
2. $G_2 \le 2\frac{\sqrt{n}}{d}(K^2+K)\gamma_2$.

(In the literature, γ_2 is called the "H_∞-norm" of (34) and is usually denoted by $\|W\|_\infty$, where $W(s)$ is the transfer matrix for system (34).)

Proof. For each $u \in L^p([0,\infty),\mathbb{R}^m)$, let x be the solution of (35) corresponding to u. Let $\tilde{x} = B'x + u$.

For the case $p=1$, consider the derivative of $V = \|x\|^2/2$ along the trajectories of (35). We get

$$\begin{aligned}\dot{V}(x) &= -\tilde{x}'\sigma(\tilde{x}) + u'\sigma(\tilde{x}) \\ &\le -\tilde{x}'\sigma(\tilde{x}) + K\|u\|\,.\end{aligned}$$

Integrating the above inequality from 0 to ∞, we obtain

$$\int_0^\infty \tilde{x}'(s)\sigma\left(\tilde{x}(s)\right) ds \le K\|u\|_{L^1}. \tag{36}$$

Let

$$v(t) = -\tilde{x}(t) + D^{-1}\sigma\left(\tilde{x}(t)\right) + u(t).$$

Then, we have

$$\begin{aligned}\int_0^\infty \|v(s)\| ds &\le \int_0^\infty \Big\{\|D^{-1}\|\,\|D\tilde{x}(s) - \sigma\left(\tilde{x}(s)\right)\| + \|u(s)\|\Big\} ds \\ &\le \int_0^\infty \left\{\frac{K}{d}\tilde{x}'(s)\sigma\left(\tilde{x}(s)\right) + \|u(s)\|\right\} ds \\ &\le \left(\frac{K^2}{d}+1\right)\|u\|_{L^1}\,.\end{aligned}$$

Now (35) can be written as

$$\begin{aligned}\dot{x} &= \tilde{A}x - BDv(t)\,, \\ x(0) &= 0\,.\end{aligned}$$

By the definition of γ_1 we have $\|x\|_{L^1} \le \gamma_1\|v\|_{L^1} \le (\frac{K^2}{d}+1)\gamma_1\|u\|_{L^1}$. Therefore

$$G_1 \le \left(\frac{K^2}{d}+1\right)\gamma_1,$$

and Conclusion 1 is then proved.

Now we show Conclusion 2. Since $\tilde{A}$ is Hurwitz, we take

$$V_2(x) = \frac{c\|x\|^3}{3} + x'Px$$

with $c = 2K\|PB\|$ and P is the positive definite symmetric matrix satisfying

$$\tilde{A}'P + P\tilde{A} = -I\,. \tag{37}$$

Then, rewriting (35) as

$$\dot{x} = \tilde{A}x + B\left(D\tilde{x} - \sigma(\tilde{x}) - Du\right)$$

and similar to the proof of Lemma 3.1, we have

$$\begin{aligned}\dot{V}_2(x) &= -c\|x\|\tilde{x}'\sigma(\tilde{x}) + c\|x\|u'\sigma(\tilde{x}) \\ &\quad -\|x\|^2 + 2x'PB\,(D\tilde{x} - \sigma(\tilde{x}) - Du) \\ &\le -\|x\|^2 + 2(\|D\| + K^2)\,\|PB\|\,\|x\|\,\|u\|\,.\end{aligned}$$

From this we can get

$$G_2 \le 2(\|D\| + K^2)\|PB\| \le 2(K^2 + K)\|PB\|\,. \tag{38}$$

Next we want to compare $\|PB\|$ with γ_2. First, let us compare $\|PBD^{1/2}\|$ with $\widehat{\gamma}_2$, where $\widehat{\gamma}_2$ is the L^2-gain of

$$\begin{aligned}\dot{x} &= \tilde{A}x + BD^{1/2}u\,, \\ x(0) &= 0\,.\end{aligned} \tag{39}$$

Notice that $\widehat{\gamma}_2 \le \|D^{-1/2}\|\gamma_2$. We now consider the *Hankel norm* $\|W\|_{\text{hankel}}$ for system (39). Note that the matrix P is the *observability Gramian* for (39) (the output is just the state in our case). The *controllability Gramian* for system (39) is defined to be the symmetric matrix $Q \ge 0$ that satisfies

$$\tilde{A}Q + Q\tilde{A}' + BDB' = 0\,. \tag{40}$$

We know that the Hankel norm for (39) is equal to

$$\|W\|_{\text{hankel}} = (\lambda_{\max}(PQ))^{\frac{1}{2}}\ , \tag{41}$$

where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue, cf. [1]. We also know that the H_∞-norm $\widehat{\gamma}_2$ for (39) is related to the Hankel norm by the following inequalities:

$$\widehat{\gamma}_2 \le (2n+1)\|W\|_{\text{hankel}} \le (2n+1)\widehat{\gamma}_2\,. \tag{42}$$

Now in our case, since $\tilde{A} = A - BDB'$ and $\tilde{A}' = -A - BDB'$, the controllability Gramian Q is equal to $\frac{1}{2}I$. Therefore the Hankel norm for (39) is just

$$\|W\|_{\text{hankel}} = (\lambda_{\max}(P/2))^{\frac{1}{2}}\ .$$

Since P satisfies

$$(A' - BDB')P + P(A - BDB') + I = PA - AP - BDB'P - PBDB' + I = 0\,,$$

multiplying both sides by P on the right, we get

$$PAP - APP - BDB'PP - PBDB'P + P = 0\,. \tag{43}$$

Now taking trace to both sides of (43), we get that

$$\|PBD^{1/2}\|_F^2 = Tr(P/2)\,.$$

On the other hand we know that $Tr(P/2)$ is equal to the sum of all the eigenvalues of $P/2$. Therefore $Tr(P/2) \le n\lambda_{\max}(P/2)$. Finally we get $\|PB\| \le \|D^{-1/2}\|\,\|PBD^{1/2}\| \le \sqrt{n}\|D^{-1/2}\|\lambda_{\max}^{\frac{1}{2}}(P/2) \le \sqrt{n}\|D^{-1/2}\|\widehat{\gamma}_2 \le \sqrt{n}\|D^{-1}\|\gamma_2$. Thus

$$G_2 \le 2\frac{\sqrt{n}}{d}(K^2 + K)\gamma_2\,,$$

and this completes the proof. □

Remark 6. The dimension of the state space does not appear in the bound of the estimate in Conclusion 1 of Corollary 3. We suspect also that the estimate for G_2 should be independent of the dimension of the state space.

5. Nonzero Initial States. We now turn to nonzero initial states. We start with an easy observation.

Remark 7. Consider systems as in Theorem 2, but without controls, that is, any system (S) given by $\dot{x} = Ax + B\sigma(Fx)$, where A, B, σ are as in Theorem 2 and F is chosen as in its proof. It is well-known that the origin is globally asymptotically stable, assuming for instance controllability of the matrix pair (A, B). It is interesting to see that this fact also can be shown as a consequence of our arguments. From the proof of Theorem 2, it is enough to show that the system $(\widehat{S})$ $\dot{x} = Ax - B\sigma(B'x)$, with A skew-symmetric and (A, B) controllable, is globally asymptotically stable with respect to the origin. But this follows trivially from (28), since we have along the trajectories of $(\widehat{S})$ that $dV_2(x(t))/dt \leq -\|x(t)\|^2$. Thus V_2 is a strict Lyapunov function for this system without controls.

The previous remark suggests the study of relationships between L^p-stability and global asymptotic stability of the origin. We prove below that, even for nonlinear feedback laws, L^p-stability for finite p implies asymptotic stability.

5.1. Relations Between State-Space Stability and L^p-Stability. We consider initialized control systems of the type (1). If this system is L^p-stable for some $p \in [1, \infty)$ and if, in addition, f satisfies some growth or regularity assumptions, we are able to draw conclusions regarding the asymptotic behavior of the solutions of

$$\dot{x} = f(x, 0). \tag{44}$$

We next define the various alternative properties of f under which we will be able obtain several such conclusions:

$(H_{1,p})$: there exist $\alpha \in [0, p]$, $\delta > 0$, $K_1, K_2 \geq 0$, such that for all $x \in \mathbb{R}^n$ with $\|x\| < \delta$ and for all $u \in \mathbb{R}^m$ we have:

$$\|f(x, u)\| \leq K_1(\|x\| + \|u\|) + K_2(\|x\|^\alpha + \|u\|^\alpha);$$

$(H_{2,p})$: there exist $\alpha \in [0, p]$, $K_1, K_2 \geq 0$, such that for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ we have:

$$\|f(x, u)\| \leq K_1(\|x\| + \|u\|) + K_2(\|x\|^\alpha + \|u\|^\alpha);$$

(H_3) : the function f is differentiable at $(0, 0)$ with $A \overset{def}{=} D_x f(0, 0)$ and $B \overset{def}{=} D_u f(0, 0)$.

Then we have the following lemma:

Lemma 5.1. *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a locally Lipschitz function. Assume that the system*

$$\dot{x} = f(x, u), \ \ x(0) = 0 \tag{45}$$

is L^p-stable for some $p \in [1, \infty)$ with L^p-gain G_p. For each $u \in L^p([0, \infty), \mathbb{R}^m)$, let x_u denote the corresponding solution of (45). We have:

1) *if f satisfies $(H_{1,p})$, then, for each u, $\lim_{t\to\infty} x_u(t) = 0$;*

2) *if f satisfies $(H_{2,p})$, then there exists a constant $C > 0$ so that, for each u,*

$$\|x_u\|_{L^\infty} \leq C \max\left(\|u\|_{L^p}, \|u\|_{L^p}{}^{\frac{p}{p+1-\alpha}}\right); \tag{46}$$

3) *if f satisfies (H_3), then the linearized system*

$$\dot{x} = Ax + Bu, \quad x(0) = 0$$

is L^p-stable with L^p-gain $\gamma_p \leq G_p$ (so, in this case, if (A, B) is controllable, then A must be Hurwitz and the system (44) is locally exponentially stable).

Note that if system (45) is L^p-stable, then $f(0,0)=0$.

Proof. In the sequel we write x_u simply as x, when the control is clear from the context.

1) Assume that the conclusion is not true for some $u \in L^p([0,\infty),\mathbb{R}^m)$. Then there exists $\delta_1 > 0$ so that $\limsup_{t\to\infty} \|x(t)\| \geq 2\delta_1$. Without loss of generality, we may assume that $\delta_1 \leq \min(1,\delta)$.

Take $\varepsilon > 0$ and fix a time $T_0 > 0$ so that

$$\|u\|_{L^p[T_0,\infty)} \leq \varepsilon, \quad \|x\|_{L^p[T_0,\infty)} \leq \varepsilon.$$

Since $\liminf_{t\to\infty} x(t) = 0$, there exist $T_1, T_2 > T_0$ such that

(a) $\frac{\delta_1}{2} \leq \|x(t)\| \leq \delta_1$ for $t \in [T_1, T_2]$;

(b) $\|x(T_2) - x(T_1)\| \geq \frac{\delta_1}{2}$.

Then using $(H_{1,p})$ and applying Hölder's inequality, we obtain

$$\begin{aligned} \frac{\delta_1}{2} \leq \|x(T_2)-x(T_1)\| &\leq \int_{T_1}^{T_2} \|f(x(s),u(s))\| ds \\ &\leq 2K_1\varepsilon(T_2-T_1)^{\frac{p-1}{p}} + 2K_2\varepsilon^{\alpha}(T_2-T_1)^{\frac{p-\alpha}{p}}, \end{aligned} \tag{47}$$

$$(T_2-T_1)(\frac{\delta_1}{2})^p \leq \int_{T_1}^{T_2} \|x(t)\|^p dt \leq \varepsilon^p. \tag{48}$$

Using (47) and (48), we get

$$\frac{\delta_1}{2} \leq 2\left(\frac{K_1}{(\frac{\delta_1}{2})^{p-1}} + \frac{K_2}{(\frac{\delta_1}{2})^{p-\alpha}}\right)\varepsilon^p.$$

Since ε is arbitrary, we obtain a contradiction.

2) For each $T > 0$, let $\beta_T = \sup_{t\in[0,T]} \|x(t)\|$ and fix an interval $[T_1,T_2]$ in $[0,T]$ such that

(a) $\frac{\beta_T}{2} \leq \|x(t)\| \leq \beta_T$ for $t \in [T_1,T_2]$;

(b) $\|x(T_2)-x(T_1)\| = \frac{\beta_T}{2}$.

Since $(H_{2,p})$ holds, we obtain, using the L^P-stability of (45) and Hölder's inequality, that

$$\frac{\beta_T}{2} \leq C_1(T_2-T_1)^{\frac{p-1}{p}}\|u\|_{L^p} + C_2(T_2-T_1)^{\frac{p-\alpha}{p}}\|u\|_{L^p}^{\alpha} \tag{49}$$

for appropriate constants C_1, C_2 and

$$(T_2-T_1)(\frac{\beta_T}{2})^p \leq C_3\|u\|_{L^p}^p \tag{50}$$

for some constant $C_3 > 0$. From (49) and (50) we can easily conclude

$$\beta_T \leq C\max\left(\|u\|_{L^p}, \|u\|_{L^p}^{\frac{p}{p+1-\alpha}}\right), \tag{51}$$

where $C > 0$ is a constant independent of T. Since T is arbitrary, (46) holds.

3) For each control u and $\varepsilon \neq 0$, let x_ε be the trajectory of (45) corresponding to εu. Then it is easy to see that $z_\varepsilon(t) \stackrel{def}{=} \frac{x_\varepsilon(t)}{\varepsilon}$ converges, for each t as $\varepsilon \to 0$, to the solution $z(t)$ of

$$\dot{z} = Az + Bu, \quad z(0) = 0.$$

We have $\|z_\varepsilon\|_{L^p} \leq G_p\|u\|_{L^p}$. From this we can prove that $\|z\|_{L^p} \leq G_p\|u\|_{L^p}$, which implies that $\gamma_p \leq G_p$, cf. also [18]. ∎

Remark 8. One can notice that the finiteness of G_p was not used in the proof of 1). Only the fact that inputs in L^p produce state trajectories in L^p is used.

If we assume reachability conditions on (45), together with L^p-stability of the system for some $p \in [1, \infty)$ and a hypothesis as in Lemma 5.1, we can obtain information on the asymptotic stability of system (45). We will focus on a special class of systems described by (45) and our results are contained in the next lemma.

LEMMA 5.2. *Let A be an $n \times n$ matrix, B an $n \times m$ matrix, σ an $\mathbb{R}^m$-valued S-function and f a locally Lipschitz function from $\mathbb{R}^n$ to $\mathbb{R}^m$. We assume that (A, B) is controllable. Consider the system of differential equations*

$$\dot{x} = Ax + B\sigma(f(x)) \tag{52}$$

and the control system

$$\begin{aligned} \dot{x} &= Ax + B\sigma(f(x) + u)\,, \\ x(0) &= 0\,. \end{aligned} \tag{53}$$

We have the following conclusions:

(i) if system (53) is L^p-stable for some $p \in [1, \infty)$, then system (52) is locally asymptotically stable with respect to the origin;

(ii) if the reachable set from 0 of (53) is equal to $\mathbb{R}^n$ and if system (53) is L^p-stable for some $p \in [1, \infty)$, then system (52) is globally asymptotically stable with respect to the origin.

Proof. We first show (i). Note that the system (53) satisfies $(H_{2,p})$ (with $\alpha = 0$). Fix a $u \in L^p([0, \infty), \mathbb{R}^m)$. Let x_u be the solution of (53) corresponding to u. From Lemma 5.1 we know that $x_u(t) \to 0$ as $t \to \infty$.

To prove stability, we need some elementary reachability results for linear systems. By our assumption we know that the system

$$\dot{x} = Ax + Bu \tag{54}$$

is controllable. Any point $x_0 \in \mathbb{R}^n$ can be reached from 0 by trajectories of (54) at time 1. Moreover we can choose a u_{x_0} on $[0, 1]$ that steers 0 to x_0 and satisfies $\|u_{x_0}\|_{L^\infty[0,1]} \leq C\|x_0\|$, where $C > 0$ is a constant depending on A, B (cf. e.g. [13]). By a measurable selection it is also true that there is a measurable control v that steers 0 to x_0 for the system (S) $\dot{x} = Ax + B\sigma(v)$, provided that x_0 is small enough. Moreover $\|v\|_{L^\infty[0,1]}$ can be made small if $\|x_0\|$ is small. So if we let $u = v(t) - f(x(t))$ on $[0, 1]$, where x is the solution of (S), then u steers 0 to x_0 for (S) at time 1. Let U be an open neighborhood of 0. For each $\delta > 0$, let $\theta(\delta) > 0$ be small enough such that, for each x_0 with $\|x_0\| \leq \theta(\delta)$, there exists a u_{x_0} that steers 0 to x_0 for (53) with $\|u_{x_0}\|_{L^p[0,1]} < \delta$. If x is the solution of (52) starting at x_0, and if we let $u(t) = u_{x_0}(t)$ on $[0, 1]$ and $u(t) = 0$ on $(1, \infty)$, then the solution x_u of (53) satisfies $x_u(t) = x(t-1)$ on $[1, \infty)$. By (46) we can take a $\delta > 0$ small enough such that for any x_0 with $\|x_0\| \leq \theta(\delta)$, the solution x of (52) starting at x_0 stays in U. So system (52) is locally stable.

We next show (ii). Local stability follows as in (i). To prove global attraction, note that the reachability assumption implies that any trajectory x of (52) can be seen as a part of a trajectory of (53) corresponding to a control in L^p. Now Lemma 5.1 provides that $x(t) \to 0$. □

5.2. Dissipation Inequality and Input to State Stability. Next we give a slightly different proof of Theorem 2, which results in a weaker statement (we now allow ε to depend on p) but which is somewhat simpler. Moreover, it results in a simple dissipation-type inequality, from which conclusions about nonzero initial states will be evident. We will only sketch the steps, as they parallel to those in the previous proofs.

Assume that A is skew-symmetric and $A - BB'$ is Hurwitz. Fix a $1 \leq p < \infty$ first. Let τ, a, b, K, $V_{0,p}, V_{1,p}$ be as in the proof of Lemma 3.1. Let

$$\lambda_p \;=\; K\|B\|c_p\,,$$

$$\varepsilon_p = \frac{1}{2K\lambda_p}.$$

Consider the system

$$\dot{x} = Ax - B\sigma(B'x + u) + \varepsilon_p\theta(v) \tag{55}$$

where the initial states are now arbitrary. Write $\tilde{x}(t) = B'x(t) + u(t)$.

Along the trajectories of (55), we have

$$\begin{aligned} \dot{V}_{0,p}(x(t)) &= -\|x(t)\|^{p-1}\tilde{x}'(t)\sigma\left(\tilde{x}(t)\right) \\ &\quad + \|x(t)\|^{p-1}\left(\varepsilon_p x'(t)\theta\left(v(t)\right) + u'(t)\sigma\left(\tilde{x}(t)\right)\right) \\ &\leq -\|x(t)\|^{p-1}\tilde{x}'(t)\sigma\left(\tilde{x}(t)\right) \\ &\quad + K\|x(t)\|^{p-1}\|u(t)\| + K\varepsilon_p\|x(t)\|^p. \end{aligned} \tag{56}$$

(Compare this with (22).) Similar to (25) we can get (for $p > 1$)

$$\begin{aligned} \dot{V}_{1,p}(x(t)) &\leq -\|x(t)\|^p + Kc_p\|B\|\,\|x(t)\|^{p-1}\tilde{x}'(t)\sigma\left(\tilde{x}(t)\right) \\ &\quad + c_p\,K\|x(t)\|^{p-1}\left(\|B\|\,\|u(t)\| + \varepsilon_p\|v(t)\|\right). \end{aligned} \tag{57}$$

Again letting $V_p(x) = \lambda_p V_{0,p}(x) + V_{1,p}(x)$, we obtain

$$\begin{aligned} \dot{V}_p(x(t)) &\leq -(1 - K\lambda_p\varepsilon_p)\|x(t)\|^p + \|x(t)\|^{p-1}\left((K+1)\lambda_p\|u(t)\| + c_pK\varepsilon_p\|v(t)\|\right) \\ &= -\frac{1}{2}\|x(t)\|^p + \|x(t)\|^{p-1}\left((K+1)\lambda_p\|u(t)\| + c_pK\varepsilon_p\|v(t)\|\right). \end{aligned}$$

Let

$$\kappa_p = \max\{(K+1)\lambda_p, c_pK\varepsilon_p\}.$$

Thus, for $p > 1$,

$$\dot{V}_p(x(t)) \leq -\frac{1}{2}\|x(t)\|^p + \kappa_p\|x(t)\|^{p-1}(\|u(t)\| + \|v(t)\|). \tag{58}$$

Arguing as in the proof of Lemma 3.1, this provides L^p-stability provided that $x(0) = 0$. But we also note in this case that it is possible to rewrite (58) in a "dissipation inequality" form, as follows. First, by Young's inequality, we have for any $\alpha, \mu, \nu > 0$ and $p > 1$,

$$\mu^{p-1}\nu \leq \frac{p-1}{p}\alpha^{\frac{p}{p-1}}\mu^p + \frac{\nu^p}{p\alpha^p}.$$

Let

$$\alpha_p = \left[\frac{p}{4(p-1)\kappa_p}\right]^{\frac{p-1}{p}}.$$

Then (58) can be written as

$$\dot{V}_p(x(t)) \leq -\frac{1}{4}\|x(t)\|^p + \frac{\kappa_p}{p\alpha_p^p}(\|u(t)\| + \|v(t)\|)^p.$$

So if we let $\tilde{V}_p = 4V_p$, $r_p = \frac{4\kappa_p}{p\alpha_p^p}$, we finally conclude, along all solutions of (55):

$$\dot{\tilde{V}}_p(x(t)) \leq -\|x(t)\|^p + r_p(\|u(t)\| + \|v(t)\|)^p \,. \tag{59}$$

This is sometimes called a *dissipation inequality*; see [7].

Take in particular $p = 2$ and write $V = \tilde{V}_2$. The estimate (59) shows that $V(x(t))$ must decrease if $\|x(t)\|$ is larger than $\sqrt{r_2}$ times the input magnitude. Thus, irrespective of the initial state, the state trajectory is ultimately bounded, assuming that the inputs u and v are bounded, and this asymptotic bound depends on an asymptotic bound on u and v. One way to summarize this conclusion is by means of the estimate

$$\|x(t)\| \ \leq \ \beta(\|x(0)\|, t) + \gamma\left(\|(u, v)\|_{L^\infty[0,t]}\right) \tag{60}$$

valid for all $x(0)$, all $t \geq 0$, and all essentially bounded u, v, where γ is a function of class K and β is a class-KL function (that is, $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is so that for each fixed $t \geq 0$, $\beta(\cdot, t)$ is a class-K function, and for each fixed $s \geq 0$, $\beta(s, \cdot)$ is decreasing to zero as $t \to \infty$). This is the notion of *ISS-stability* discussed in e.g. [11, 10, 17, 15]; Equation (60) is a consequence of (59), which says that V is a *Lyapunov-ISS* function. In fact, in our case one can say more about the function γ, namely: it can be taken to be *linear*. Indeed, from the proof (p. 441) in [11] one can take any $\gamma \geq \alpha_1^{-1} \circ \alpha_2 \circ \alpha_4$, where $\alpha_4(l) = \sqrt{r_2} l$ and where the α_i's are class-K functions so that

$$\alpha_1(\|x\|) \ \leq V(x) \leq \ \alpha_2(\|x\|)$$

for all $x \in \mathbb{R}^n$. Here we can choose $\alpha_2 = c\alpha_1$, for some $c > 1$, where α_1 is of the form $\alpha(l) = a_1 l^2 + a_2 l^3$, and is thus a convex function. Since for any increasing convex function α and $c > 1$, and any $d > 0$, $\alpha^{-1}(c\alpha(dl)) \leq cdl$ for all l, this gives a linear γ as claimed.

6. More General Input Nonlinearities. Now we consider a broader class of input nonlinearities, allowing unbounded functions as well. The main result will be extended to this case.

DEFINITION 4. We call $\Sigma : \mathbb{R} \to \mathbb{R}$ an *$\widetilde{S}$-function* if it can be written as $\Sigma(t) = \alpha t g(t) + \sigma(t)$, where

- $\alpha \geq 0$ is a constant;
- $g : \mathbb{R} \to [a, b]$ is measurable and a, b are strictly positive real numbers; and
- $\sigma : \mathbb{R} \to \mathbb{R}$ is an S-function.

We say that $\Sigma = (\Sigma_1, \ldots, \Sigma_m)'$ is an $\mathbb{R}^m$-valued $\widetilde{S}$-function if each Σ_i is an $\widetilde{S}$-function. As before if $\xi = (\xi_1, \ldots, \xi_m)' \in \mathbb{R}^m$, then $\Sigma(\xi) = (\Sigma_1(\xi_1), \ldots, \Sigma_m(\xi_m))'$.

With this definition we have the following generalization of Theorem 1.

THEOREM 4. *Let A, B be $n \times n, n \times m$ matrices respectively and Σ be an $\mathbb{R}^m$-valued $\widetilde{S}$-function. Assume that A is neutrally stable. Then there exists an $m \times n$ matrix F such that the system*

$$\begin{aligned} \dot{x} &= Ax + B\Sigma(Fx + u)\,, \\ x(0) &= 0\,, \end{aligned} \tag{61}$$

is L^p-stable for all $1 \leq p \leq \infty$.

Proof. As in the proof of Theorem 2, we can assume without loss of generality that A is skew-symmetric and (A, B) is controllable.

Assume that $\Sigma = (\Sigma_1, \ldots, \Sigma_m)'$ with $\Sigma_i(t) = \alpha_i t g_i(t) + \sigma_i(t)$. Let $\sigma = (\sigma_1, \ldots, \sigma_m)'$ and $G = \mathrm{diag}(\alpha_1 g_1, \ldots, \alpha_m g_m)$ with $G(\xi) = \mathrm{diag}(\alpha_1 g_1(\xi_1), \ldots, \alpha_m g_m(\xi_m))$ for $\xi \in \mathbb{R}^m$. Then $\Sigma(\xi) = G(\xi)\xi + \sigma(\xi)$.

The α_i's split into two sets, $\Lambda_1 = \{\alpha_i, \alpha_i > 0\}$ and $\Lambda_2 = \{\alpha_i, \alpha_i = 0\}$. We can assume without loss of generality that

$$\Lambda_1 = \{\alpha_1, \dots, \alpha_r\} \text{ and } \Lambda_2 = \{\alpha_{r+1}, \dots, \alpha_m\}, \quad r \leq m.$$

Therefore system (61) becomes

$$\dot{x} = Ax + B \left(\begin{array}{cccc|c} \alpha_1 g_1 & & & & \\ & \ddots & & O & O \\ O & & \ddots & & \\ & & & \alpha_r g_r & \\ \hline & & O & & O \end{array} \right) (Fx + u) + B\sigma(Fx + u).$$

Write $B = (B_1, B_2)$, $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$, $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\sigma = \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix}$, and let $G_1 = \text{diag}(\alpha_1 g_1, \dots, \alpha_r g_r)$ with $G_1(\xi) = \text{diag}(\alpha_1 g_1(\xi_1), \dots, \alpha_r g_r(\xi_r))$ for $\xi \in \mathbb{R}^r$. The sizes of the matrices B_1, B_2, F_1, F_2 are respectively, $n \times r, n \times (m - r), r \times n, (m - r) \times n$. As for u_1, u_2, they are respectively elements of $\mathbb{R}^r$ and $\mathbb{R}^{m-r}$. The S-functions σ^1, σ^2 are respectively $\mathbb{R}^r$ and $\mathbb{R}^{m-r}$-valued. We rewrite (61) as

$$\begin{aligned} \dot{x} &= Ax + B_1 G_1(F_1 x + u_1)(F_1 x + u_1) \\ &\quad + B_1 \sigma^1(F_1 x + u_1) + B_2 \sigma^2(F_2 x + u_2), \\ x(0) &= 0. \end{aligned} \tag{62}$$

Let $R(A, B_1) : \mathbb{R}^{rn} \to \mathbb{R}^n$ be the reachability matrix of (A, B_1). (Here and below we will identify matrices with the corresponding linear maps.)

Let $D = \text{Im}R(A, B_1)$ and $H = D^\perp$. We have $D \oplus H = \mathbb{R}^n$. Clearly the subspace D is invariant under A and $\text{Im}(B_1) \subseteq D$. Since A is skew-symmetric, the subspace H is also invariant under A. So there exists an orthogonal $n \times n$ matrix U such that

$$UAU' = \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix} \tag{63}$$

where A_1 and A_2 are skew-symmetric and are restrictions of A to G and H respectively. So, up to an orthonormal change of basis, we can assume that A is already of the form (63). According to this decomposition, $D = \text{Im } R(A, B_1)$. Let $s = \dim D = \text{rank } R(A, B_1)$. Consider now

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} B_{11} \\ B_{12} \end{pmatrix}, \qquad B_2 = \begin{pmatrix} B_{21} \\ B_{22} \end{pmatrix},$$
$$F_1 = (F_{11},\ F_{12}), \qquad \text{and } F_2 = (F_{21},\ F_{22}).$$

Here, $x_1 \in \mathbb{R}^s$, $x_2 \in \mathbb{R}^{n-s}$ and the sizes of $B_{11}, B_{12}, B_{21}, B_{22}$ and $F_{11}, F_{12}, F_{21}, F_{22}$ are respectively $s \times r, (n - s) \times r, s \times (m - r), (n - s) \times (m - r)$ and $r \times s, r \times (n - s), (m - r) \times s, (m - r) \times (n - s)$.

Since $\text{Im}B_1 \subset D$, we have $B_{12} = 0$. Now system (62) becomes

$$\begin{aligned} \dot{x}_1 = A_1 x_1 &+ B_{11} G_1(F_{11} x_1 + F_{12} x_2 + u_1)(F_{11} x_1 + F_{12} x_2 + u_1) \\ &+ B_{11} \sigma^1(F_{11} x_1 + F_{12} x_2 + u_1) + B_{21} \sigma^2(F_{21} x_1 + F_{22} x_2 + u_2), \\ \dot{x}_2 = A_2 x_2 &+ B_{22} \sigma^2(F_{21} x_1 + F_{22} x_2 + u_2)\ . \end{aligned}$$

Choose now $F_{12} = F_{21} = 0$, $F_{11} = -B'_{11}$, and $F_{22} = -B'_{22}$. We obtain

$$\begin{aligned} \dot{x}_1 &= \left(A_1 - B_{11} G_1(-B'_{11} x_1 + u_1) B'_{11}\right) x_1 + B_{11} G_1(-B'_{11} x_1 + u_1) u_1 \\ &\quad + B_{11} \sigma^1(-B'_{11} x_1 + u_1) + B_{21} \sigma^2(-B'_{22} x_2 + u_2), \\ \dot{x}_2 &= A_2 x_2 + B_{22} \sigma^2(-B'_{22} x_2 + u_2). \end{aligned}$$

In the above system, replacing $\sigma(\cdot)$ by $-\sigma(-\cdot)$ (still denoted by σ), the system becomes

$$\begin{aligned}\dot{x}_1 &= \left(A_1 - B_{11}G_1(-B'_{11}x_1 + u_1)B'_{11}\right)x_1 + B_{11}G_1(-B'_{11}x_1 + u_1)u_1 \\ &\quad -B_{11}\sigma^1(B'_{11}x_1 - u_1) - B_{21}\sigma^2(B'_{22}x_2 - u_2), \\ \dot{x}_2 &= A_2x_2 - B_{22}\sigma^2(B'_{22}x_2 - u_2).\end{aligned}$$

Since (A, B) is controllable, (A_2, B_{22}) is also controllable. It follows from Theorem 2 that the x_2-subsystem is L^p-stable for all $1 \le p \le \infty$. So there exists $C_p^1 > 0$ such that $\|x_2\|_{L^p} \le C_p^1\|u_2\|_{L^p}$.

For $i = 1, \ldots, r$, let $d_i(t) = \sigma_i^1(t)/t$ if $t \neq 0$ and $d_i(t) = 0$ if $t = 0$. Let $\tilde{G}_1(\xi) = \text{diag}(d_1(\xi_1), \ldots, d_r(\xi_r))$. Then we can rewrite the x_1-subsystem as

$$\dot{x}_1 = \left[A_1 - B_{11}\left(G_1(-B'_{11}x_1 + u_1) + \tilde{G}_1(B'_{11}x_1 - u_1)\right)B'_{11}\right]x_1 + v,$$

where

$$v = B_{11}G_1(-B'_{11}x_1 + u_1)u_1 + B_{11}\tilde{G}_1(B'_{11}x_1 - u_1)u_1 - B_{21}\sigma^2(B'_{22}x_2 - u_2).$$

We have

$$\|v\| \le C(\|u_1\| + \|x_2\| + \|u_2\|)$$

for some $C > 0$.

If we let $\widetilde{D}(t) = G_1(-B'_{11}x_1(t) + u_1(t)) + \tilde{G}_1(B'_{11}x_1(t) - u_1(t))$, then the above equation can be written as

$$\dot{x}_1(t) = \left(A_1 - B_{11}\widetilde{D}(t)B'_{11}\right)x_1(t) + v(t)\,.$$

By definition of an S-function and an $\widetilde{S}$-function, there exist two real numbers δ_1 and δ_2 such that $0 < \delta_1 \le \delta_2$ and if we write $\widetilde{D}(t) = \text{diag}(\tilde{d}_1(t), \cdots, \tilde{d}_r(t))$, then

$$\delta_1 \le \tilde{d}_i(t) \le \delta_2$$

for $i = 1, \cdots, r$. Since (A, B) is controllable, (A_1, B_{11}) is controllable too. Then it follows from Corollary 2 that

$$\|x_1\|_{L^p} \le \bar{C}_p^2\|v\|_{L^p}\,,$$

for some $\bar{C}_p^2 > 0$ depending on $A_1, B_{11}, \delta_1, \delta_2$ and p. But we know that

$$\|v\|_{L^p} \le C(\|u_1\|_{L^p} + \|u_2\|_{L^p} + \|x_2\|_{L^p}) \le C\|u_1\|_{L^p} + C(1 + C_p^1)\|u_2\|_{L^p}\,.$$

Therefore we have $\|x_1\|_{L^p} \le C_p^2\|u\|_{L^p}$ for some constant $C_p^2 > 0$. □

7. Counterexample: The nth Order Scalar Integrator. The next result is a negative one, and it concerns systems such as those in equation (2), except that the matrix A is not neutrally stable but instead is assumed to have a non-simple Jordan block for the zero eigenvalue. In that case, we show that for any possible F which stabilizes the corresponding linear control system

$$\begin{aligned}\dot{x} &= Ax + B(Fx + u)\,, \\ x(0) &= 0\,,\end{aligned}$$

the resulting system (Σ_u) is *not* in general L^p-stable for any $1 \le p \le \infty$. We first consider the simplest case, namely the double integrator. The proof is of interest because the origin of the corresponding system without inputs (but *with* the saturation) *is* globally asymptotically stable. Thus the result is quite surprising. In the end we discuss the n-integrator for $n \geq 3$.

PROPOSITION 1. *Let $1 \leq p \leq \infty$. Consider the following 2-dimensional initialized control system*

$$(S_{a,b}) \qquad \begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -\sigma(ax+by+u)\,, \\ x(0) &= y(0)=0\,, \end{aligned}$$

where $a,b>0$, σ is a scalar S-function and inputs u belong to $L^p([0,\infty),\mathrm{I\!R})$. Then $(S_{a,b})$ is not L^p-stable.

Proof. Up to a reparametrization of the time and a linear change of variables, it is enough to show that the initialized control system

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -\lambda\sigma(x+y+u)\,, \\ x(0) &= y(0)=0\,, \end{aligned}$$

where $\lambda>0$, is not L^p-stable. Now replacing $\lambda\sigma$ by σ (note $\lambda\sigma$ is still an S-function) we may assume that $\lambda=1$. Therefore all needed is to show that the system

$$(S) \qquad \begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -\sigma(x+y+u)\,, \\ x(0) &= y(0)=0 \end{aligned}$$

is not L^p-stable. The proof is quite technical, but the idea is not difficult to understand. It is based on the fact that the feedback $u=-y$ makes the system (S) have periodic trajectories, with a control u whose norm is proportional to that of the y coordinate. But the x coordinate is the integral of y, so the ratio between the p-norms of x and u can be made to be large for $p<\infty$. (For $p=\infty$, one modifies the argument to reach states of large magnitude.)

Let us first fix a p in $[1,\infty)$. Assume that (S) is L^p-stable. Then the following holds: There exists $C_p>0$ so that, if $u\in L^p([0,\infty),\mathrm{I\!R})$, then

$$\|y_u^2\|_{L^p} \leq C_p\ \|u\|_{L^p}\,, \tag{64}$$

where y_u is the second coordinate of (x_u,y_u), the solution of (S) associated to u.

To see this, let $q=2(p-1)\geq 0$ and

$$V_q(x,y) = -\frac{xy|y|^q}{q+1}\,.$$

Then along the trajectory (x_u,y_u) of (S) we have

$$\dot{V}_q = -\frac{1}{q+1}\ |y_u|^{q+2} + x_u\sigma(x_u+y_u+u)|y_u|^q.$$

Therefore,

$$\dot{V}_q + \frac{1}{q+1}\ |y_u|^{q+2} \leq K|x_u||y_u|^q\,, \tag{65}$$

where K is an S-bound for σ. From Lemma 5.1 we know that $\lim_{t\to\infty}(x_u,y_u)=(0,0)$. Integrating (65) from 0 to t and letting $t\to\infty$, we end up with

$$\frac{1}{q+1}\int_0^\infty |y_u|^{q+2} \leq K\int_0^\infty |x_u||y_u|^q.$$

Therefore if $p = 1$ we get that $\|y_u^2\|_{L^1} \leq K\|x\|_{L^1}$. If $p > 1$, applying Hölder's inequality, we get

$$\frac{1}{q+1}\int_0^\infty |y_u|^{2p} \leq K\|x_u\|_{L^p}\left(\int_0^\infty |y_u|^{\frac{qp}{p-1}}\right)^{\frac{p-1}{p}}.$$

But $q\frac{p}{p-1} = 2(p-1)\frac{p}{p-1} = 2p$. Therefore

$$\|y_u^2\|_{L^p} \leq (2p-1)K\|x_u\|_{L^p}. \tag{66}$$

Since (S) is L^p-stable, $\|x_u\|_{L^p} \leq G_p\|u\|_{L^p}$, where G_p is the L^p-gain of (S). So (64) indeed holds.

Now we will construct trajectories of (S) which contradict (64).

We consider the level sets of the following Lyapunov function:

$$V(x,y) = y^2 + G(x),$$

where $G(x) = 2\int_0^x \sigma(s)ds$.

Let $\rho_1 = 2\inf_{|t|\geq 1}|\sigma(t)| > 0$ and define $H : \mathbb{R} \to \mathbb{R}$ by

$$H(x) = \begin{cases} 0 & \text{if } |x| \leq 1, \\ \rho_1(|x|-1) & \text{if } |x| > 1. \end{cases}$$

We have

$$y^2 + H(x) \leq V(x,y) \leq y^2 + 2K|x|. \tag{67}$$

Note that along trajectories of

$$(\widehat{S}) \quad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\sigma(x), \end{aligned}$$

V is constant.

Let us fix a constant $V_0 \geq \max\{1, 2K\}$ and let $x^- < 0$ and $x^+ > 0$ be such that $G(x^+) = G(x^-) = V_0$. Since (S) is controllable, there exist a $T_1 > 0$ and a u_0 in $L^p([0,T_1],\mathbb{R})$ such that $(x_{u_0}(T_1), y_{u_0}(T_1)) = (0, \sqrt{V_0})$. We can also assume that $u_0(t) = 0$ for $t > T_1$. For $t \geq 0$, consider $(\bar{x}_0(t), \bar{y}_0(t))$, the solution of $(\widehat{S})$ with $(\bar{x}_0(0), \bar{y}_0(0)) = (0, \sqrt{V_0})$. Note that $V(\bar{x}_0(t), \bar{y}_0(t)) \equiv V_0$. Clearly this trajectory is periodic, since it lies in the closed curve $V(x) \equiv V_0$ and there are no equilibria there. Assume that the period is T.

Consider the sequence $\{u_n\}_{n=1}^\infty$ of inputs defined as follows,

$$u_n(t) = \begin{cases} u_0(t) & \text{on } [0,T_1], \\ -\bar{y}_0(t-T_1) & \text{on } (T_1, T_1+nT], \\ 0 & \text{on } (T_1+nT, \infty). \end{cases}$$

Then if (x_n, y_n) denotes the solution of (S) associated to u_n, we have for $t \in [T_1, T_1+nT]$,

$$(x_n(t), y_n(t)) = (\bar{x}_0(t-T_1), \bar{y}_0(t-T_1)).$$

In this case (note that $y_n(t) = y_{u_0}(t)$ for $t \in [0,T_1]$ and $y_n(t) = y_{u_0}(t-nT)$ for $t \in [T_1+nT, \infty)$)

$$\begin{aligned} \int_0^\infty |u_n(s)|^p\,ds &= \int_0^{T_1} |u_0(s)|^p\,ds + n\int_0^T |\bar{y}_0(s)|^p\,ds, \\ \int_0^\infty |y_n^2(s)|^p\,ds &= \int_0^\infty |y_{u_0}^2(s)|^p\,ds + n\int_0^T |\bar{y}_0^2(s)|^p\,ds. \end{aligned}$$

We conclude that

$$\lim_{n\to\infty} \frac{\|y_n^2\|_{L^p}}{\|u_n\|_{L^p}} = \frac{\left(\int_0^T |\bar{y}_0^2(s)|^p ds\right)^{\frac{1}{p}}}{\left(\int_0^T |\bar{y}_0(s)|^p ds\right)^{\frac{1}{p}}} \stackrel{def}{=} L_{p,V_0}.$$

According to (64), this quotient should be bounded independently of the choice of V_0. We next derive a contradiction by showing that this is not so.

Notice that for any $r \geq 1$, since $\dot{\bar{x}}_0(t) = \bar{y}_0(t)$, we have

$$\int_0^T |\bar{y}_0(s)|^r ds = \int_0^T |\bar{y}_0(s)|^{r-1} |\dot{\bar{x}}_0(s)| ds.$$

Since $V(x, y) = V(x, -y)$, we have

$$\int_0^T |\bar{y}_0(s)|^r ds = \int_0^T |\bar{y}_0(s)|^{r-1} |\dot{\bar{x}}_0(s)| ds = 2 \int_{x^-}^{x^+} |\bar{y}(x)|^{r-1} dx\,, \tag{68}$$

where $|\bar{y}(x)| = \sqrt{V_0 - G(x)}$ for x between x^- and x^+. (Note that the curve $V(x) = V_0$ can be written as the union of the graphs of the functions $y(x) = \pm\sqrt{V_0 - G(x)}$. Thus we can reparameterize the orbit in each of these two parts in terms of the variable x.)

Considering (67), we have $V_0/(2K) \leq |x^-|, x^+ \leq V_0/\rho_1 + 1$. Then it follows from (68) that

$$\begin{aligned} \int_0^T |\bar{y}_0(s)|^p ds &\leq 2V_0^{\frac{p-1}{2}} (x^+ - x^-) \leq 4V_0^{\frac{p-1}{2}} \left(\tfrac{V_0}{\rho_1} + 1\right) \leq C_1 V_0^{\frac{p+1}{2}}\,, \\ \int_0^T |\bar{y}_0^2(s)|^p ds &\geq 4 \int_0^{V_0/(2K)} (V_0 - 2Kx)^{p-\frac{1}{2}}\, dx \geq C_2 V_0^{p+\frac{1}{2}}\,, \end{aligned}$$

where $C_1, C_2 > 0$ are some constants. Finally, we get $L_{p,V_0} \geq C V_0^{1/2}$ for some $C > 0$. But according to (64), $L_{p,V_0} \leq C_p$. Therefore, for V_0 large enough we get a contradiction. So (S) cannot be L^p-stable for $1 \leq p < \infty$.

There remains to establish the special case $p = \infty$. We use again the level sets of V. Let u^0 on $[0, T_0]$ for some $T_0 > 0$ be an input such that $(x_{u^0}(T_0), y_{u^0}(T_0)) = (0, \sqrt{V_0})$, for some $V_0 > 0$ that will be fixed below.

From $(0, \sqrt{V_0})$, follow the trajectory of

$$\begin{aligned} (I)\quad \dot{x} &= y, \\ \dot{y} &= \rho_2, \end{aligned}$$

on $[T_0, T_0+1]$, where $\rho_2 = -\sigma(-1) > 0$. The trajectory (x, y) of (I), hence, reaches $(\sqrt{V_0}+\rho_2/2, \sqrt{V_0}+\rho_2)$. Let

$$V_1 = (\sqrt{V_0} + \rho_2)^2 + G(\sqrt{V_0} + \rho_2/2) \geq V_0 + 2\rho_2\sqrt{V_0}.$$

Note that also $V_1 \leq V_0 + C(\sqrt{V_0} + 1)$ for some $C > 0$. Furthermore, the trajectory of (I) can be viewed as a trajectory of (S) with $u^1(t) = -1 - x(t) - y(t)$ for $T_0 < t \leq T_0 + 1$. Let

$$u_1 = -u^1(T_0 + 1) = 1 + \sqrt{V_0} + \rho_2/2 + \sqrt{V_0} + \rho_2 = 2\sqrt{V_0} + 3/2\rho_2 + 1.$$

Then, for $T_0 + 1 < t \leq T_1$, follow the trajectory $(\bar{x}, \bar{y})$ of $(\widehat{S})$ from $(\sqrt{V_0} + \rho_2/2, \sqrt{V_0} + \rho_2)$ until the resulting trajectory reaches $(0, \sqrt{V_1})$ at $t = T_1$. This trajectory can also be considered as a trajectory of (S) with $u^1(t) = -\bar{y}(t)$ on $(T_0 + 1, T_1]$. Note that $|u^1(t)| \leq \sqrt{V_1}$ for $T_0 + 1 < t \leq T_1$. Fix V_0 such that $\sqrt{V_1} \leq u_1 \leq 3\sqrt{V_0}$. It is clear that on $[T_0, T_1]$ $|u^1(t)| \leq u_1$.

If we iterate the above construction, we can build three sequences $\{V_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=0}^{\infty}$ such that

(1): $V_{n+1} = (\sqrt{V_n} + \rho_2)^2 + G(\sqrt{V_n} + \rho_2/2) \geq V_n + 2\rho_2\sqrt{V_n}$;
(2): $u_n = 2\sqrt{V_{n-1}} + 3/2\rho_2 + 1 \leq 3\sqrt{V_{n-1}}$;
(3): on $[T_n, T_{n+1}]$, there exists an input u^n such that $\sup\{|u^n(t)| : t \in [T_n, T_{n+1}]\} = u_n$ and the trajectory of (S) associated to u^n goes from $(0, \sqrt{V_n})$ to $(0, \sqrt{V_{n+1}})$.

Clearly $\lim_{n\to\infty} V_n = \infty$ and then $\lim_{n\to\infty} u_n = \infty$. Furthermore let $x_n^- < 0$ be such that $G(x_n^-) = V_n$. Then $|x_n^-| \geq 1/(2K)V_n$ for n large enough, which implies that $\lim_{n\to\infty} |x_n^-| = \infty$.

Let $\{\bar{u}^n\}_{n=0}^{\infty}$ be the sequence of inputs which equals to the concatenation of $u^0, u^1, \cdots, u^n$ on $[0, T_n]$ and 0 for $t > T_n$. For n large enough, we have

$$\begin{aligned} \|(x_{\bar{u}^n}, y_{\bar{u}^n})\|_\infty &\geq |x_n^-|, \\ \|\bar{u}^n\|_\infty &= u_n\ . \end{aligned}$$

Since $\left|\frac{x_n^-}{u_n}\right| \geq 1/(2K)\sqrt{V_n}$ for n large enough, (S) is not L^∞-stable. □

For n integrators and $n > 2$, the proof that L^p-stabilization is not possible is simpler (but the result is far less interesting). We can argue as follows. Let σ be a scalar S-function. It was proved in [4, 21] that, if $n \geq 3$, the n-integrator

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ &\vdots \\ \dot{x}_{n-1} &= x_n\,, \\ \dot{x}_n &= -\sigma(u) \end{aligned}$$

is not globally asymptotically stabilizable by any possible linear feedback. With this, it follows from Lemma 5.2 that, if $n \geq 3$, the system

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ &\vdots \\ \dot{x}_{n-1} &= x_n\,, \\ \dot{x}_n &= -\sigma(Fx + u)\,, \\ x(0) = 0 & \end{aligned}$$

is not L^p-stable for any $1 \leq p < \infty$ and any row vector F.

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