Robust Stabilization via Saturated Feedback

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***Abstract*—In this paper, we deal with the problem of stabilization of uncertain systems in the presence of input constraint. First algebraic conditions are derived for input-to-state stability of linear system with saturated linear feedback of low dimension. Then a recursive design procedure is derived for robust stabilization of block upper triangular nonlinear systems with feedforward structure.**

***Index Terms*—Feedforward nonlinear systems, input-to-state stability (ISS), robust control, saturated feedback, small gain.**

I. Introduction

STABILIZATION of linear and nonlinear (possibly uncertain) systems subject to constraints is a central topic in control theory which has relevance both from a practical and theoretical point of view. There are several approaches that one can take when dealing with control in presence of constraints. Model predictive control is one possibility and has received a great deal of attention in the past years [10]. The main idea, roughly speaking, is to convert the control problem into an optimization problem to be solved (either online or offline) so that constraints are satisfied and stability (or performance) is preserved. Nonlinearity of the system usually affects the optimization algorithm that may become nonconvex, whereas uncertainty, in most approaches, is treated according to a worst-case approach which usually entails an exponential growth of the computational burden as the control horizon increases.

Among all, input constraints are a relevant class for which specific techniques have been developed in recent years. In the linear setting stabilizability under arbitrary input saturation has been extensively studied with constructive algebraic approaches (see, i.e., [6], [8], and [14]) or with receding horizon techniques (see [2]). In this respect, it is well-known that global asymptotic stability can be achieved only if the open loop system does not exhibit exponential instability. In particular, for neutrally stable systems, saturated linear feedback can be used to achieve global asymptotic stability and robustness to persistent actuator noise by exploiting passivity ideas as shown in [8]. More in general, the presence of multiple poles on the imaginary axis may prevent the existence of a single saturated linear controller achieving global stability results [5]. Under these circumstances saturated linear feedback can only achieve semiglobal stabilization (see [6]) and "nested saturation" are needed for global convergence ([14]). To the best knowledge of the authors, most of the techniques in the current literature deal with exactly known system parameters and leave open the question of robustness.

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Alternative approaches include the so-called "anti-windup" methods which originally developed as "ad hoc" techniques for specific applications and which have been recently developed in a more rigorous and general perspective (see [16] and [17]).

In the nonlinear setting the use of saturation functions has been shown to be an effective tool in order to globally stabilize the important class of "feedforward systems" (see [12]). In particular, it has been shown in [15] (see also [7]) how recursive stabilization of the linear approximation of the system by means of saturation functions in combination with small gain ideas succeeds in achieving global asymptotic stability. This approach, while capable of handling large uncertainties in the nonlinear higher order terms, is affected by the same limitations encountered for design of saturated linear feedback if the linear approximation of the system is uncertain. An attempt in the direction of achieving robustness with respect to (possible time-varying) uncertain linear approximation, has been studied in [11] for a certain class of nonlinear feedforward systems. However the problem of systematic robust design of feedforward nonlinear system is in general an open research field.

Our goal is to add a contribution in the context of robust state feedback stabilization both for linear and nonlinear systems subject to input constraints. More specifically our contribution is two-fold. First algebraic conditions for single-input linear systems up to dimension three are given which ensure Input-to-State Stability with respect to external exogenous signals of sufficiently small amplitudes. This allows us to treat in a simpler way constant parametric uncertainties affecting the model of the system. Our criterion is indeed a generalization of the so-called "Kalman conjecture" to the special case of saturation nonlinearities. This criterion is then instrumental to extend the design procedure proposed in [15]. In particular a constructive design procedure yielding a nested saturation control law is proposed able to deal with feedforward system with uncertain linear approximation.

The paper is organized as follows. In the next section the problem of robust stabilization of saturated linear feedback is precisely stated and the algebraic conditions for robust input-to-state stability (ISS) are given. Section III reviews the design procedure for feedforward systems by saturation functions and presents the extension of it to the case of nonlinear systems with uncertain linear approximation. Section IV presents a simulative example while Section V concludes with final remarks. Some technical proofs are deferred to the Appendix.

Notations: The vector norm of $x \in \mathbb{R}^n$ is simply denoted by $|x|$. Let $\mathcal{L}_\infty$ be the space of essentially bounded real functions over $[0, \infty)$ equipped with the standard $\mathcal{L}_\infty$-norm $\|x\|_\infty = \operatorname{ess\,sup}_{0 \leq t < \infty} |x(t)|$. Asymptotic signal amplitudes are denoted by $\|x\|_a := \limsup_{t \to \infty} |x(t)|$.

We say that a system $\dot{x} = f(x, u)$ with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ is Input-to-State Stable (in short ISS) with restrictions Δ on the inputs if for all essentially bounded inputs u such that $\|u\|_\infty \leq \Delta$ and for all initial conditions $x^0 \in \mathbb{R}^n$, the following holds:

$$|x(t)| \leq \beta\left(|x^0|, t\right) + \gamma\left(\|u\|_\infty\right)$$

with $\beta(\cdot,\cdot)$ of class $\mathcal{KL}$ and $\gamma(\cdot) \in \mathcal{K}_\infty$.

We call $\sigma : \mathbb{R} \to \mathbb{R}$ a *saturation function* if σ is of class $\mathcal{C}^1$, increasing and if there exist two real numbers $0 < a \leq K_\sigma$ such that for all $t, t' \in \mathbb{R}$

i) $|\sigma(t) - \sigma(t')| \leq K_\sigma \inf(1, |t - t'|)$;

ii) $|\sigma(t) - at| \leq K_\sigma t\sigma(t)$.

A constant K_σ defined as above is called an S-bound for σ. From now on, for simplicity, we assume that $a = K_\sigma = 1$. Finally for a given square matrix L we denote by $\chi_L(s)$, $\mathrm{tr}(L)$ and $\det(L)$ respectively its characteristic polynomial, its trace and its determinant.

II. Algebraic Criteria for ISS of Systems With Saturated Linear Feedback

A. Preliminaries and Main Objective

In this section, we are interested to study the robust design of saturated linear feedback for uncertain linear systems. More specifically, let μ be an uncertain constant parameter ranging within a known compact set $\mathcal{P}$ and suppose we are given a pair $(A(\mu), G(\mu))$, $A(\mu) : \mathcal{P} \to \mathbb{R}^{n\times n}$ and $G(\mu) : \mathcal{P} \to \mathbb{R}^n$, of *continuous* matrix-valued functions which is stabilizable for all $\mu \in \mathcal{P}$. Moreover, let w and v be $\mathcal{L}_\infty$ exogenous signals with $w \in \mathbb{R}$, $v \in \mathbb{R}^n$. Then, we address the problem of designing a state feedback F such that the saturated linear system

$$S_{(w,v)} : \dot{x} = A(\mu)x - G(\mu)\sigma(F^T x + w) + v \qquad (1)$$

(with $x \in \mathbb{R}^n$, $w \in \mathbb{R}$, $v \in \mathbb{R}^n$) is *input-to-state stable* (ISS) with respect to the exogenous inputs (w, v) with suitable restrictions on their amplitude and linear gain, *uniformly* in $\mu \in \mathcal{P}$. Specifically, the adjective *uniform* regards the asymptotic gains and the amplitude restrictions which are required to be independent of the uncertain parameters. In a more formal way we look for an F which guarantees the existence of real numbers $\Delta_w \geq 0$, $\Delta_v \geq 0$, $\gamma_v > 0$, $\gamma_w > 0$ (all independent of μ) such that for all $x^0 \in \mathbb{R}^n$, and any measurable inputs w and v with $\|w\|_\infty \leq \Delta_w$ and $\|v\|_\infty \leq \Delta_v$, the trajectory of the system starting from $x(0) = x^0$ satisfies

$$\|x\|_\infty \leq \max\left\{\gamma_0\left(|x^0|\right),\ \gamma_w\|w\|_\infty,\ \gamma_v\|v\|_\infty\right\}$$
$$\|x\|_a \leq \max\left\{\gamma_w\|w\|_a,\ \gamma_v\|v\|_a\right\}$$

for some class-$\mathcal{K}$ function $\gamma_0(\cdot)$ possibly depending on μ.

Motivated by the results presented in the deterministic scenario (see [8], [14]), we focus on a pair $(A(\mu), G(\mu))$ which is *asymptotically null controllable with bounded inputs (ANCBI)* (using the terminology introduced in [14]) for all μ in the given compact set. This means that, the linear system (1) can be controlled to the origin, for each fixed value of the parameter μ, by applying (arbitrarily small) bounded controls. Algebraically, this is equivalent to the pair $(A(\mu), G(\mu))$ being stabilizable and $A(\mu)$ having all eigenvalues with non negative real part.

The idea pursued in this paper is to reformulate the problem of designing the state feedback F achieving uniformly ISS into a problem of *robust pole placement* regarding the linear approximation of system (1) at the origin in the absence of disturbances. As a matter of fact we show that, under suitable limitations on the dimension n of the state space, the design of F which renders uniformly ISS the saturated linear system is equivalent to the design of a feedback gain F so that (see Theorem 1 in the next subsection)

$$A - \alpha G F^T \quad \text{is Hurwitz } \forall \alpha \in (0, 1] \qquad \forall\, \mu \in \mathcal{P}. \qquad (2)$$

This allows to identify algebraic conditions for the design of F satisfying (2) and to propose a constructive design procedure (see Section II-C). In this respect, it is interesting to note how this result can be seen as a case in which a stronger version of the Kalman conjecture holds. We recall that the latter states that given a nonlinear differentiable function $\sigma(\cdot)$ such that $0 \leq d\sigma(s)/ds \leq b$ for all $s \in \mathbb{R}$ then system $S_{0,0}$ is globally asymptotically stable if

$$A(\mu) - \alpha G(\mu) F^T \quad \text{is Hurwitz for all } \alpha \in [0, b]. \qquad (3)$$

As a matter of fact, the result we are going to show does not rely upon asymptotic stability of the state matrix A within the sector (required in Kalman conjecture as one considers (3) for $\alpha = 0$) but just asks for "at most" polynomial instability of the matrix A (as imposed by the continuity of the spectrum of $A - \alpha G F^T$ as α approaches zero in (2)). Furthermore, while the Kalman conjecture states conditions just for global asymptotic stability of the saturated linear system $S_{(0,0)}$, we are interested in studying input to state stability with respect to exogenous $\mathcal{L}_\infty$ signals. The motivation for dealing with exogenous inputs, besides the fact that it represents a more general perspective, is that this will allow to employ the result also for stabilization of *nonlinear uncertain feedforward systems* as shown in Section III. Before illustrating the main result it is worth stressing that, in accordance with known results about the Kalman conjecture, we are able to prove our result only for $n \leq 3$. In fact, the Kalman conjecture (and our result) is in general false for $n \geq 4$ as it is possible to build a four-dimensional system which satisfies the Hurwitz condition (3) but presents a stable limit cycle (see [9]).

B. Extended Kalman Conjecture

We focus on (1) with $n \leq 3$. The main result of this section is the next extended Kalman conjecture which is given in terms of the positive root locus relative to the polynomials $\chi_A(s)$ and $\chi_{A-GF^T}(s)$.

Theorem 1: Consider the system $S_{(v,w)}$ in (1) with $x \in \mathbb{R}^n$ with $n \in \{1, 2, 3\}$ and σ a saturation function. Assume that the matrix

$$A(\mu) - \alpha G(\mu) F^T \ \text{ is Hurwitz } \forall \alpha \in (0, 1], \forall \mu \in \mathcal{P} \qquad (4)$$

and the corresponding root locus be tranversal to the imaginary axis for $\alpha = 0$ and all $\mu \in \mathcal{P}$. Then there exist positive numbers $\gamma_v, \gamma_w, \Delta_w$ and Δ_v such that system (1) is ISS with restrictions (Δ_w, Δ_v) on the inputs (w, v) and asymptotic linear gains (γ_v, γ_w), uniformly over $\mu \in \mathcal{P}$. ∎

We consider now the proof of Theorem 1. To this end the following definition of *ultimate boundedness* will play a key role.

Definition 2.1: Consider the nonlinear system $\dot{x} = f(x,u)$ with $u(t) \in \mathbb{R}^m$ a vector of $\mathcal{L}_\infty$ signals. Then such a system enjoys the *ultimate boundedness* (UBND) property with restriction Δ on the inputs if there exists a nondecreasing function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for any initial condition $x(0) \in \mathbb{R}^n$ and any essentially bounded $u(\cdot)$ with $\|u\|_\infty \leq \Delta$, it holds

$$\|x\|_a \leq \eta\left(\|u\|_\infty\right). \tag{5}$$

□

We just note that a similar property, with η of class $\mathcal{K}_\infty$, was introduced in [13] under the name of *asymptotic gain*. The notion considered here is much weaker, as we allow $\eta(0) > 0$.

It turns out that all the proof of theorem 1 reduces to show that system (1) has the UBND property with suitable restriction on the inputs (w, v). The reason why is clear by the following two results whose proofs are presented in the Appendix.

Lemma 2.2: A system is ISS with restrictions on the inputs if and only if it is[1] 0-GAS and enjoys the Ultimate Boundedness property with suitable restrictions. □

Proposition 2.3: Consider the n-dimensional system, $n \in \{1,2,3\}$,

$$\dot{x} = Ax - G\sigma(F^T x) \tag{6}$$

where $A - \alpha G F^T$ Hurwitz for all $\alpha \in (0,1]$ and $\sigma(\cdot)$ is a $\mathcal{C}^1$ saturation function with $d\sigma(r)/dr \in (0,1]$ for all $r \in \mathbb{R}$. Then the system is GAS provided that all positive semi-orbits of (6) be bounded. □

As a matter of fact we have that if system (1) enjoys the UBND property with suitable restrictions on (w,v) then the system is 0-GAS by Proposition 2.3 and hence it is ISS with suitable restrictions by Lemma 2.2. Hence, to show ISS we are left to show the UBND property. This will be done considering several cases depending on n, the spectral properties of A and, finally, on G, F.

1) Case $n = 1$: Consider first the scalar case, viz. $x \in \mathbb{R}$. Then, system equations read as

$$\dot{x} = -\lambda x - g\sigma(fx + w) + v \tag{7}$$

with $\lambda \geq 0$. In case $\lambda > 0$ the UBND property trivially follows without restrictions on w and v by the definition of saturation function. In case $\lambda = 0$, by stabilizability $g \neq 0$ and $fg > 0$ by (4). From this the result in [8] shows that system (7) is ISS (and thus satisfies the UBND property) without restrictions on w and nonzero restriction on v.$\triangle$

2) Case $n = 2$: Consider now the bidimensional case. The case of two uncontrollable modes (viz. $G = 0$) is trivial, as A is then Hurwitz and this immediately gives UBND. Assume now that an uncontrollable eigenvalue (necessarily real) exists. Then, by a suitable change of coordinates, the system can be brought in the following form

$$\begin{aligned} \dot{x}_1 &= -\lambda_1 x_1 - \sigma(f_1 x_1 + f_2 x_2 + w) + v_1 \\ \dot{x}_2 &= -\lambda_2 x_2 + v_2 \end{aligned} \tag{8}$$

[1]We use the standard notation "0-GAS" to mean global asymptotic stability in case no inputs are present.

with $\lambda_2 > 0$ and $\lambda_1 + \alpha f_1 > 0$, for all $\alpha \in (0,1]$ [by (4)]. By the result previously proved for the scalar case, the x_1-subsystem is ISS without restrictions with respect to (x_2, w) and with non zero restrictions with respect to v_1. Therefore, Input-to-State stability of the system (8) without restriction on w and with nonzero restrictions on $v = (v_1, v_2)$ follows by a standard cascade argument as the x_2-subsystem is ISS without restriction with respect to v_2. Finally, we address the case of a completely controllable system. By a linear change of coordinates the system can be brought in the canonical form

$$\begin{aligned} \dot{x}_1 &= x_2 + v_1 \\ \dot{x}_2 &= -\alpha_1 x_1 - \alpha_2 x_2 - \sigma(F^T x + w) + v_2 \end{aligned} \tag{9}$$

with $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$. For convenience, we consider separately all the possible cases according to α_i, $i = 1,2$, positive or zero.

Case 2-1: $\alpha_1 > 0$ and $\alpha_2 > 0$: In this case the Ultimate Boundedness without restrictions easily follows since A is Hurwitz and $\sigma(\cdot)$ is uniformly bounded.

Case 2-2: $\alpha_1 = 0$ and $\alpha_2 > 0$: Let $z = \alpha_2 x_1 + x_2$ and rewrite the system as

$$\begin{aligned} \dot{z} &= -\sigma(\bar{f}_1 z + \bar{f}_2 x_2 + w) + v_1 \\ \dot{x}_2 &= -\alpha_2 x_2 - \sigma(-\bar{f}_1 z + \bar{f}_2 x_2 + w) + v_2 \end{aligned} \tag{10}$$

where $\bar{f}_1 := f_1/\alpha_2$ and $\bar{f}_2 := f_2 - f_1/\alpha_2$. Since $\alpha_2 > 0$ and $\sigma(\cdot) < 1$ we have that

$$\|x_2\|_a \leq \frac{2}{|\alpha_2| \max\{1, \|v_2\|_\infty\}}. \tag{11}$$

Moreover, as $\alpha_1 = 0$, condition (4) yields $f_1 > 0$. This and the fact that $\alpha_2 > 0$ imply ISS of the z-subsystem without restrictions with respect to the inputs x_2 and w and with nonzero restriction with respect to v_1. Hence, UBND of (10) without restriction on w and nonzero restriction on $v = (v_1, v_2)$ follows by standard cascade arguments.

Case 2-3: $\alpha_1 > 0$ and $\alpha_2 = 0$: By a linear change of coordinates and up to a linear time rescaling, the system can always be brought in the following form:

$$\begin{aligned} \dot{x} &= -y - \gamma_1 \sigma(y + w) + v_1 \\ \dot{y} &= x - \gamma_2 \sigma(y + w) + v_2 \end{aligned} \tag{12}$$

with $\gamma_1 = f_1/\alpha_1$ and $\gamma_2 = f_2$. In particular, condition (4) yields $1 + \gamma_1 > 0$ and $\gamma_2 > 0$. To show that system (12) enjoys the UBND property without restriction on w and non zero restriction on $v = (v_1, v_2)$ requires a bit of computations which for clarity are reported in Appendix B.

Case 2-4: $\alpha_1 = 0$, $\alpha_2 = 0$: This is the case of the double integrator for which conditions (4) is equivalent to $f_1 > 0$ and $f_2 > 0$. In such a case the transversality condition is never satisfied. However, ISS with respect to w without restriction for the double-integrator with saturated linear feedback was proved in [3] (see Remark 4), where it has been also shown that zero tolerance is accepted for external signals $v = (v_1, v_2)$ (i.e., the restriction $\Delta_v = 0$). Specifically in [3] it has been shown how to design an $\mathcal{L}_\infty$ signal v, with arbitrarily small amplitude, capable of destabilizing the system.$\triangle$

3) Case $n=3$: Without loss of generality we assume that the triplet (A,G,F) is given in the canonical controllable form

$$A=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\alpha_1 & -\alpha_2 & -\alpha_3\end{pmatrix}$$
$$G=\begin{pmatrix}0\\0\\1\end{pmatrix}\quad F=\begin{pmatrix}f_1\\f_2\\f_3\end{pmatrix}. \tag{13}$$

We first identify all the relevant cases we have to deal with. To this regard let $m(0)$ be the multiplicity of a possible eigenvalue of A at the origin. If $m(0)=3$, then A is zero or similar to either J or J_2 with

$$J=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\quad J_2=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}.$$

If $A=0$ or similar to J, then for every $\alpha\in(0,1]$ the rank of $A-\alpha GF^T$ is less or equal than two, then (4) cannot be satisfied. Similarly if A is similar to J_2, a simple application of the Routh–Hurwitz test on $A-\alpha GF^T$, yields (4) is equivalent to

$$f_3>0\quad \alpha f_2f_3-f_1>0\quad f_1>0\qquad \forall\alpha\in(0,1]. \tag{14}$$

From this, it is immediately seen that if $\alpha\to 0$ the second inequality of (14) yields $f_1\le 0$ which contradicts $f_1>0$. Thus, also in this case the condition (4) cannot be satisfied.

If $m(0)=2$, then $\dim\{\mathrm{Ker}(A)\}$ can be either 1 or 2. In case $\dim\{\mathrm{Ker}(A)\}=2$ then $\mathrm{rank}\{A\}=1$ and for every $\alpha\in(0,1]$ we have $\mathrm{rank}\{A-\alpha GF^T\}\le 2$. Therefore, (4) cannot be satisfied.

By the previous considerations we are left to consider just the case in which $m(0)=2$ and $\dim\{\mathrm{Ker}(A)\}=1$, $m(0)=1$ and finally the case in which A is invertible (i.e., $m(0)=0$). For convenience, all these cases are enumerated and discussed in the following.

Case 3-1: $m(0)=2$ and $\dim\{\mathrm{Ker}(A)\}=1$: This case corresponds to the presence of double poles at the origin and a negative real pole, viz. $\alpha_1=\alpha_2=0$ and $\alpha_3>0$. In this case the transversality condition is never satisfied, however, as in the case of dimension 2, it is possible to ensure ISS with respect to disturbances entering within the saturation. Up to linear changes of variables, a time rescaling and neglecting trivial cases, it is possible to assume that the system is described by (pick to this end the change of variables: $x_1=(f_1/\alpha_3)(\alpha_3z_1+z_2)$, $x_2=((\alpha_3f_2-f_1)/\alpha_3^2)(\alpha_3z_2+z_3)$, $x_3=((f_3\alpha_3^2+f_1-f_2\alpha_3)/\alpha_3^2)z_3$, where z are meant here as the original coordinates).

$$\begin{aligned}\dot x_1&=x_2+v_1\\ \dot x_2&=-b\sigma(x_1+x_2+x_3+w)+v_2\\ \dot x_3&=-\alpha_3x_3-c\sigma(x_1+x_2+x_3+w)+v_3\end{aligned} \tag{15}$$

where $b=((f_1-f_2\alpha_3)/\alpha_3^2)>0$ by condition (4) and $c=((-f_1-f_3\alpha_3^2+f_2\alpha_3)/\alpha_3^2)$.

Now note that as $\alpha_3>0$ and as $\sigma(\cdot)$ is uniformly bounded the $|x_3|$ variable is ultimately bounded by $|c|+\|v_3\|_\infty/\alpha_3$. Moreover the first two equations of (15) reduces to Case 2-4 above addressed. Specifically, the (x_1,x_2) subsystem turns out to be ISS without restrictions on the inputs x_3, w and zero restrictions on (v_1,v_2). From this the UBND property of (15) without restriction on (w,v_3) and zero restriction on (v_1,v_2) easily follows.

Case 3-2: $m(0)=1$, $\alpha_3=\alpha_1=0$, $\alpha_2>0$: The first scenario characterized by $m(0)=1$ is for A with three poles on the imaginary axis (one in 0 and 2 complex conjugates). Modulo change of coordinates and after time rescaling system (1)–(13) can be rewritten as

$$\begin{aligned}\dot x_1&=-a\sigma(x_1+x_3+w)+v_1\\ \dot x_2&=-x_3-b\sigma(x_1+x_3+w)+v_2\\ \dot x_3&=x_2-c\sigma(x_1+x_3+w)+v_3\end{aligned} \tag{16}$$

with $a=1/\alpha_2f_1$, $b=f_2/\alpha_2$ and $c=(f_3-f_1/\alpha_2)/\sqrt{\alpha_2}$. This system can be shown to enjoy the UBND property without restriction on w and non zero restriction on $v=(v_1,v_2,v_3)$ *provided* that $c>0$. To this regard it is worth noting that condition (4) yields $c\ge 0$. Hence, this case motivates the transversality assumption which indeed rules out the possibility of having $c=0$. For convenience all the mathematical steps needed to prove the result are deferred to Appendix B.

Case 3-3: $m(0)=1$, $\alpha_1=0$, α_2, $\alpha_3>0$: The second scenario characterized by $m(0)=1$ is for A having one pole at the origin and two poles with negative real part (namely $\alpha_1=0$, $\alpha_2,\alpha_3>0$ in (13)). To deal with this case, define $z_1:=x_1+x_2/\alpha_2+\alpha_3x_2/\alpha_2$ so that in the new coordinates the system reads

$$\begin{aligned}\dot z_1&=-\frac{1}{\alpha_2}\sigma(f_1z_1+\bar f_2x_2+\bar f_2x_3+w)+v_1\\ \dot x_2&=x_3+v_2\\ \dot x_3&=-\alpha_2x_2-\alpha_3x_3+\\ &\quad-\sigma(f_1z_1+\bar f_2x_2+\bar f_2x_3+w)+v_3\end{aligned}$$

where $\bar f_2=f_2-f_1\alpha_3/\alpha_2$ and $\bar f_3=f_3-f_1/\alpha_2$. Since (4) implies $f_1>0$, it turns out that the z_1-subsystem is ISS without restrictions on the inputs (x_2,x_3,w) and nonzero restrictions on v_1. From this, it is easy to conclude that the whole system enjoys the UBND properties without restriction on w and non zero restrictions on $v=(v_1,v_2,v_3)$ since the (x_2,x_3) subsystem behaves as an asymptotically stable linear system driven by the uniformly bounded function $\sigma(\cdot)$ and by the $\mathcal{L}_\infty$ signal v_3.

Case 3-4: $m(0)=0$, $\alpha_1=\alpha_2\alpha_3$: The first scenario in which A is invertible is the one in which there are two poles on the imaginary axis and one negative real pole (which corresponds to consider $\alpha_1=\alpha_2\alpha_3$, $\alpha_i>0$, $i=1,2,3$, in (13)). A linear change of coordinates bringing the matrix A to its real Jordan form changes system (1) to

$$\begin{aligned}\dot x_1&=-\lambda x_1-\sigma(a_0x_1+x_3+w)+v_1\\ \dot x_2&=-kx_3-b_0\sigma(a_0x_1+x_3+w)+v_2\\ \dot x_3&=kx_2-c_0\sigma(a_0x_1+x_3+w)+v_3\end{aligned} \tag{17}$$

with $\lambda,k>0$. Equating the expressions giving the characteristic polynomial of $A-\alpha GF^T$ with the two sets of coordinates leads to the relation $c_0=(\alpha_2f_3+\alpha_3f_2-f_1)/(\alpha_2+\alpha_3^2)$. Without loss of generality, a_0 can be taken nonzero. Then, after a new linear change of coordinates and a time rescaling, we may assume that system (1) is described by

$$\begin{aligned}\dot{x}_1 &= -\lambda x_1 - a\sigma(x_1 + x_3 + w) + v_1\\ \dot{x}_2 &= -x_3 - b\sigma(x_1 + x_3 + w) + v_2\\ \dot{x}_3 &= x_2 - c\sigma(x_1 + x_3 + w) + v_3\end{aligned} \tag{18}$$

with $c = c_0/k$. Moreover, condition (4) implies that $c \geq 0$ while the transversality condition rules out the case $c = 0$. Hence, we can study system (18) with $c > 0$ for which the UBND property clearly holds. Indeed, as $\lambda > 0$ and σ is uniformly bounded, we have

$$\|x_1\|_a \leq \frac{|a| + \|v_1\|_\infty}{\lambda}. \tag{19}$$

and hence UBND of (18) without restriction on w and non zero restriction on $v = (v_1, v_2, v_3)$ follows by dealing with the (x_2, x_3)-subsystem as in the case 2-4.

Case 3-5: $m(0) = 0$, A Hurwitz: This case is trivial as the UBND property without restriction on w and v follows by the fact the saturation function is uniformly bounded and A is Hurwitz.$\triangle$

All the previous analysis has shown that indeed conditions (4) and transversality of the root-locus imply ISS without restrictions with respect to w and v. Moreover, under assumption (4), ISS with nonzero restriction Δ_w on w can still be achieved whenever the matrix A has two poles at the origin. To complete the proof of the result we have to show that the gain is indeed linear. This is a consequence of the following general lemma.

Lemma 2.4: Consider a $\mathcal{C}^1$-system of the form $\dot{x} = f(x, u, \mu)$, in which μ are uncertain constant parameters ranging in a compact set $\mathcal{P}$, and assume that $\forall \mu \in \mathcal{P}$ this system is ISS with respect to the input u and locally exponentially stable for $u \equiv 0$. Then the system in question is ISS with uniform *linear* asymptotic gains and restrictions on u. □

Proof: By local exponential stability, there exist $r > 0$ and $\Delta_u > 0$ small enough so that, for all initial conditions in the ball $B_r(0)$ and for all measurable $u(\cdot)$ with $\|u\|_\infty \leq \Delta_u$, we have

$$|x(t)| \leq Me^{-\lambda t}|x_0| + K\|u\|_\infty \tag{20}$$

where M, λ, K are positive reals. Equation (20) establishes the conclusion of the lemma for initial conditions x_0 with $|x_0| \leq r$. In particular note that standard linear arguments can be used to show that K is a continuous function of the parameter μ. This and the fact that μ ranges in a compact set yield that K can be upper bounded by a function independent of μ, namely that the asymptotic gain is uniform with respect to μ.

We now consider the case $|x_0| > r$. Let γ be the ISS gain of the system. We let

$$\tilde{\Delta}_u := \min\left\{\Delta_u, \gamma^{-1}\left(\frac{r}{4(1+M)}\right)\right\}.$$

As for all u, it holds

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(\|u\|_\infty)$$

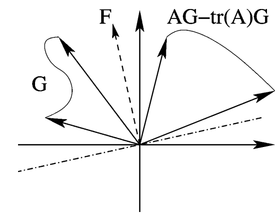

Fig. 1. Graphical design for two-dimensional systems.

there exists a nondecreasing function of $T(\cdot)$ so that $x(t) \leq r$ for all $t \geq T(|x_0|)$ and all u with $\|u\|_\infty \leq \tilde{\Delta}_u$. Therefore, for any u with $\|u\|_\infty \leq \tilde{\Delta}_u$ we have for $t \geq T(|x_0|)$

$$\begin{aligned}|x(t)| &\leq Me^{-\lambda(t-T(|x_0|)}\,|x(T(|x_0|))| + K\|u\|_\infty\\ &\leq Me^{-\lambda(t-T(|x_0|))}|x_0| + K\|u\|_\infty.\end{aligned} \tag{21}$$

Thus, the lemma follows by combining (20) and (21). ■

C. Algebraic Criterion for Robust Design

In the previous subsection, we have reformulated the problem of rendering robustly ISS system (1) as that of robustly stabilize the linear approximation. Specifically, theorem 1 states that robust ISS is automatically achieved if the Hurwitz condition (4) and the transversality condition on the root locus of $A - \alpha GF^T$ are satisfied. The advantage of stating the problem in these terms is that it is possible to develop synthesis tools for the design of the state feedback F which satisfy the requirements of Theorem 1. This is indeed the goal of this part which presents some constructive criterion for checking the condition of Theorem 1 in the two- and three-dimensional cases.

1) Two-Dimensional Systems: In this case it is simple to check, by means of root locus arguments, that the transversality condition is always fulfilled unlike for the double integrator which always violates it regardless the choice of the feedback F. Hence, the problem reduces to investigate algebraic conditions for having (4) fulfilled, namely to design a simple linear feedback, $u = F^T x$ with $F \in \mathbb{R}^2$ so that the matrix $A(\mu) - \alpha G(\mu)F^T$ is Hurwitz for all $\alpha \in (0, 1]$. The next proposition presents algebraic conditions for such a design.

Proposition 2.5: Assume that the couple $(A(\mu), G(\mu))$ be ANCBI for each $\mu \in \mathcal{P}$. Then a robust saturated linear controller for the planar system exists provided that there exists $F \in \mathbb{R}^2$ so that for any $\mu \in \mathcal{P}$

$$\begin{aligned}F^T G(\mu) &> 0\\ F^T\left(A(\mu)G(\mu) - \mathrm{tr}\,(A(\mu))\,G(\mu)\right) &> 0.\end{aligned} \tag{22}$$

Moreover, in case G is independent of μ, $A(\mu) : \mathcal{P} \to \mathbb{R}^{2\times 2}$ a continuous function and $\mathcal{P}$ an arcwise connected compact set, robust controllability of the couple $(A(\mu), G)$ is equivalent to condition (22). □

Remark 2.6: It is interesting to note that, from a graphical point of view, condition (22) is equivalent to ask that the cones spanned by the vectors $G(\mu)$ and $A(\mu)G(\mu) - \mathrm{tr}(A(\mu))G(\mu)$, as μ ranges in $\mathcal{P}$, all lie in the same half plane (see Fig. 1).

Remark 2.7: Note that the second part of the statement stresses the fact that in case the input vector G does not depend on uncertain parameters, then the existence of a robust stabilizer is implied by controllability of the pair $(A(\mu), G)$ for all $\mu \in \mathcal{P}$.

In order to prove the proposition we use the intermediate result stated here.

Proposition 2.8: The following facts are equivalent for two-dimensional systems:

1) the matrix $A - \alpha G F^T$ is Hurwitz for all $\alpha \in (0, 1]$;
2) the couple (A, G) is ANCBI and $A - GF^T$ is Hurwitz.

□

Proof: We simultaneously show the two directions of the implication $1 \Leftrightarrow 2$. Consider the characteristic polynomial $\chi(s)$ of the matrix $A + \alpha G F^T$ which is given by

$$\chi(s) = s^2 - \left[\mathrm{tr}(A) - \alpha F^T G\right] s + \det(A) + \alpha \left[\det(A - GF^T) - \det(A)\right]. \quad (23)$$

From this, the Hurwitz condition in 1 is equivalent to

$$\mathrm{tr}(A) - \alpha F^T G < 0 \quad (24)$$

$$\det(A) + \alpha \left[\det(A - GF^T) - \det(A)\right] > 0 \quad (25)$$

for all $\alpha \in (0, 1]$. By linearity of the previous expressions with respect to α, these conditions are equivalent to

$$\mathrm{tr}(A) \leq 0 \quad \det(A) \geq 0 \quad (26)$$

which is easily obtained as $\alpha \to 0$ and

$$\mathrm{tr}(A - GF^T) < 0 \quad \det(A - GF^T) > 0 \quad (27)$$

obtained evaluating the above conditions for $\alpha = 1$. Finally, note that the two conditions in (26) amount to asking all eigenvalues of A in the left closed half-plane while (27) are equivalent to ask for $A - GF^T$ Hurwitz. Then, conditions (26), (27) are equivalent to (A, G) being ANCBI and $A - GF^T$ Hurwitz. ■

This result allows to forget the parameter α and to focus just on the robust stabilizability of the ANCBI pair $(A(\mu), G(\mu))$. Hence, in the following part we prove that the state feedback F satisfying (22) robustly stabilize the pair $(A(\mu), G(\mu))$. This indeed implies that system $\dot{x} = A(\mu)x - G(\mu)\sigma(F^T x + w) + v$ is uniformly ISS with respect to w and v with suitable restriction.

Proof (Proposition 2.5): The system is ANCBI for all $\mu \in A(\mu)$, therefore $\mathrm{tr}(A(\mu)) \leq 0$ and $\det(A(\mu)) \geq 0$. Therefore, by virtue of (22)

$$\begin{aligned}\mathrm{tr}\left(A(\mu) - G(\mu)F^T\right) &= \mathrm{tr}\left(A(\mu)\right) - F^T G(\mu) \\ &< 0 \\ \det\left(A(\mu) - G(\mu)F^T\right) &= \det\left(A(\mu)\right) \\ &\quad + F^T\left[A(\mu)G(\mu) - \mathrm{tr}\left(A(\mu)\right)G(\mu)\right] \\ &> 0 \end{aligned} \quad (28)$$

which proves that $A(\mu) - G(\mu)F^T$ is Hurwitz. Let now assume G independent of μ and full rank of $[A(\mu)G, G]$ for any $\mu \in \mathcal{P}$. Then, for any $\mu \in \mathcal{P}$ the vectors G and $A(\mu)G$ are linearly independent; consequently, the same holds true for G and $A(\mu)G - \mathrm{tr}(A(\mu))G$. We claim that, this is equivalent to (22)). As $\mathcal{P}$ is arcwise connected and compact, the same holds true for the image $\mathrm{Im}_{\mu \in \mathcal{P}}[A(\mu)G - \mathrm{tr}(A(\mu))G] := \mathcal{I}$. In particular then, the cone generated by $\mathcal{I}$, $\mathcal{C} := \bigcup_{\lambda \geq 0} \lambda \mathcal{I}$, is also arcwise connected and spanning a compact circular sector. Therefore, linear independence of any vector of $\mathcal{C}$ with respect to the constant vector G is equivalent to $\mathcal{C}$ and G being strictly contained in some half plane. ■

2) Three-Dimensional Systems: In this case the transversality condition can be affected by the design of the state feedback F and hence must be explicitly considered. To this end, define the invariant $\gamma(A)$ as follows:

$$\gamma(A) = a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} - a_{13}a_{31} + a_{22}a_{33} - a_{23}a_{32}$$

where a_{ij} denotes the (i, j) entry of A. Moreover, let the coefficients p_1, p_2 and p_3 be defined as

$$\begin{aligned} p_1 &:= (F^T G)^2 F^T \left(\mathrm{tr}(A)I - A\right) G \\ p_2 &:= -F^T G \gamma(A) - \mathrm{tr}(A)(F^T G)F^T \left(\mathrm{tr}(A)I - A\right) G \\ &\quad + \det(A) - \det(A - GF^T) \\ p_3 &:= \mathrm{tr}(A)\gamma(A) - \det(A). \end{aligned} \quad (29)$$

Then, the Hurwitz condition (4) and the transversality condition can be translated into algebraic conditions as stated in the next proposition (where for convenience we dropped the dependence on the uncertain parameter μ).

Proposition 2.9: The matrix $A - \alpha G F^T$ is Hurwitz for all $\alpha \in (0, 1]$ if and only if

a) $\mathrm{tr}(A) \leq 0$, $\det(A) \leq 0$, $p_3 \leq 0$;
b) $\mathrm{tr}(A - GF^T) < 0$, $\det(A - GF^T) < 0$;
c) $p_1 + p_2 + p_3 < 0$;

and, alternatively

d1) $p_1 \geq 0$;
d2) $p_1 < 0$ and $p_3 < (p_2^2/4p_1)$;
d3) $p_1 < 0$ and $|p_2| < 2|p_1|$.

Moreover, the transversality condition of the root locus of $A - \alpha G F^T$ with respect to the imaginary axis is satisfied for $\alpha = 0$ if and only if

$$p_3 \leq 0 \quad \text{and} \quad (p_3 = 0 \;\Rightarrow\; p_2 < 0). \quad (30)$$

Proof: Note that the characteristic polynomial of $A - \alpha G F^T$ can be written as

$$\begin{aligned}\chi(s) = s^3 &- \left[\mathrm{r}(A) - \alpha F^T G\right] s^2 \\ &+ \left[\gamma(A) - \alpha (F^T G) F^T \left[\mathrm{tr}(A)\mathrm{I} - A\right] G\right] s + \\ &- \det(A) + \alpha\left[-\det(A - GF^T) + \det(A)\right] \end{aligned} \quad (31)$$

from which a simple application of the Routh criterion yields that $A - \alpha G F^T$ is Hurwitz if and only if for all $\alpha \in (0, 1]$

$$\begin{aligned} \mathrm{tr}(A) - \alpha F^T G &< 0 \\ \det(A) + \alpha\left[\det(A - GF^T) - \det(A)\right] &< 0 \end{aligned} \quad (32)$$

and

$$\left[\mathrm{tr}(A) - \alpha F^T G\right]\left[\gamma(A) - \alpha(F^T G)F^T\left[\mathrm{tr}(A)\mathrm{I} - A\right] G\right] + - \det(A) + \alpha\left[-\det(A - GF^T) + \det(A)\right] < 0. \quad (33)$$

Since the inequalities in (32) are linear in α they can equivalently be stated as the first two conditions in item **a)** and the ones in item **b)**. As far as inequality (33) is concerned note that it is quadratic in α and can be rewritten, rearranging the terms, as

$$P(\alpha) := p_1\alpha^2 + p_2\alpha + p_3 < 0 \quad \forall\alpha \in (0,1] \tag{34}$$

where the coefficients p_i have been defined in (29). In view of this, the last condition in item **a)** and the condition in item **b)** express the fulfillment of the inequality respectively for $\alpha = 0$ (non strictly) and for $\alpha = 1$ (strictly). These conditions are indeed sufficient for having (34) fulfilled for any $\alpha \in (0,1]$ if the condition in item **d1)** is satisfied. On the other hand in case $p_1 < 0$ the items **d2)** or **d3)** yields the extra conditions for having (34) fulfilled. Finally, the fact that (30) implies the transversality condition follows by simple linear arguments. ■

Remark 2.10: The extra condition (27) due to transversality required to the state feedback F can be interpreted as the requirement that $P(0) = p_3$ and $P'(0) = p_2$ should not simultaneously be equal to zero.

Moreover, note that, from its definition, the polynomial $P(\alpha)$ does not depend on the choice of coordinates in which system (1) is expressed. □

Remark 2.11: Note that while the conditions expressed by **b)**, **c)**, and **d)** in the previous proposition are affected by choice of the state feedback F, those in item **b** only depend on the controlled system. In particular, it is easy to check that they are automatically fulfilled if the pair (A, G) is ANCBI. □

We conclude this section providing a result which aims to translate the transversality condition (30) in the frequency domain. This indeed is a further tool which can be used in order to check if the root locus of $A - \alpha GF^T$ is transversal to the imaginary axis for $\alpha = 0$ or not.

Proposition 2.12: Let A be neutrally stable with simple roots on the imaginary axis. Then the root locus of $A - \alpha GF^T$ is transversal to the imaginary axis for $\alpha = 0$ if and only if $\chi_A(j\omega^\star) = 0$ implies

$$\arg\left(\chi_{A-GF^T}(j\omega^\star)\right) - \arg\left(\chi'_A(j\omega^\star)\right) \neq \frac{2\nu+1}{2}\pi \tag{35}$$

with $\nu \in \mathbb{Z}$.

Proof: Let $\sigma : [0,\epsilon] \to \mathbb{C}$ be a continuous root function of $\chi_{A-\lambda GF^T}$, i.e., for every $\lambda \in [0,\epsilon]$, $\sigma(\lambda)$ is a root of $\chi_{A-\lambda GF^T}$. Moreover, assume that $\sigma(0) = j\omega^\star$. Then σ can be made as a function of class $\mathcal{C}^1$. The sufficient part of the proposition reduces to show that if (35) holds then

$$Re\frac{\partial\sigma(\lambda)}{\partial\lambda}\bigg|_{\lambda=0} \neq 0.$$

To this end note that, since GF^T is rank 1,

$$\chi_{A-\lambda GF^T}(s) = \det(A - \lambda GF^T - sI) = P_0(s) + \lambda P_1(s)$$

with

$$P_0(s) = \chi_A(s) \quad P_1(s) = \chi_{A-GF^T}(s) - \chi_A(s).$$

Since $j\omega^\star$ is a simple root, then $\chi'_A(j\omega^\star) \neq 0$ and we may therefore assume that σ is of class C^1. Deriving the relation $\chi_{A-\lambda GF^T}(\sigma(\lambda)) = 0$ leads to

$$\begin{aligned}0 = & \frac{\partial\chi_{A-\lambda GF^T}(\sigma(\lambda))}{\partial\lambda}\bigg|_{\lambda=0}\\ =& P'_0(\sigma(0))\frac{\partial\sigma(\lambda)}{\partial\lambda}\bigg|_{\lambda=0} + P_1(\sigma(0))\\ &- \lambda P'_1(\sigma(0))\frac{\partial\sigma(\lambda)}{\partial\lambda}\bigg|_{\lambda=0}\\ =& \chi'_A(\sigma(0))\frac{\partial\sigma(\lambda)}{\partial\lambda}\bigg|_{\lambda=0} + \chi_{A-GF^T}(\sigma(0))\end{aligned}$$

we get

$$\frac{\partial\sigma(\lambda)}{\partial\lambda}\bigg|_{\lambda=0} = -\frac{\chi_{A-GF^T}(j\omega^\star)}{\chi'_A(j\omega^\star)}.$$

Now note that

$$\frac{\partial Re(\sigma(\lambda))}{\partial\lambda}\bigg|_{\lambda=0} = Re\frac{\partial\sigma(\lambda)}{\partial\lambda}\bigg|_{\lambda=0}$$

from which it is clear that the claim is true if

$$Re\frac{\chi_{A-GF^T}(j\omega^\star)}{\chi'_A(j\omega^\star)} \neq 0$$

which is equivalent to (35). ■

III. Robust Stabilization of Nonlinear Feedforward Systems

A. A Stabilization Procedure for Nonlinear Feedforward Systems

This part is devoted to briefly present the design procedure proposed in [15] to globally asymptotically stabilize by state feedback the class of *feedforward* nonlinear systems described by

$$\begin{aligned}\dot{x}_1 &= A_1(\mu)x_1 + f_1(x_2,\ldots,x_n,u,\mu)\\ \dot{x}_2 &= A_2(\mu)x_2 + f_2(x_3,\ldots,x_n,u,\mu)\\ &\vdots\\ \dot{x}_{n-1} &= A_{n-1}(\mu)x_{n-1} + f_{n-1}(x_n,u,\mu)\\ \dot{x}_n &= A_n(\mu)x_n + f_n(u,\mu)\end{aligned} \tag{36}$$

where $x_i \in \mathbb{R}^{n_i}$, μ is an uncertain parameter taking values in a known compact set $\mathcal{P}$, $A_i(p)$ are critically stable matrices and $f_i(\cdot)$ are differentiable nonlinear functions vanishing at the origin for all $\mu \in \mathcal{P}$. Our final goal is to show how, in the framework proposed in [15], the result regarding robust saturated feedback of linear systems illustrated in the previous section can be successfully employed for achieving robust global stabilization of system (36). To this end, we first briefly review the procedure in [15] which shows how the problem reduces to a saturated linear stabilization problem.

Consider the nonlinear system

$$\begin{aligned}\dot{x}_{p-1} &= A_{p-1}(\mu)x_p + f_{p-1}(x_p,\mu,u)\\ \dot{x}_p &= f_p(x_p,\mu,u)\end{aligned} \tag{37}$$

for which we assume the following:

A1) $A_{p-1}(\mu)$ is a critically stable matrix for all $\mu \in \mathcal{P}$;

A2) the linear approximation of the system, denoted by the pair (the symbol $\star$ denotes a generic term of compatible dimension without any additional structure)

$$A = \begin{pmatrix} A_{p-1}(\mu) & \star(\mu) \\ 0 & A_p(\mu) \end{pmatrix} \quad G = \begin{pmatrix} G_{p-1}(\mu) \\ G_p(\mu) \end{pmatrix} \tag{38}$$

is ANCBI for all $\mu \in \mathcal{P}$;

A3) the x_p dynamics is *uniformly ISS* with respect to u with linear gain γ_p, restriction r_p on u, and it is 0-LES (i.e., A_p Hurwitz).

As far as the global stabilization of (37) is concerned it has been shown in [15] that it is equivalent to the design of a "good saturated linear controller" for the pair (38), namely to a saturated linear controller for the linear approximation able to achieve ISS with suitable restrictions with respect to inputs entering inside and outside the saturation. In the following proposition, which is an easy extension of Theorem 4 in [15], we show that indeed this feature is preserved when the uncertain parameter μ enters into the picture. Specifically, global robust stabilization of (37) is equivalent to the design of a "robust good saturated linear controller" for the linear approximation.

Proposition 3.1: Let $u := -\sigma(F^T x + w) + v$ and $x := (x_p^T, x_{p-1}^T)^T$ be a "robust good saturated linear controller" for the observable critically stable pair $(A(\mu), G(\mu))$, namely let

$$\dot{x} = A(\mu)x - G(\mu)\sigma(F^T x + w) + v$$

be uniformly ISS with respect to (w, v) with linear gain γ_l, and nonzero restrictions (Δ_w, Δ_v) for all $\mu \in \mathcal{P}$. Then, there exists $\lambda^\star > 0$ such that for all positive $\lambda \leq \lambda^\star$ the closed loop system given by (37) with the state feedback controller

$$u = -\lambda\sigma\left(\frac{F^T x + w}{\lambda}\right)$$

is uniformly ISS with respect w with linear gain and nonzero restrictions. □

Proof: The proof treads upon that of theorem 4 in [15]. The only difference is to keep track of the uncertain parameter μ. Specifically, the idea in [15] is to look at system (37) as an higher dimensional system given by the interconnection of the system

$$\begin{aligned} \dot{x} &= A(\mu)x - G(\mu)\lambda\sigma\left(\frac{F^T x + w}{\lambda}\right) + y_2 \\ y_1 &= \lambda\sigma\left(\frac{F^T x + w}{\lambda}\right) \end{aligned}$$

with inputs w and y_2 and output y_1, with the system

$$\begin{aligned} \dot{x}_p &= f_p(x_p, y_1, \mu) \\ y_2 &= f_h(x_p, y_1, \mu) \end{aligned}$$

with input y_1 and output y_2, where the function $f_h(\cdot,\cdot,\cdot)$ collects the higher order terms of the vector field in (37), and to show that all the small gain conditions reported in [15. Th. 1] are indeed fulfilled by suitably decreasing λ. It is trivial to check that all these reasoning can be repeated *mutatis mutandis* in this context, since $f_h(x_p, y_1, \mu)$ is a continuous function of the arguments and μ ranges within a compact set. ■

Clearly, the result of the previous proposition can be iterated, to deal with the stabilization of an arbitrary nonlinear feedforward system of the form (36) via a bottom-up procedure, as the system with state (x_p, x_{p-1}), under the feedback defined in Section III-A, recovers the same ISS properties of the x_p subsystem with u replaced by v. The final control law turns out to be a nested saturated control law with the saturation level of each stage tuned according to the previous proposition. Moreover, note that at each step of the previous design procedure the dimension for the critically stable dynamics A_{p-1} in (38) is fixed (as it coincides with the A_i in (36)), while the Hurwitz matrix A_p has an increasing dimension.

By means of this analysis we conclude that the stabilization of nonlinear feedforward systems reduces to a problem of designing a "robust good saturated controller" for the ANCBI linear systems described by (38) with the dimension of the asymptotically stable part A_p which has an arbitrarily large dimension (dictated by the number n of subsystems in (36)) and the critically stable part A_{p-1} whose size may vary from step to step but never grows above 3. The problem of designing a robust saturated controller for such a pair will be addressed in the next subsection.

B. The Robust Good Saturated Linear Controller

Our goal is to employ the result stated in Sections II-A–II-C, namely the design procedure for rendering robustly ISS an uncertain ANCBI system of dimension $n \leq 3$, for the design of a robust good saturated linear controller for a controllable pair described by (38). To this end we shall focus on feedforward systems (36) with $n_i \leq 3$, $i = 1, \ldots, n$, where n [the number of subsystems in (36)] is an arbitrarily large number. In view of this the problem is to design a robust good saturated linear controller for a pair

$$A(\mu) = \begin{pmatrix} A_0(\mu) & A_j(\mu) \\ 0 & A_s(\mu) \end{pmatrix} \quad G(\mu) = \begin{pmatrix} G_1(\mu) \\ G_2(\mu) \end{pmatrix} \tag{39}$$

in which $A_0(\mu)$ is a critically stable matrix with dimension $n_0 \leq 3$ while $A_s(\mu)$ is an Hurwitz matrix with arbitrary dimension n_s. With this in mind we state the next proposition which represents the link between the results in Sections II-B and II-C and the solution of the problem. Specifically, it shows that the design of a robust good saturated linear controller for the pair (39) can be reduced to an analogous problem involving just the critically stable dynamics $A_0(\mu)$ (for which the design procedure illustrated in Sections II-B and II-C can be resumed). In the following we partition the state x as $x = (x_1, x_2)$, $x_1 \in \mathbb{R}^{n_0}$, $x_2 \in \mathbb{R}^{n_s}$ accordingly to (39).

Proposition 3.2: Assume that the spectrum of A_0 and A_s are disjoint and let $X(\mu)$ be the $(n_0 \times n_s)$-matrix solution of the following Sylvester equation:

$$A_0(\mu)X(\mu) - X(\mu)A_s(\mu) = -A_j(\mu) \tag{40}$$

and denote by $G_X(\mu)$ the vector

$$G_X(\mu) := G_1(\mu) - X(\mu)G_2(\mu).$$

Then

a) if the pair $(A(\mu), G(\mu))$ is ANCBI for all $\mu \in \mathcal{P}$ then also the pair $(A_0(\mu), G_X(\mu))$ is such;

b) if the system

$$\dot{x}_1 = A_0(\mu)x_1 - G_X(\mu)\sigma(F^T x_1 + w) + v$$

is uniformly ISS with respect to the inputs (w, v) with nonzero restrictions, then also the system

$$\dot{x} = A(\mu)x - G(\mu)\sigma\left(F^T x_1 - F^T X(\mu)x_2 + w\right) + v$$

is uniformly ISS with linear gains and nonzero restrictions;

c) let A_0 be neutrally stable. Then if

c1) the state feedback F is such that for all $\mu \in \mathcal{P}$ and for all $\epsilon \in (0, 1]$ the system

$$\dot{x}_1 = A_0(\mu)x_1 - G_X(\mu)\sigma(\epsilon F^T x_1 + w) + v \qquad (41)$$

is uniformly[2] ISS with linear gain $N(\epsilon)$ and non zero restrictions Δ_v, $\Delta_w(\epsilon)$;

c2) for all $\mu \in \mathcal{P}$ the root locus corresponding to $A_0(\mu) - \alpha G_X(\mu)F^T$ is transversal to the imaginary axis for $\alpha = 0$;

then there exists $\epsilon^\star > 0$ such that for all $\epsilon \leq \epsilon^\star$ and for all $\mu \in \mathcal{P}$ the closed-loop system

$$\dot{x} = A(\mu)x - G(\mu)\sigma(\epsilon F^T x_1 + w) + v$$

is uniformly ISS with linear gains and non zero restrictions. □

Before proving the result few remarks are in order.

Remark 3.3: The aim in the previous proposition [in particular, of the claims b) and c)] is to set conditions under which the problem of designing a robust good saturated controller for the pair (39) can be reduced to study a design problem of lower dimension involving just the critically stable dynamics. The advantage of ruling out from the design the Hurwitz matrix A_s is that, since we are limiting the dimension of A_0 to be $n_0 \leq 3$, all the constructive algebraic conditions of Section II-C can be used for the robust design regardless the number of subsystems considered in the feedforward structure (36). □

Remark 3.4: It is worth noting that claim b) in the previous proposition yields a constructive result for design in all the cases in which $X(\mu)$, solution of the Sylvester equation, does not depend on the uncertain parameter μ. This for instance happens in case the state matrix A is not affected by μ and just the input vector G is uncertain or in case the uncertainties enter linearly in the state matrix A. In all the other cases in which $X(\mu)$ depends on μ, claim c) must be used to design the stabilizing robust control law. □

Remark 3.5: Note that conditions c1) and c2) in claim **c)** are automatically satisfied if we adopt the result proposed in Theorem 1 for the design of the robust state feedback F for the pair $(A_0(\mu), G_X(\mu))$. As a matter of fact, while c2) corresponds to the transversality condition already present in Theorem 1, the fact that $A_0 - \alpha G_X F^T$ is Hurwitz for all $\alpha \in (0, 1]$ as required by theorem implies that also $A_0 - G_X \alpha\epsilon F^T$ is such for all $\alpha \in (0, 1]$ and $\epsilon \in (0, 1]$. Furthermore, if the root locus of $A_0 - \alpha G_X F^T$ is transversal to the imaginary axis for $\alpha = 0$ also that of $A_0 - \alpha\epsilon G_X F^T$ is such for all $\epsilon \in (0, 1]$. From this and Theorem 1 we conclude that c1) holds. □

[2]Uniform with respect to μ not with respect to ϵ.

Proof (Prop. 3.2): We show that there exists a change of coordinates of the form

$$T = \begin{pmatrix} I & T_{12}(\mu) \\ 0 & T_{22}(\mu) \end{pmatrix}$$

such that the system in the new coordinates is block diagonal. As a matter of fact let T_{22} be any nonsingular matrix and choose T_{12} as

$$T_{12}(\mu) = X(\mu)T_{22}(\mu)$$

with $X(\mu)$ solution of the Sylvester equation (40). The system in the new coordinates reads as (from now on we drop for convenience the parameter μ)

$$\begin{aligned} \bar{A} &= \begin{pmatrix} A_0 & A_0T_{12} + A_jT_{22} - T_{12}T_{22}^{-1}A_sT_{22} \\ 0 & T_{22}^{-1}A_sT_{22} \end{pmatrix} \\ &= \begin{pmatrix} A_0 & 0 \\ 0 & T_{22}^{-1}A_sT_{22} \end{pmatrix} \\ \bar{G} &= T^{-1}G \\ &= \begin{pmatrix} G_1 - T_{12}T_{22}^{-1}G_2 \\ T_{22}^{-1}G_2 \end{pmatrix} \\ &= \begin{pmatrix} G_X \\ T_{22}^{-1}G_2 \end{pmatrix}. \end{aligned}$$

From this, the fact that the pair (A_0, G_X) is ANCBI (and specifically that it is controllable) easily follows from controllability of the original pair.

Choosing now

$$u = \sigma(F^T x_1 - F^T X x_2 + w) + v = \sigma(F^T \tilde{x}_1 + w) + v$$

where

$$\tilde{x}_1 = x_1 - T_{12}T_{22}^{-1}x_2$$

is the vector with the first n_0 state components in the new coordinates, it turns out that the system is described by the series connection of the system

$$\dot{\tilde{x}}_1 = A_0\tilde{x}_1 - G_X\lambda\sigma(F^T\tilde{x}_1 + w) + v \qquad (42)$$

with the system

$$\dot{\tilde{x}}_2 = T_{22}^{-1}A_sT_{22}\tilde{x}_2 - T_{22}^{-1}G_2\sigma(F^T\tilde{x}_1 + w) + v$$

where $\tilde{x}_2 = T_{22}x_2$. Since A_s is Hurwitz, the claim *b)* easily follows from standard cascade arguments.

Claim *c)* is a bit more involved and can be proved as follows. Under the feedback $u = \sigma(\epsilon F^T x_1 + w) + v$ and after the change of coordinates

$$\tilde{x}_1 \quad \to \quad \tilde{z}_1 := \epsilon\tilde{x}_1$$

the overall system reads as the feedback interconnection of the system

$$\begin{aligned}\dot{\tilde{z}}_1 &= A_0\tilde{z}_1 - G_X\epsilon\sigma(F^T\tilde{z}_1 + y_2 + w) + \epsilon v\\ y_1 &= F^T\tilde{z}_1\end{aligned} \tag{43}$$

with the system

$$\begin{aligned}\dot{\tilde{x}}_2 &= T_{22}^{-1}A_sT_{22}\tilde{x}_2 + \\ &\quad - T_{22}^{-1}G_2\sigma(\epsilon F^T T_{12}\tilde{x}_2 + y_1 + w) + v\\ y_2 &= \epsilon F^T T_{12}\tilde{x}_2.\end{aligned} \tag{44}$$

The idea is to prove that for a sufficiently small ϵ this feedback interconnection satisfies a suitable "small gain" condition. To this end, we first need to investigate the ISS features of both the systems.

The subsystem (44), regarded as a nonlinear system with state $\tilde{x}_2$ and exogenous inputs y_1, w and v, can be easily shown, for sufficiently small ϵ, to be uniformly ISS with linear gain independent of ϵ and no restrictions on the inputs. To this regard, let $P > 0$ such that

$$T_{22}^{-1}A_sT_{22}P + PT_{22}^TA_s^TT_{22}^{-T} = -I$$

and consider the candidate ISS-Lyapunov function $V = \tilde{x}_2^TP\tilde{x}_2$. Notice that P is a continuous function of μ over $\mathcal{P}$. Taking derivatives and keeping in mind the definition of saturation function we get

$$\begin{aligned}\dot{V} &= -\tilde{x}_2^T\tilde{x}_2 + 2\tilde{x}_2T_{22}^{-1}B_2\sigma(\epsilon F^TT_{12}\tilde{x}_2 + y_1 + w) + 2\tilde{x}_2v\\ &\leq -|\tilde{x}_2|^2 + \epsilon q_2|\tilde{x}_2|^2 + q_1|\tilde{x}_2||y_1| + q_1|\tilde{x}_2||w| + 2|\tilde{x}_2||v|\end{aligned}$$

where q_1 and q_2 are positive numbers not dependent on ϵ. Taking $\epsilon_2^\star < 1/2q_2$ it turns out that, for all $\epsilon \leq \epsilon_2^\star$

$$\begin{aligned}\dot{V} &\leq -\frac{|\tilde{x}_2|^2}{2} + q_1|\tilde{x}_2|\left(|y_1| + |w| + \frac{2}{q_1}|v|\right)\\ &\leq -\frac{|\tilde{x}_2|^2}{2} + 3q_1|\tilde{x}_2|\max\left\{|y_1|, |w|, \frac{2}{q_1}|v|\right\}\end{aligned}$$

which represents a classical ISS dissipation inequality. Hence, by continuity of $P(\mu)$ and compactness of $\mathcal{P}$ there exists γ_2 (independent of μ and ϵ) such that

$$\|\tilde{x}_2\|_a \leq \gamma_2 \max\{\|y_1\|_a, \|w\|_a, \|v\|_a\}.$$

Moreover, note that, since A_s is Hurwitz and by definition of saturation function, there exists $\Gamma > 0$ so that for any for all $\mu \in \mathcal{P}$ the output corresponding to any solution satisfies

$$|y_2(t)| \leq \Gamma\epsilon \qquad \forall t \geq T$$

with T sufficiently large (possibly depending on the initial conditions but independent of ϵ and μ).

As far as the subsystem (43) is concerned we claim that under assumptions c1)–c2) there exist positive $\epsilon_1^\star$, γ_1 and Δ_w and a function $\Delta_v(\cdot) : (0, \epsilon_1^\star] \to \mathbb{R}$ such that for all $\epsilon \leq \epsilon_1^\star$ system (43) is uniformly ISS with respect to the inputs $(y_2 + w, v)$ with linear asymptotic gain γ_1 and restrictions $(\Delta_w, \Delta_v(\epsilon))$ on the inputs (*with* γ_1 and Δ_w *not dependent on* ϵ). Specifically

$$\begin{aligned}&\|y_2 + w\|_\infty < \Delta_w \quad \|v\|_\infty < \Delta_v(\epsilon) \Rightarrow\\ &\qquad \|\tilde{z}_1\|_a \leq \gamma_1 \max\{\|y_2 + w\|_a, \|v\|_a\}.\end{aligned}$$

In fact, let $N(\epsilon)$ and $(\Delta_w, \Delta_v(\epsilon))$ be as in assumption c1). By applying the change of coordinates $\tilde{z}_1 \to \zeta := \tilde{z}_1/\epsilon$ and using assumption c_1) it is easy to see that (43) is uniformly ISS with linear asymptotic gain $\epsilon N(\epsilon)$ and restrictions $(\Delta_w, \Delta_v(\epsilon))$ with respect to the inputs $(y_1 + w, v)$. In view of this, our claim follows if the asymptotic linear gain $N(\epsilon)$ of (41) can be bounded as $N(\epsilon) \leq \gamma_1/\epsilon$ for all $\epsilon \in (0, 1]$, where $\gamma_1 > 0$ is a sufficiently large real independent of ϵ. This problem amounts to study the linear approximation of system (41) given by

$$\dot{x}_1 = (A_0 - \epsilon G_XF^T)x_1 - G_Xw - v$$

and, in particular, the condition under which there exists a positive definite and symmetric matrix $P(\epsilon)$ solution of the Lyapunov equation

$$(A_0 - \epsilon G_XF^T)P(\epsilon) + P(\epsilon)(A_0 - \epsilon G_XF^T)^T \leq -\epsilon I \tag{45}$$

such that

$$\|P(\epsilon)\| \leq \bar{p} \quad \text{for all } \epsilon \in (0, 1]. \tag{46}$$

As a matter of fact in such a case, considering the quadratic candidate ISS Lyapunov function $V = x^TP(\epsilon)x$, it is possible to obtain

$$\begin{aligned}\dot{V} &= 2x^TP(\epsilon)(A_0x - \epsilon G_XF^Tx) - 2x^TP(\epsilon)(G_Xw + v)\\ &\leq -\epsilon|x|^2 + 2\bar{p}\|G_X\||x||w| + 2\bar{p}|x||v|\end{aligned}$$

which is a dissipation inequality yielding by some standard manipulations (i.e., completion of squares) the desired asymptotic gain γ_1/ϵ by choosing γ_1 sufficiently large. Hence, our problem reduces to study the existence of a $P(\epsilon)$ satisfying (45) and (46). The existence of such a matrix is a consequence of the "transversality condition" c_2) expressed by the Proposition 8 reported in Appendix C.

With all this in mind we can study the feedback interconnection of the systems (43), (44) by small gain arguments. To this end, note that taking ϵ so that

$$\epsilon < \min\left\{\frac{\Delta_w}{\Gamma}, \frac{1}{\gamma_1\gamma_2\|F^TT_{12}\|\|F\|}, \epsilon_1^\star, \epsilon_2^\star\right\}$$

it is easy to check that all the conditions of the small gain Theorem 1 in [15] are fulfilled for $T \geq 0$. This and the fact that the system can not have finite escape time in the interval $[0, T)$ (since it behaves as a linear system driven by bounded inputs) yield the result. ∎

IV. EXAMPLE

Consider the four-dimensional uncertain system

$$\begin{aligned}\dot{\mathbf{x}} &= A_0\mathbf{x} + Bx_4 + h(x_4)\\ \dot{x}_4 &= u\end{aligned} \tag{47}$$

where

$$\mathbf{x} := (x_1 \quad x_2 \quad x_3)^T$$

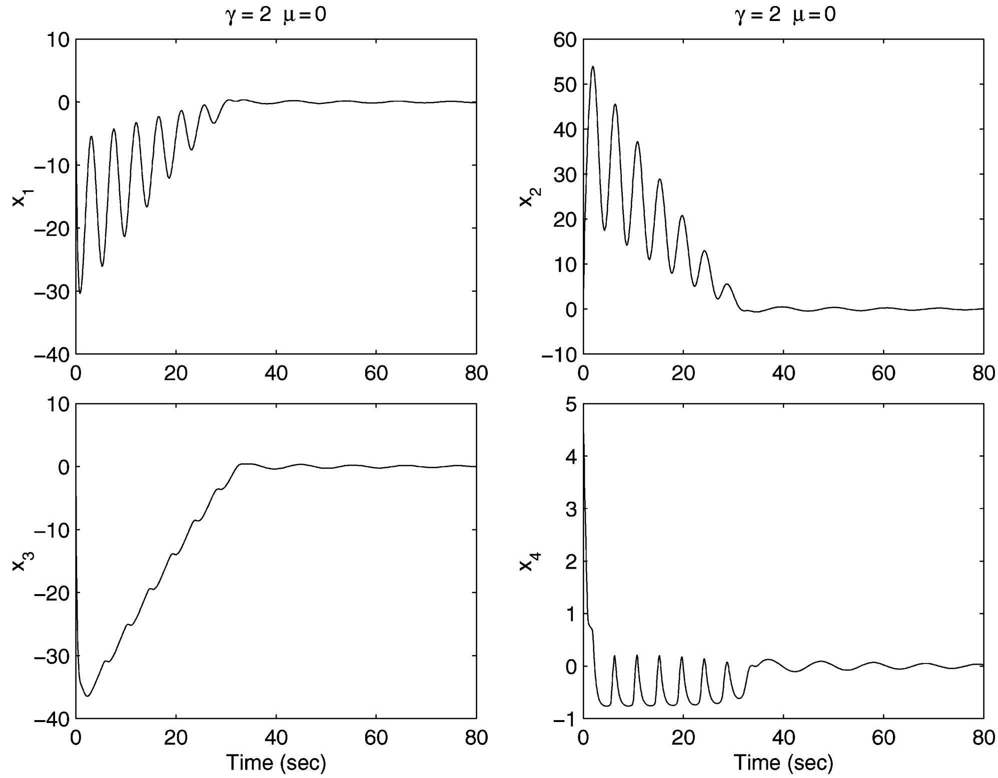

Fig. 2. Behavior of $\mathbf{x}(t)$, $x_4(t)$ in case $\gamma = 2$ and $\mu = 0$ (eigenvalues of A_0 in $\{0, j2, -j2\}$).

$h(x_4)$ collects higher order terms and

$$A_0 = \begin{pmatrix} 0 & 1 & 1 \\ -\gamma^2 & 0 & 1 \\ \mu & 0 & -\mu \end{pmatrix} \quad B = \begin{pmatrix} -1 \\ -1 \\ \mu - 2 \end{pmatrix}$$

with γ and μ uncertain parameters ranging within the compact sets $0 \leq \mu \leq 1$ and $2 \leq \gamma \leq 10$. A simple computation shows that the pair (A_0, B) is ANCBI for all possible values of γ and μ in the given compact sets. In particular, it turns out that

- for $\mu = 0$ the matrix A_0 presents three eigenvalues on the imaginary axis at $\lambda = 0$, $\lambda = \pm j\gamma$;
- for $\mu = 1$ the matrix A_0 presents one stable eigenvalues at $\lambda = -1$ and two eigenvalues on the imaginary axis at imaginary a $\lambda = \pm j\sqrt{\gamma^2 - 1}$;
- for $0 < \mu < 1$ the matrix A_0 is Hurwitz.

The design of a (saturated) state feedback able to globally asymptotically stabilize the previous system for all possible values of μ and γ in the given compact sets can be achieved following the steps presented in Section III-A.

We start from the bottom scalar subsystem with state x_4 for which the good saturated linear controller

$$u = -\lambda_1 \sigma \left(\frac{K x_4 + v}{\lambda_1} \right) \tag{48}$$

with K any positive number, trivially yields ISS with respect to the inputs v for any possible value of $\lambda_1 > 0$, without restrictions on the initial state and nonzero restrictions (proportional to λ_1) on the input.

Then, we study the overall (four dimensional) system with the preliminary feedback (48) using statement **c)** of Proposition 3.2 which claims that the design of a robust good saturated linear controller for the linear approximation of such a system boils down to that of the reduced ANCBI pair (A_0, G_X) where G_X is defined as in Proposition 3.2 (with $G_1 = (0, 0, 0)^T$, $G_2 = 1$, $A_s = -K$, $A_J = B$). In particular, choosing $K = 2$ it turns out that the spectrum of A_0 and A_s are disjoint for all possible values of the uncertain parameters in the given compact sets and the Sylvester equation (40) have the following simple solution $X(\gamma, \mu) = (0\, 0\, {-1})^T$ which in turn yields $G_X(\gamma, \mu) = (0\ 0\ 1)^T$. Bearing in mind the discussion in Section II-C, we now proceed to the design of the feedback $F = (f_1\ f_2\ f_3)^T$ such that the conditions of Proposition 2.9 are fulfilled. A simple computation shows that while the conditions reported in **a)** in the proposition are automatically satisfied due to the fact that the pair $(A_0(\gamma, \mu), G_X(\gamma, \mu))$ is ANCBI for all possible values of the parameters, the conditions in **b)** specialize as

$$\begin{aligned} \mathrm{tr}(A_0 - G_X F^T) &= -\mu - f_3 < 0 \\ \det(A_0 - G_X F^T) &= -\gamma^2 \mu - \gamma^2 f_3 + \gamma^2 f_2 + \mu - f_1 < 0. \end{aligned}$$

Furthermore, it turns out that a sufficient condition for having the **c)** and **d1)** fulfilled is to set $p_1 = 0$ and $p_2 + p_3 < 0$ namely, for the specific pair (A_0, G_X)

$$\begin{aligned} f_3^2(f_1 + f_2) &= 0 \\ f_3(1 - f_2 - f_1)\mu - \gamma^2 f_2 + f_1 + \mu^2 - \mu &< 0. \end{aligned}$$

It is easy to check that a possible solution of the above sets of inequalities is

$$f_3 = c \quad f_2 = \frac{c}{2} \quad f_1 = -\frac{c}{2}$$

for any positive c. By virtue of the arguments in Section II-C the matrix $A_0(\gamma, \mu) - \alpha G_X(\gamma, \mu) F^T$ is Hurwitz for any $\alpha \in (0, 1]$.

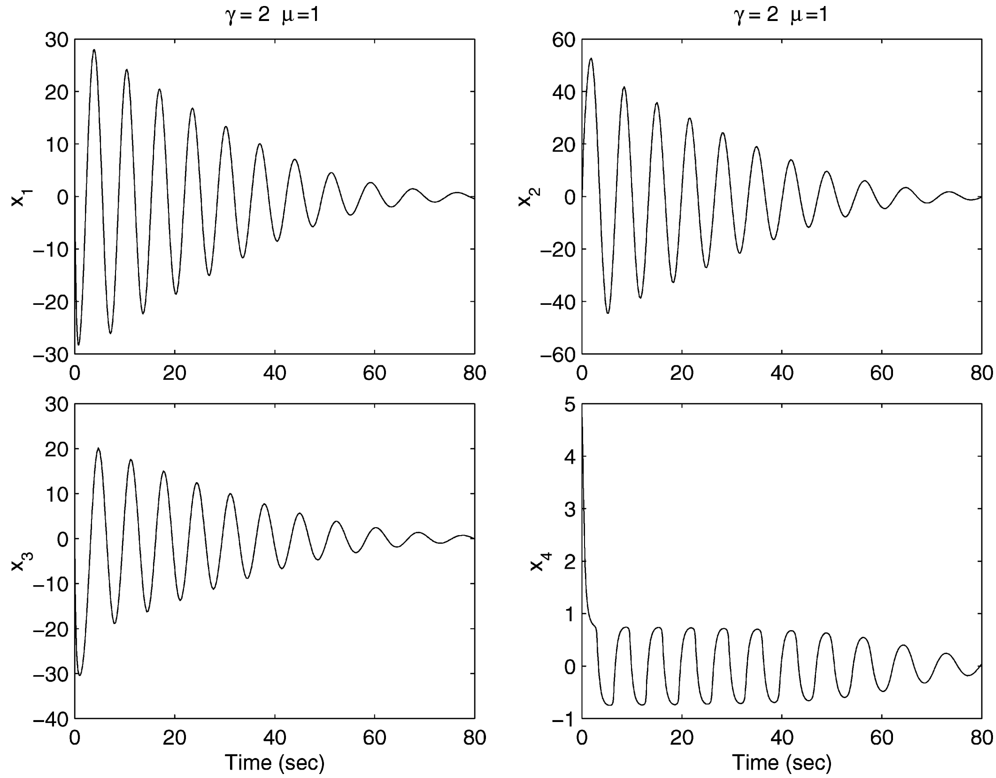

Fig. 3. Behavior of $\mathbf{x}(t)$, $x_4(t)$ in case $\gamma = 2$ and $\mu = 1$ (eigenvalues of A_0 in $\{-1, j\sqrt{3}, -j\sqrt{3}\}$).

Therefore, application of theorem 1 implies ISS of system (41) without restriction on the initial state and non zero restrictions on the inputs for any $\epsilon \in (0, 1]$. Notice that the transversality condition is fulfilled in the present case. Hence, by Proposition 3.2 claim **c)**, the system

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} A_0 & B \\ 0 & -K \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ x_4 \end{pmatrix} - \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \sigma(\epsilon F^T \mathbf{x} + w) + v$$

is ISS for a sufficiently small value of ϵ without restriction on the initial state, nonzero restrictions on the inputs and linear gains. Combining all the previous results and running the forwarding procedure described in Section III-A, it follows that the control law

$$u = -\lambda \sigma \left(\frac{K x_4 - \lambda_1 \sigma \left(\frac{\epsilon F^T \mathbf{x} + w}{\lambda_1} \right)}{\lambda} \right)$$

yields ISS for the system (47) without restriction on the initial state and non zero restriction on the input w if the design parameter λ_1 is taken sufficiently small. In particular for $w = 0$ the control law achieves robust global asymptotic stability for all possible values of the uncertain parameters in the given compact sets.

We have simulated the closed-loop system fixing the nonlinear higher order terms in (47) as

$$h(x_4) = (-1 \quad 1 \quad -1)^T x_4^3$$

and tuning the previous controller by choosing

$$\lambda = 5 \quad \lambda_1 = 1 \quad c = 2 \quad \epsilon = 1.$$

In particular, while the value of λ is arbitrary, the value of λ_1 has been fixed by trial and error in order to get rid of the higher order term $h(x_4)$ as described in Section III-A. The initial conditions of the system have been set to $(\mathbf{x}, x_4) = (5, -5, 0, 5)$ and the closed-loop system has been simulated with four different settings of the uncertain parameters $(\gamma, \mu) = \{(\sqrt{2}, 0); (\sqrt{2}, 1); (\sqrt{2}, 0.5); (\sqrt{10}, 0)\}$ within the allowed range. Note that the four settings have been chosen in order to have four completely different dynamics for the linear approximation A_0. In the first case (the associated state behaviors are plotted in Fig. 2) A_0 is a neutrally stable matrix with all the eigenvalues on the imaginary axis in $\{0, j2, -j2\}$. Then in the second scenario (see Fig. 3) the two oscillatory modes are shifted in $\{j\sqrt{3}, -j\sqrt{3}\}$ and the pole at the origin is moved in -1. In the third scenario (see Fig. 4) all the controlled modes are in the left half plane (A_0 Hurwitz) while in the last case (see Fig. 5) the matrix A_0 is characterized by two strongly oscillatory modes $\{j10, -j10\}$ and one pole at the origin. It is seen that in spite of all these so different uncontrolled modes the feedback is able to achieve robust asymptotic stabilization.

V. CONCLUSION

In this paper, we addressed the issue of robust feedback design for uncertain linear and nonlinear systems subject to input saturations. Our contribution is two-fold. A sufficient algebraic condition for ISS of low-dimensional single-input ($\leq$3) linear systems with saturated linear state-feedback was derived, similarly to the well-known Kalman conjecture. Then, the technique is recursively applied for stabilization of block-triangular systems with possible uncertain higher order nonlinearities in

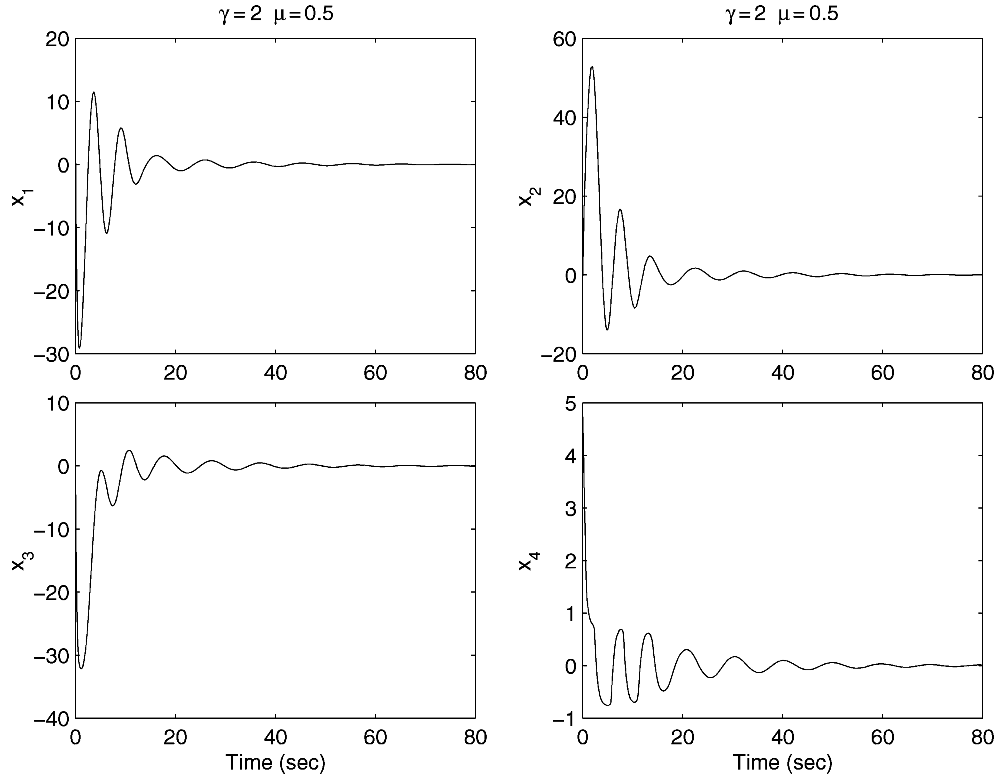

Fig. 4. Behavior of $\mathbf{x}(t)$, $x_4(t)$ in case $\gamma = 2$ and $\mu = 0.5$ (A_0 Hurwitz).

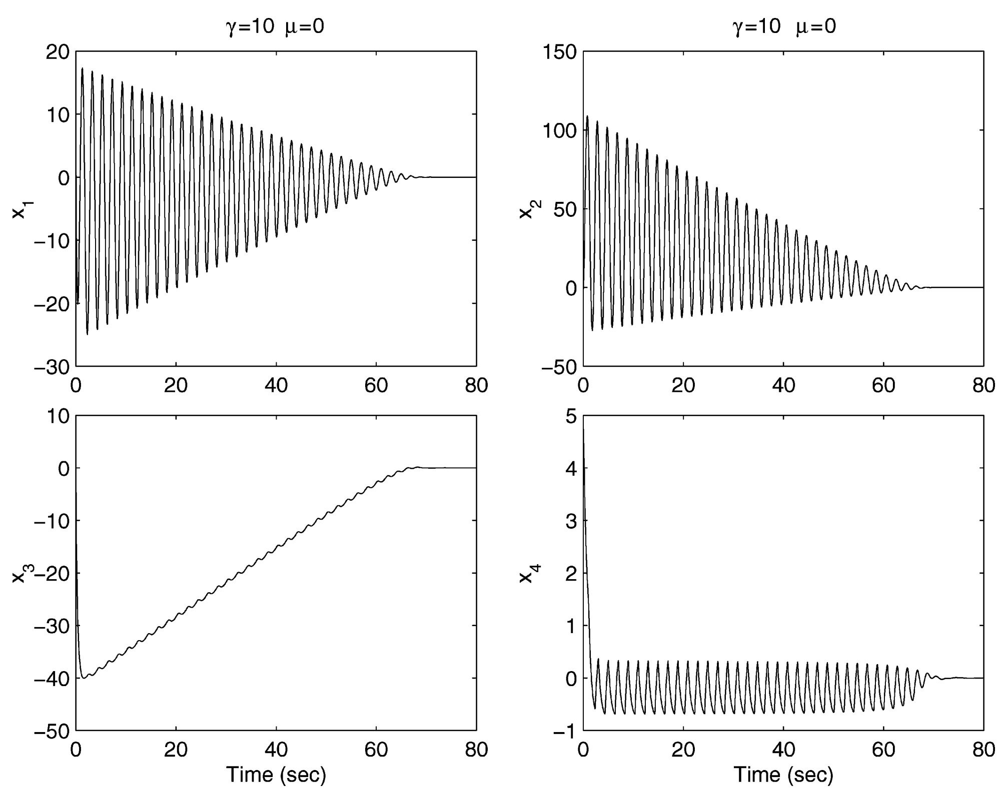

Fig. 5. Behavior of $\mathbf{x}(t)$, $x_4(t)$ in case $\gamma = 10$ and $\mu = 0$ (eigenvalues of A_0 in $\{0, j10, -j10\}$).

feed-forward form. The paper is intended as a first contribution in the field of robust saturated control which needs to be improved in several directions. One possible extension is to look for tight sufficient conditions which would allow treatment of higher dimensional uncertain blocks. More in general it is still a theoretically relevant question the determination of the class of uncertain (even linear) systems which can be robustly stabilized subject to arbitrary input constraints.

APPENDIX

PROOF OF LEMMA 2.2

One direction of the implication is obvious. The converse implication (namely 0-GAS plus UBND $\rightarrow$ ISS) is well-known to hold if ultimate boundedness is replaced by the asymptotic gain property. Therefore, the lemma follows provided that we can show the following implication:

$$0 - GAS \quad \& \quad UBND \Rightarrow AG.$$

By [1, Lemma 4.10], 0-GAS implies the existence of a smooth function $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, positive definite and radially unbounded such that, along trajectories of the system we have

$$DV(x)f(x,u) \leq -\alpha(|x|) + \gamma(|x|)\,\delta(|u|) \tag{49}$$

for some α, γ, δ of class $\mathcal{K}_\infty$. Let us consider input signals with $\|u\|_\infty \leq 1$. Then for any initial condition $x(0)$, $|x(t,\xi,u)| \leq \eta(2)$ for all t sufficiently large. In particular, for all such t's the following holds:

$$\dot{V}(x(t)) \leq -\alpha(|x(t)|) + \gamma(\eta(2))\,\delta(|u(t)|). \tag{50}$$

Expression (50) is an ISS dissipation inequality and, therefore, for all trajectories with $\|u\|_\infty \leq 1$ there exists a $\mathcal{K}_\infty$ asymptotic gain $\tilde{\gamma}$, viz.

$$\|x\|_a \leq \tilde{\gamma}(\|u\|_\infty). \tag{51}$$

Combining (51) with the estimate (5) we have that the asymptotic gain property holds with any gain $\hat{\gamma} \in \mathcal{K}_\infty$ such that $\hat{\gamma}(r) \geq \gamma(r)$ for $r \in [0,1]$ and $\hat{\gamma}(r) \geq \eta(r)$ for $r \geq 1$. This concludes the proof of the lemma.■

PROOF OF PROPOSITION 2.3

The system is clearly locally asymptotically stable. Therefore, we only need to show global attractivity. Let $\xi \in \mathbb{R}^n$ be arbitrary. By assumption there exists $M > 0$ such that $|x(t)| < M$ for any $t \geq 0$. Let $R := \max_{|x| \leq M} |F^T x|$. We define the function $\tilde{\delta}_1$ as

$$\tilde{\delta}_1(r) = \begin{cases} \delta'(-R) & r < -R \\ \delta'(r) & r \in [-R, R] \\ \delta'(R) & r > R. \end{cases} \tag{52}$$

Notice that, by continuity of δ', there exists $\varepsilon > 0$ such that $\tilde{\delta}_1(r) \in [\varepsilon, 1]$ for all $r \in \mathbb{R}$. Define $\tilde{\delta}(r) = \int_0^r \tilde{\delta}_1(s)ds$ and consider the auxiliary system

$$\dot{z} = Az - G\tilde{\delta}(F^T z). \tag{53}$$

Trajectories of (53) exists as $\tilde{\delta}$ is of class $\mathcal{C}^1$. Moreover, as $\tilde{\delta}(r) = \delta(r)$ for all $r \in [-R, R]$, we have $x(t) = z(t)$. Notice that (53) satisfies all the assumptions of the Kalman conjecture (the interval $[\varepsilon, 1]$ is in fact a stability sector), therefore (53) is GAS and hence so is (6) as $x(t) = z(t) \rightarrow 0$.■

In this section, we prove the ultimate boundedness of a 2×2 system and a 3×3 system. Here σ is simply assumed to be an increasing globally Lipschitz saturation function. Set $\Sigma(x) := \int_0^x \sigma(t)dt$. Note that Σ is positive definite and unbounded with a linear growth at infinity.

Lemma 1.1: Consider the control system $(SS)_2$

$$\begin{aligned} \dot{x} &= -y - \gamma_1\sigma(y+d) + v_1 \\ \dot{y} &= x - \gamma_2\sigma(y+d) + v_2 \end{aligned} \tag{54}$$

with $(\gamma_1, \gamma_2) \neq (0,0)$. If $1 + \gamma_1 > 0$ and $\gamma_2 > 0$, then $(SS)_2$ has the ultimate boundedness property with restrictions on v_1 and v_2 and no restrictions on d. ■

Lemma 1.2: Consider the control system $(SS)_3$

$$\begin{aligned} \dot{x}_1 &= -a\sigma(x_1+x_3+d) + v_1 \\ \dot{x}_2 &= -x_3 - b\sigma(x_1+x_3+d) + v_2 \\ \dot{x}_3 &= x_2 - c\sigma(x_1+x_3+d) + v_3 \end{aligned} \tag{55}$$

with $(b,c) \neq (0,0)$. If $c > 0$, then $(SS)_3$ has the ultimate boundedness property with restrictions on v_1 and v_2 and no restrictions on d. ■

In the sequel, we will provide arguments for both lemmas in the case where there is no external disturbance, i.e., $v_1 = v_2 = v_3 = 0$. Once this is done, then the general case can easily be deduced by adapting the proofs given below according to the argument of [8, Lemma 2]. Therefore, for the rest of the paper, we assume that $v_1 = v_2 = v_3 = 0$.

A. Proof of Lemma 1.1

Let $\xi(t) = (x(t), y(t))$ be a trajectory of $(SS)_2$ at time t. We consider $V(\xi) := \rho\|\xi\|^3 + \xi^T P\xi$, with $\rho > 0$ chosen later. Then, V is a positive–definite function, radially unbounded, $V(t)$ is its evaluation along $(SS)_2$ at time t and $\dot{V}$ its time derivative along $(SS)_2$. The result will result as a combination of the following lemmas.

Lemma 1.3: With the previous notations, the system $(SS)_2^0$

$$\begin{aligned} \dot{x} &= -y - \gamma_1\sigma(y) \\ \dot{y} &= x - \gamma_2\sigma(y) \end{aligned} \tag{56}$$

is GAS if and only if $1 + \gamma_1 > 0$ and $\gamma_2 > 0$. □

(Applying Lasalle's with the Lyapunov function $x^2 + y^2 + 2\gamma_1\Sigma(y)$ immediately provides an argument for the previous lemma.)

Lemma 1.4: With the previous notations, there exist positive constants D_0, C_0, C_1, C_2, such that for every $\|d\|_\infty \geq D_0 \gg 1$ and every $t \geq 0$ so that $V^{1/3}(t) \geq C_0\|d\|_\infty$, there exists $3\pi/2 \leq T(t) \leq 5\pi/2$ for which

$$V(t + T(t)) \leq V(t) - C_1 V^{\frac{1}{3}}(t)\left(V^{\frac{1}{3}}(t) - \frac{C_0}{2}\|d\|_\infty\right) \tag{57}$$

and $\max_{t \leq s \leq t+T(t)} V^{1/3}(s) \leq V^{1/3}(t) + C_2$. □

Assuming Lemma 1.4, let us first prove that there exists D_0, $C_0 > 0$ such that

$$\|(x,y)\|_a \leq C_0 \max(D_0, \|d\|_\infty). \tag{58}$$

Note that this equation implies, first, that Lemma 1.1 holds true for inputs d with $\|d\|_\infty > D_0$ and, secondly, that trajectories of $(SS)_2$ are ultimately bounded by an universal constant D_0 for inputs d with $\|d\|_\infty \leq D_0$. Moreover, there exists C_0 only depending on the problem parameters and ρ such that if $\|\xi\| \geq C_0$ then $(\rho/2)\|\xi\| \leq V^{1/3}(\xi) \leq 2\rho\|\xi\|$. Next, note that there exists a time t_1 such that $V^{1/3}(t_1) \leq C_0\|d\|_\infty$, otherwise we can build a sequence of time (t_n) tending to ∞ and defined by $t_{n+1} = t_n + T(t_n)$ such that the corresponding sequence (V_n), with $V_n = V(t_n)$, goes to $-\infty$. On the time interval $[t_1, t_2]$, $V^{1/3}(t)$ is clearly bounded by $2C_0\|d\|_\infty$. If $V^{1/3}(t_2) \geq C_0\|d\|_\infty$, we get that $V^{1/3}(t) \leq 2C_0\|d\|_\infty$ for $t \in [t_2, t_3]$ and $V^{1/3}(t_3) \leq V^{1/3}(t_2)$. We iterate the construction until $V^{1/3}$ gets smaller than $C_0\|d\|_\infty$. If $V^{1/3}(t_2) < C_0\|d\|_\infty$, we take t_3 to be the first time greater than t_2 for which $V^{1/3}(t_3) \geq C_0\|d\|_\infty$ (if it ever happens). By the new definition of t_3, we also have $V^{1/3}(t_3) \leq C_0\|d\|_\infty + C_2$. With t_3 playing now the role of t_1, we easily show inductively that for every $t \geq t_1$, we have $V^{1/3}(t) \leq 2C_0\|d\|_\infty$. That concludes the proof of (58).

It remains to prove that Lemma 1.1 holds true for inputs d with $\|d\|_\infty \leq D_0$. In fact, it is enough to prove the result for any positive number D_0' smaller than D_0 (with a possibly different bound $C_0' \geq C_0$). We intend next to show the existence of such a D_0'.

Since $(SS)_2^0$ is GAS, it admits a strict Lyapunov function V_2. By computing the time derivative of V_2 along trajectories of $(SS)_2$ and by taking into account (58), we get that $(SS)_2$ is in fact ISS for inputs d with $\|d\|_\infty \leq D_0$. It implies that, for small inputs d, the dynamics of $(SS)_2$ is asymptotically a perturbation of the linearized system at (0,0) and, therefore, Lemma 1.1 holds true for small inputs d. The proof of Lemma 1.1 is now complete.

B. *Proof of Lemma 1.4*

Fix $t > 0$ such that $V^{1/3}(t) \geq C_0\|d\|_\infty$ and set $D := \|d\|_\infty \gg 1$ and $R(t) := \|\xi(t)\|$. Thanks to the first observation given before, (57) will follow from

$$V\left(t+T(t)\right) \leq V(t) - C_1 R(t)\left(R(t) - \frac{C_0}{2}D\right) \tag{59}$$

and $\max_{t\leq s\leq t+T(t)} R(s) \leq R(t) + C_2$. We have, for $s \geq t$

$$\begin{aligned}\dot{V} = &-3\rho R(s)\left(\gamma_1 x(s) + \gamma_2 y(s)\right)\sigma\left(y(s)+d(s)\right) + \\ &-R^2(s) - 2\left(a_1x(s)+a_2y(s)\right)\left(y(s)-\sigma\left(y(s)+d(s)\right)\right)\end{aligned} \tag{60}$$

where $a_1x(s)+a_2y(s) := \xi(s)^T P(\gamma_1\ \gamma_2)^T$. By using property *(ii)* of the saturation σ (applied to $y+d-\sigma(y+d)$) and taking into account that $\gamma_2 > 0$, we can get rid of the term $-a_2y(y-\sigma(y+d))$ with an appropriate choice of ρ. Equation (60) yields

$$\begin{aligned}\dot{V}(s) \leq &-R^2(s) + C_0R(s)D + b_1'x(s)y(s) \\ &+b_1x(s)R(s)\sigma\left(y(s)+d(s)\right)\end{aligned} \tag{61}$$

for some constants b_1, b_1' (not necessarily positive). Writing the system (12) in polar coordinates, we get

$$\begin{aligned}\dot{R} &= -\left(\gamma_1\cos(\theta) + \gamma_2\sin(\theta)\right)\sigma\left(R\sin(\theta)+d\right) \\ \dot{\theta} &= 1 - \frac{\gamma_2\cos(\theta) - \gamma_1\sin(\theta)}{R}\sigma\left(R\sin(\theta)+d\right).\end{aligned} \tag{62}$$

Equation (61) can be written

$$\begin{aligned}\dot{V} \leq &-R^2 + C_0RD + b_1'R^2\cos(\theta)\sin(\theta) \\ &+b_1R^2\cos(\theta)\sigma\left(R\sin(\theta)+d\right).\end{aligned} \tag{63}$$

Note that there exists $C_0 > 0$ such that for $s \geq t$

$$|\dot{R}| \leq C_0 \quad |\dot{\theta} - 1| \leq \frac{C_0}{R(s)}.$$

It is then clear that there exists $T(t) \in [3\pi/2, 5\pi/2]$ and $C_2 > 0$ for which $\theta(t+T(t)) = \theta(t) + 2\pi$ and $\max_{t\leq s\leq t+T(t)} R(s) \leq R(t) + C_2$. Set $R_0 := R(t)$ and $\theta_0 := \theta(t)$. Then we also have $s \in [t, t+T(t)]$

$$|R(s) - R_0| \leq C_2|\dot{\theta} - 1| \leq \frac{C_0}{R_0} \ll 1.$$

Reparameterizing by the angle $\theta \in [\theta_0, \theta_0 + 2\pi]$ (assuming also that $\theta(t) = 0$) we can rewrite (63) as

$$\begin{aligned}\frac{dV}{d\theta} \leq &-\frac{R_0^2}{2} + C_0R_0D + b_1'R_0^2\cos(\theta)\sin(\theta) \\ &+b_1R_0^2\cos(\theta)\sigma\left(R_0\sin(\theta)+d'\right)\end{aligned} \tag{64}$$

since $|(R^2/\dot{\theta}) - R_0^2| + |(R/\dot{\theta}) - R_0|$ is of the magnitude R_0. Note also that $d' := d + (R - R_0)\sin(\theta)$ and we will assume it bounded by D. After integrating (64) between θ_0 and $\theta_0 + 2\pi$, it is clear that Lemma 1.4 will follow if the integral I defined by

$$I = R_0^2 \int_{[\theta_0,\theta_0+2\pi]} \cos(\theta)\sigma\left(R_0\sin(\theta)+d'\right)d\theta \tag{65}$$

is bounded by C_1DR_0 for some positive constant C_1. Indeed, note that

$$\int_{[\theta_0,\theta_0+2\pi]} \sin(\theta)\cos(\theta)d\theta = 0.$$

We write $I = I_1 + I_2$ with

$$\begin{aligned}I_1 &= R_0^2\int_0^{2\pi} \cos(\theta)\sigma\left(R_0\sin(\theta)+d'\right)d\theta \\ I_2 &= R_0^2\int_{\theta_0}^{0} \cos(\theta)\sigma\left(R_0\sin(\theta)+d'\right)d\theta \\ &\quad + R_0^2\int_{2\pi}^{2\pi+\theta_0} \cos(\theta)\sigma\left(R_0\sin(\theta)+d'\right)d\theta\end{aligned}$$

and this last integral can be written

$$I_2 = R_0^2\int_0^{\theta_0} \cos(\theta)\left[\sigma\left(R_0\sin(\theta)+d_1\right) - \sigma\left(R_0\sin(\theta)+d_2\right)\right]d\theta$$

where d_1, d_2 are disturbances bounded by D.

We first cut the interval $[0, 2\pi]$ in the four intervals $[(i\pi/2), ((i+1)\pi/2)]$, $i = 0, 1, 2, 3$, then we rearrange the four integrals in such a way that performing the change of variable $v = R_0 \sin(\theta)$, we get

$$I_1 = 2R_0 \int_0^{R_0} [\sigma(v + d_1) - \sigma(v + d_2)]\, dv \tag{66}$$

where d_1 and d_2 are disturbances bounded by D. Thanks to the fact that σ is increasing, we have, for $v \in [0, R_0]$

$$|\sigma(v + d_1) - \sigma(v + d_2)| \leq \sigma(v + D) - \sigma(v - D).$$

We deduce that

$$\begin{aligned} |I_1| &\leq 2R_0 \int_0^{R_0} [\sigma(v + D) - \sigma(v - D)]\, dv \\ &= 2R_0 \left[\int_D^{R_0+D} \sigma(v)dv - \int_{-D}^{R_0-D} \sigma(v)dv \right] \\ &= 2R_0 \left[\Sigma(-D) + \int_{R_0-D}^{R_0+D} \sigma(v)dv \right] \leq 8R_0 D. \end{aligned}$$

For I_2, we cut $[0, \theta_0]$ using as many intervals $[i\pi/2, ((i+1)\pi/2)]$ contained in it plus the last one of the form $[i\pi/2, \theta_0]$. On each of them we proceed as done previously and we deduce for $|I_2|$ a similar estimate as the one obtained for $|I_1|$. This ends the proof of Lemma 1.4.

C. Proof of Lemma 1.2

We rewrite the system (55) in coordinates $x = (X, x_2, x_3) \in \mathbb{R}^3$, with $X := x_1 + x_3$. We have

$$\begin{aligned} \dot{X} &= x_2 - (a + c)\sigma(X + d) \\ \dot{x}_2 &= -x_3 - b\sigma(X + d) \\ \dot{x}_3 &= x_2 - c\sigma(X + d). \end{aligned} \tag{67}$$

Moreover, we have $a, c > 0$ and $\rho := \min(c, c + b(a + c)) > 0$. Consider the Lyapunov function V

$$V(x) = \frac{1}{2}\left[\frac{c}{a}(X - x_3)^2 + x_2^2 + x_3^2\right] + b\Sigma(X).$$

Let $\dot{V}$ be the time derivative of V along (67). Then

$$\dot{V} = -\left[c + \frac{\sigma(X)}{X} b(a+c)\right] X\sigma(X+d) - bx_2[\sigma(X+d) - \sigma(X)]. \tag{68}$$

It follows immediately from (68) that the undisturbed system associated to $(SS)_3$ is GAS. Following the same lines of the proof Lemma 1.1, we get that an argument for Lemma 1.2 reduces to obtain the following: there exists $D_0, C_0 > 0$ such that if, $\|d\|_\infty > D_0$, then

$$\|(x, y)\|_a \leq C_0 \|d\|_\infty. \tag{69}$$

The rest of the paragraph is devoted to the proof of (69).

We assume that

$$\begin{aligned} &D := \|d\|_\infty \gg 1 \\ &\delta_D := \sup_{|x|,|y| \geq \frac{2D}{3}, xy>0} |\sigma(x) - \sigma(y)| \ll 1 \\ &\max(\sigma(D), -\sigma(-D)) \geq \frac{3}{4} S. \end{aligned}$$

where $S := \max(\sigma_+, -\sigma_-)$.

We will prove the existence of $C_0, C_1, C_2 > 0$, $C_0 \gg 1$ such that for every $t \geq 0$ with $V(t) \geq C_0^2 D^2$, there exists $T(t) \geq 1$ with

$$\begin{aligned} \max_{s \in [t, t+T(t)]} V^{\frac{1}{2}}(s) &\leq V^{\frac{1}{2}}(t) + C_1 \\ V(t + T(t)) &< V(t) - C_2. \end{aligned} \tag{70}$$

It is easy to deduce (69) from (70): reasoning by contradiction as in the proof of Lemma 1.1 above, we have that there exists $t_0 > 0$ such that $V(t_0) < C_0^2 D^2$. If $t_1 > t_0$ is the first time for which $V(t_1) = C_0^2 D^2$, then by applying (70), $V(t_1 + T(t_1)) < V(t_1) - C_2 < C_0^2 D^2$ and

$$\max_{s \in [t_1, t_1+T(t_1)]} V^{\frac{1}{2}}(s) \leq V^{\frac{1}{2}}(t_1) + C_1 \leq 2C_0 D.$$

Now $t_1 + T(t_1)$ plays the role of t_0. This shows (69).

Let us prove (70). Set $L := (1 + a + c + |b| + \rho)^2$ and, for $t \geq 0$, $r(t) := (x_2^2(t) + x_3^2(t))^{1/2}$. It is enough to prove the result for $t = 0$. We may then assume that $V_0 := V(0) \geq C_0^2 D^2$. Note that $|\dot{V}^{1/2}| \leq C_1$. Therefore, the first part of (70) will follow if $T(0)$ to be determined is also bounded by some constant independent of D.

Along with (67), let us consider the dynamics of (x_2, x_3) written in polar coordinates. We get

$$\dot{r} = -(b\cos(\theta) + c\sin(\theta))\,\sigma(X + d) \tag{71}$$

$$\dot{\theta} = 1 - \frac{c\cos(\theta) - b\sin(\theta)}{r}\sigma(X + d). \tag{72}$$

Note that $|\dot{r}| \leq L$. First assume that $r(0) < \mu C_0 D$ for some positive $\mu < (1/5)$. For an appropriate choice of μ (only depending on a), b, c, we have $|X(0)| \geq (C_0/2D)$. For $s \in [0, 1]$, we have

$$\begin{aligned} |x_2(s)| &\leq r(s) \leq \mu C_0 D + L \leq 2\mu C_0 D \\ |X(s)| &\geq \frac{C_0}{3} D \gg D. \end{aligned}$$

Then $\dot{V} \leq -\rho S(C_0/3)D + 2bS\mu C_0 D \leq -C_0' D$. Then (70) follows. Note that μ is independent of C_0.

Next we suppose that $r_0 := r(0) \geq \mu C_0 D$. Set $\theta(0) = \theta_0$. With no loss of generality, we assume $C_0 \mu \gg 1$. Define T_0 such that $\theta(T_0) - \theta_0 = 2\pi$. Set $I_0 := [0, T_0]$. It is clear that $3\pi/2 \leq T_0 \leq 5\pi/2$. We reparameterize all the dynamics of I_0 by the angle $\theta \in [\theta_0, \theta_0 + 2\pi]$. Then, for $\theta \in [\theta_0, \theta_0 + 2\pi]$, an easy computation shows that

$$|x_2(\theta) - r_0\cos(\theta)| \quad |x_3(\theta) - r_0\sin(\theta)| \leq C_1 \tag{73}$$

where $C_1 > 0$ is a constant independent of r_0, D, V_0. Equation (73) implies that, for $\theta \in [\theta_0, \theta_0 + 2\pi]$,

$$|X(\theta) - X(0) - r_0 (\sin(\theta) - \sin(\theta_0))| \le C_1. \tag{74}$$

The integration of (68) over $[\theta_0, 2\pi + \theta_0]$ amounts to consider the two following integrals:

$$J_1 = \int_{\theta_0}^{2\pi+\theta_0} \left[c + \frac{\sigma(X)}{X} b(a+c)\right] X\sigma(X+d)d\theta$$

$$J_2 = \int_{\theta_0}^{2\pi+\theta_0} b\dot{X}\left[\sigma(X+d) - \sigma(X)\right] d\theta. \tag{75}$$

Let $E_D = \{\theta \in [\theta_0, 2\pi + \theta_0], |X(\theta)| \le 2D\}$. The Lebesgue measure $|E_D|$ is less than $C_2(D/r_0)$ since $(D/r_0) \le (1/C_0\mu) \ll 1$. Taking into account (75) and the previous remark, we deduce that there exists $C_3 > 0$ such that

$$\begin{aligned} J_1 \ge \rho & \int_{\theta_0}^{2\pi+\theta_0} (X_0 + r_0 \sin(\theta))\, \sigma\left(X_0 + r_0 \sin(\theta + \theta_0)\right) d\theta \\ & - \frac{4LS}{C_0\mu} D \\ J_2 = & \int_{\theta_0}^{2\pi+\theta_0} r_0 \cos(\theta) \left[\sigma\left(X_0 + r_0 \sin(\theta + \theta_0) + d\right) d\theta \right. \\ & \left. - \int_{\theta_0}^{2\pi+\theta_0} \sigma\left(X_0 + r_0 \sin(\theta)\right)\right] d\theta \\ & + I_2 \end{aligned} \tag{76}$$

where I_2 is the integral over $[\theta_0, 2\pi + \theta_0]$ of a bounded function. Note that the function d appearing in (76) actually stands for the disturbance of (75) plus some bounded functions coming from (74). We still assume d bounded by D. We are then left with the estimation of

$$J_1' = \int_{\theta_0}^{2\pi+\theta_0} (X_0 + r_0 \sin(\theta))\, \sigma\left(X_0 + r_0 \sin(\theta)\right) d\theta$$

and

$$\begin{aligned} J_2' = \int_{\theta_0}^{2\pi+\theta_0} r_0 \cos(\theta) \cdot [\sigma\left(X_0 + r_0 \sin(\theta) + d\right) \\ - \sigma\left(X_0 + r_0 \sin(\theta)\right)] d\theta. \end{aligned}$$

We start with J_2' since its majoration is almost identical to the one of the integral defined in (65). The only difference lies in the fact that for J_2', we reach the inequality $|J_2'| \le J_+ + J_-$, where

$$\begin{aligned} J_\varepsilon := \int_{\theta_0}^{\frac{\pi}{2}+\theta_0} r_0 \cos(\theta) \cdot [\sigma\left(X_0 + r_0 \sin(\theta) + D\right) \\ - \sigma\left(X_0 + r_0 \sin(\theta) - D\right)] d\theta \end{aligned}$$

with $\varepsilon = +, -$. Performing in each the change of variable $v_\varepsilon = X_0 + \varepsilon r_0 \sin(\theta) \pm D$, we end up with $|J_2'| \le C_2 D$, with C_2 only dependent on the constants appearing in (67).

As for J_1', it is easy to see that $|J_1'| \ge C_3 \max(|X_0|, r_0)$, with C_3 only dependent on the constants appearing in (67). Note that $\max(|X_0|, r_0) \ge \eta V_0^{1/2}$, for some positive constant η.

Gathering all the estimates, we have

$$V(T_0) \le V_0 - C_4 V_0^{\frac{1}{2}} + C_5 D$$

with C_4 and C_5 positive constants depending only the constants appearing in (67). We can then adjust C_0 which is a lower bound for $V_0^{1/2}/D$ to obtain (70).

Proposition 1.5: Let A be neutrally stable and $A - \lambda GF^T$ Hurwitz for all $\lambda \in (0, 1]$. If the root locus of $A - \lambda GF^T$ is transversal to the imaginary axis for $\lambda = 0$ then there exists $P(\lambda) > 0$ such that for sufficiently small λ

$$P(\lambda)(A - \lambda GF) + (A - \lambda GF)^T P(\lambda) = -\lambda I \tag{77}$$

and

$$\|P(\lambda)\| \le \bar{p}$$

for some $\bar{p} > 0$ not dependent on λ.

Proof: Consider the explicit solution of (77) given by

$$P(\lambda) = \lambda \int_0^\infty e^{(A-\lambda GF^T)t} e^{(A-\lambda GF^T)^T t} dt. \tag{78}$$

Our goal is to show that (78) is bounded for λ sufficiently small. To this end suppose that A has m eigenvalues on the imaginary axis and $n - m$ eigenvalues with negative real parts. Since A is neutrally stable (namely its m eigenvalues with zero real part are simple), there exists $\epsilon > 0$ such that for all $\lambda \in (0, \epsilon]$

$$\begin{aligned} e^{(A-\lambda GF^T)t} = & \sum_{k=1}^{N_1} M_k(\lambda) e^{Re(\sigma_k(\lambda))t} \sin\left(\omega_k(\lambda)t + \varphi_k(\lambda)\right) \\ & + \sum_{k=N_1+1}^{N_2} M_k(\lambda) e^{Re(\sigma_k(\lambda))t} \sin\left(\omega_k(\lambda)t + \varphi_k(\lambda)\right). \end{aligned}$$

with $N_1 = \lfloor (m+1)/2 \rfloor$ and $N_2 = \lfloor (n+1)/2 \rfloor$. As for all $\lambda \in (0, \epsilon]$, $Re(\sigma_k(\lambda)) < 0$, for $k = N_1 + 1 \ldots, N_2$ and $M_k(\lambda)$, $k = 1, \ldots, N_1$, are uniformly bounded (since $A - \lambda GF^T$ is Hurwitz and A is neutrally stable), it turns out that (78) is bounded for all $\lambda \in [0, \epsilon]$ provided that

$$\lim_{\lambda \to 0} \frac{\lambda}{Re\left(\sigma_k(\lambda)\right)}, \qquad k = 1, \ldots, N_1 \quad \text{is bounded.}$$

Indeed this is true if

$$\left.\frac{\partial Re\left(\sigma_k(\lambda)\right)}{\partial \lambda}\right|_{\lambda=0} \ne 0, \qquad k = 1, \ldots, N_1$$

which is equivalent to the fact that the root locus of $A - \lambda GF^T$ is trasversal to the imaginary axis for $\lambda = 0$. This completes the proof of the result.

□

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