Time-Varying High-Gain Observers for Numerical Differentiation

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***Abstract*—In this note, we propose high-gain numerical differentiators for estimating the higher derivatives of a given signal. We consider time-varying high-gain vectors converging exponentially to the high-gain vectors introduced by Esfandiari and Khalil in an earlier paper. The dynamics of these time-varying high-gain vectors can be chosen in order to achieve specific objectives, such as peaking attenuation and low sensitivity with respect to noise disturbance. In particular, we show that the numerical differentiator introduced in an earlier paper avoids the peaking phenomenon in the sense of Sussmann and Kokotovic, i.e., there is no unbounded overshoot of the error estimate during the initial times. We also propose another numerical differentiator which filters the reference signal with respect to a very simple quadratic cost.**

***Index Terms*—Filtering, numerical differentiation, time-varying linear systems.**

I. Introduction

In the last few years, estimation of the output derivatives has rapidly increased in importance in control and observation theory; see, for instance, [1]–[5], [7]–[9], and the references therein. This note is the outgrowth of new results in estimation and filtering of the higher derivatives of any given signal with some norm-bounded derivative. Our approach follows the seminal work of Esfandiari and Khalil [5]. They introduced a high-gain continuous differentiator as the following n-dimensional dynamical system:

$$\dot{x} = Ax + H(\alpha,\gamma)(y - Cx) \tag{1}$$

where

i) $A = (\delta_{i,j-1})_{1\leq i,j\leq n}$ and $C = (\,1 \quad 0 \quad 0 \quad \cdots \quad 0\,)$;
ii) $y\colon \mathbb{R}^+ \to \mathbb{R}$ is the signal to differentiate and $\|y^{(n)}\|_\infty < \infty$;
iii) $H(\alpha,\gamma) = (\,\alpha_i\gamma^i\,)^T_{1\leq i\leq n}$ with $\gamma > 0$ and the polynomial $P_\alpha(s) \overset{def}{=} s^n + \sum_{i=1}^n \alpha_i s^{n-i}$ is Hurwitz.

We call *EK-vector* a vector $H(\alpha,\gamma)$ as defined in iii). Esfandiari and Khalil showed, among numerous important results, that for signals y subject to ii), the trajectory x of (1) is an estimator of $(y,\dot{y},\ldots,y^{(n-1)})^T$, if $\gamma > 0$ is large enough. That property can be interpreted as follows: there exists a positive function $\tau(\gamma)$ such that $\lim_{\gamma\to\infty}\tau(\gamma) = 0$ and for $j = 0,\ldots,n-1$, $\lim_{\gamma\to\infty}\sup_{t\geq\tau(\gamma)}|x_{j+1}(t) - y^{(j)}(t)| = 0$. In this note, we propose two types of differentiators, addressing the issues of peaking and filtering by constructing dynamical systems of order n which estimate the higher derivatives of a measured signal $y(t)$ up to the order $n-1$. The idea followed in this note and first considered in [8] is rather simple: we still consider high-gain continuous differentiators of the type $\dot{x} = Ax + H(t,\alpha,\gamma)(y - Cx)$ where, instead of being an EK-vector, $H(t,\alpha,\gamma)$ is now a time-dependent vector with the constraint that $H(t,\alpha,\gamma)$ converges exponentially fast at infinity to some EK-vector $H(\alpha,\gamma)$. The whole point consists in choosing the dynamics of $H(t,\alpha,\gamma)$. The convergence of $H(t,\alpha,\gamma)$ to its steady state has to be fast enough in order to preserve the properties of the observer (1), not only for differentiating the signal y, but also for stabilization applications as considered, for instance, in [5], [11], or [1]. This note is divided into four sections. Section II treats the dynamics relative to the peaking issue and, in particular, we show that the differentiator introduced in [8] is a no-peaking observer, in the sense of [10]. More precisely, the estimation error associated to the aforementioned differentiator does not exhibit an unbounded overshoot in the first instants of the simulation. Section III is devoted to the filtering issue in the continuous-time case and the discrete-time case. We show that the proposed differentiator filters the assigned signal y. SectionIV gathers concluding remarks. The main technical proofs are collected in the Appendix .

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II. Peaking

For $\gamma > 0$, let $A_\gamma \overset{def}{=} A + (\gamma/2)Id_n$. For $\alpha = (\alpha_1,\ldots,\alpha_n)^T \in \mathbb{R}^n$ and $1 \leq k \leq n$, $P_{\alpha,\gamma}(s) \overset{def}{=} s^n + \sum_{i=1}^n \alpha_i\gamma^i s^{n-i} = \prod_{i=1}^n (s + \gamma\beta_i)$, with $\alpha_k = \sum \beta_{i_1},\ldots,\beta_{i_k}$. Set $\beta \overset{def}{=} (\beta_1,\ldots,\beta_n)^T \in \mathbb{R}^n$ and $\beta_m = \min_{1\leq i\leq n} Re(\beta_i)$. Then $P_{\alpha,\gamma}(s)$ is the characteristic polynomial of $A(\alpha,\gamma)$ and it is Hurwitz for $\gamma > 0$ if and only if $\beta_m > 0$. If $M \in M_n(\mathbb{R})$ and $B \in M_{n\times m}(\mathbb{R})$, $P_M(s)$ stands for the characteristic polynomial of M and $R(M,B)$ for the controllability matrix $\left[B, MB, \ldots, M^{n-1}B\right]$ of the pair (M,B). We use $M > 0$ (≥ 0, respectively) to denote that M is symmetric positive–definite (symmetric semipositive, respectively). Let Id_n and $(c_i)_{1\leq i\leq n}$ be, respectively, the identity matrix and the canonical basis of $\mathbb{R}^n$, respectively. If $M > B > 0$, then $0 < M^{-1} < B^{-1}$. Define $K(\gamma) = \mathrm{diag}(1/\gamma, 1/\gamma^2,\ldots,1/\gamma^n)$. We have the following relations $K(\gamma)AK(\gamma)^{-1} = \gamma A$, $K(\gamma)^{-1}A^TK(\gamma) = \gamma A^T$, $K(\gamma)^{-1}C^TCK(\gamma)^{-1} = \gamma^2 C^TC$.

The following matrix ordinary differential equation (ODE) was introduced in [6] for observation purposes and in [8] in a more general setting: $\dot{L} = -A_\gamma^T L - LA_\gamma + C^TC$, $L(0) = Id_n$. Note that $P_{-A_\gamma}(s) = P_{-A_\gamma^T}(s) = (s+\gamma/2)^n$. For $t \geq 0$, we have $L(\gamma,t) = e^{-\gamma t}e^{-A^Tt}e^{-At} + S(\gamma,t)$, where $S(\gamma,t) \overset{def}{=} \int_0^t e^{-\gamma\tau}e^{-A^T\tau}C^TCe^{-A\tau}d\tau$. Set $S(t) \overset{def}{=} S(1,t)$. We have

$$\begin{aligned}\lim_{t\to\infty} L(\gamma,t) &= L_\infty(\gamma)\\ &= \int_0^\infty e^{-\gamma\tau}e^{-A^T\tau}C^TCe^{-A\tau}d\tau\\ &= \left(\frac{(-1)^{i+j}C_{i+j-2}^{i-1}}{\gamma^{i+j-1}}\right)_{1\leq i,j\leq n} > 0\\ L(\gamma,t) - L_\infty(\gamma) &= e^{-\gamma t}e^{-A^Tt}\left(Id_n - L_\infty(\gamma)\right)e^{-At}.\end{aligned}$$

Notice that $L_\infty(\gamma) = \gamma K(\gamma)L_\infty K(\gamma)$, where $L_\infty \overset{def}{=} L_\infty(1)$. Moreover $L_\infty(\gamma) > 0$ since $(-A_\gamma^T, C^T)$ is a controllable pair. Define $N(\gamma,t) \overset{def}{=} L^{-1}(\gamma,t)$. Then

$$\dot{N} = A_\gamma N + NA_\gamma^T - NC^TCN,\ N(0) = Id_n. \tag{2}$$

For γ large enough, $L(\gamma,0) = Id_n > L_\infty(\gamma)$. By standard arguments, one has $N(\gamma,t) > 0$ for $t \geq 0$, is increasing and $\lim_{t\to\infty} N(\gamma,t) = L_\infty^{-1}(\gamma)$. Passing to the limit in (2) leads to $L_\infty^{-1}(\gamma)(-A^T - \gamma Id_n)L_\infty(\gamma) = A - L_\infty^{-1}(\gamma)C^TC$. It implies that $-A^T - \gamma Id_n$ and $A - L_\infty^{-1}(\gamma)C^TC$ are similar and, therefore, $P_{-A^T-\gamma Id_n}(s) = P_{A-L_\infty^{-1}(\gamma)C^TC}(s)$. We have $P_{-A^T-\gamma Id_n}(s) = (s+\gamma)^n = s^n + \sum_{i=1}^n C_n^i\gamma^i s^{n-i}$. On the other hand, $P_{A-L_\infty^{-1}(\gamma)C^TC}(s) = s^n + \sum_{i=1}^n l_i s^{n-i}$, where $(l_i)_{1\leq i\leq n} = L_\infty^{-1}(\gamma)C^T$. Then, $l_i = C_n^i\gamma^i$ for $1 \leq i \leq n$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ be an EK-vector. If $D_\alpha \overset{def}{=} \mathrm{diag}((\alpha_1/C_n^1), \ldots, (\alpha_n/C_n^n))$, then $\lim_{t\to\infty} D_\alpha N(\gamma,t)C^T = H(\alpha,\gamma)$. We are now able to define a differential system (P) as follows:

$$\dot{x} = Ax + D_\alpha N(\gamma,t)C^T\,(y - Cx) \tag{3}$$

$$\dot{N}(\gamma,t) = \gamma N + AN(\gamma,t) + N(\gamma,t)A^T - NC^TCN(\gamma,t)$$
$$N(0) = Id_n. \tag{4}$$

For $D_\alpha = Id_n$, (P) was already introduced in [8] as a numerical differentiator. We next show that (P) defined in (3) indeed defines a continuous differentiator which converges exponentially fast to (1). Let $\widetilde{Y}(t) = (y, y', \ldots, y^{(n-1)})^T$ defined for a signal y subject to ii). We have the following.

Theorem 1: There exist $C_0 > 0$ and an integer $q > 0$ such that for $t \geq t_\gamma := q(\ln(\gamma)/\gamma)$

$$\sup_{t\geq t_\gamma} \|x(t) - \widetilde{Y}(t)\| \leq \frac{C_0}{\gamma}\left(\left\|x(t_\gamma) - \widetilde{Y}(t_\gamma)\right\| + \left\|y^{(n)}\right\|_\infty\right). \tag{5}$$

The numerical differentiator considered in (3) attenuates the peaking phenomenon occuring for small time with (1) because we can choose the EK-gain vector at the origin small (independent of γ) and, therefore, $\|\dot{x}(0)\|$ is small. This intuitive remark translates rigorously as follows: in the case where $D_\alpha = Id_n$, the corresponding differentiator does not exhibit any peaking phenomenon. We indeed show the following.

Theorem 2: Let $D_\alpha = Id_n$. There exist $C_1, C_2 > 0$ independent of γ (large enough) such that for $t \geq 0$ and $1 \leq i \leq n$

$$\begin{aligned}|x_i(t) - y^{(i-1)}(t)| \leq &\frac{C_1\|y^{(n)}\|_\infty}{\gamma^{n+1-i}} \\ &+ C_2 \min\left(1, \gamma^{2n-1}\exp(-\gamma t)\left(1 + t^{3n-2}\right)\right) \\ &\cdot\left(\|x(0) - \widetilde{Y}(0)\| + \|y^{(n)}\|_\infty\right).\end{aligned} \tag{6}$$

The proofs of both theorems are given in the Appendix . Similar results are possible on finite-time intervals by replacing $\|y^{(n)}\|_\infty$ by $\sup_{\xi\in[0,t]}\|y^{(n)}(\xi)\|$.

Remark 1: From (5) and (6), it is clear that the peaking phenomenon is attenuated *only* in the initial times of the estimation. In practical situations, additional perturbations may occur at any time. Therefore, peaking may as well appear after the first instants. This is the reason why, even though (4) can be implemented offline, the scheme for peaking prevention proposed in this note is not as suitable for control applications as those that guarantee a time-invariant cure of the peaking phenomenon (cf. [5], [1], and [9]). Indeed, a numerical study using our scheme was performed on the first numerical example in [1]. Our results are worse (w.r.t. the peaking phenomenon) than those given in [1]. However, if one wants a differentiator for theoretical purposes (for instance, [11]), then (3) and (4) can be a possible alternative since those differentiators are more "linear" than one with a saturation and their stability (as dynamical systems) is robust (cf. Remark 4 for the existence of appropriate quadratic Lyapunov functions).

III. Filtering the Reference Signal

A. The Continuous Case

In general, the signal y appearing in (1) is equal to $\bar{y} + w$, where $\bar{y}$ is the reference signal we want to differentiate and w is a white noise. We still would like to differentiate $\bar{y}$ and to limit the effect of w. For that purpose, we again consider the ODE (Od) $\dot{x} = Ax + H(t)(\bar{y} - Cx)$. Of course, we require that $H(t)$ tends to an EK-vector when $t \to \infty$. Moreover, we want (Od) to be the optimal trajectory of the following quadratic cost minimization problem:

$$\min \int v^TQ^{-1}v + (\bar{y} - Cx)^2 \text{ subject to } \dot{x} = Ax + v \tag{7}$$

for some $Q > 0$ to be determined. The solution of this Kalman filtering problem is given by $(d\bar{x}/dt) = A\bar{x} + \bar{P}(t)C^T(\bar{y} - C\bar{x})$, $\bar{x}(0) = 0$, where $\bar{P}(t)$ is solution of the Riccati equation $\dot{P} = AP + PA^T - PC^TCP + Q$, $P(0) = 0$. Since (A, C) is observable, $\bar{P}(t)$ admits a limit $P_\infty > 0$ as $t \to \infty$ satisfying

$$AP_\infty + P_\infty A^T - P_\infty C^TCP_\infty + Q = 0. \tag{8}$$

In order for (Od) to be a differentiator, we necessarily require that $P_\infty C^T$ is an EK-vector $H(\alpha,\gamma)$. Therefore, we are left with the choice of $Q > 0$ such that $P_\infty C^T = H(\alpha,\gamma)$. Since this is an under-determined problem, we look for a diagonal matrix solution $Q = \mathrm{diag}(q_1, \ldots, q_n)$ (or equivalently a vector $q = (q_1, \ldots, q_n)^T$). We must then solve the following algebraic problem $(AP)_c$: for a given EK-vector $H(\alpha,\gamma)$, find a vector $q = (q_1, \ldots, q_n)^T$ with positive coordinates such that, if $P_\infty > 0$ is the solution of (8), then $P_\infty C^T = H(\alpha,\gamma)$. It is clear that there is at most one solution for $(AP)_c$ and, as shown below, there indeed exists one if α satisfies the condition (C) given next: $Re(\beta_i^2) > 0$ for $1 \leq i \leq n$ where the $-\beta_i$'s are the roots of $P_{\alpha,1}$. We get the following.

Theorem 3: Let $H(\alpha,\gamma)$ be an EK-vector. Then $(AP)_c$ has a solution if and only if $H(\tilde{\alpha},\gamma^2)$ has positive coordinates, where $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)^T$ verifies $\tilde{\alpha}_i = \sum \beta_{k_1}^2 \ldots \beta_{k_i}^2$ for $1 \leq i \leq n$. In this case, $q = H(\tilde{\alpha},\gamma^2)$. If condition (C) holds, then $H(\tilde{\alpha},1)$ is an EK-vector and $(AP)_c$ has a solution.

Remark 2: The continuous differentiator $\bar{x}$ filters $y - Cx$, i.e., if $w \overset{def}{=} y - \bar{y}$, we have (F) $\|\bar{y} - Cx\|_2 \leq K_0 + K_1\|w\|_2$, with K_0, K_1 independent of γ and w. The proof of (F) is deferred to Appendix I.

Remark 3: There are examples for which: when (C) does not hold, q still has positive coordinates and when the algebraic problem $(AP)_c$ does not have a solution, there is still some $Q > 0$ (not diagonal) such that (8) holds. Therefore, considering (C) may seem arbitrary and restrictive but, checking it is trivial when the roots associated to α are available. In addition, if (C) holds, the solution Q given in Theorem 3 is characterized in a very simple way.

Remark 4: The dynamics of $\bar{x}$ and $\bar{P}$ define a continuous differentiator satisfying (6). The argument is identical to the one given in the proof of Theorem 2. Both differentiators x [given in (3)] and $\bar{x}$ can be written as dynamical systems $\dot{x} = Ax + R(t)C^T(y - Cx)$, where $R(t) > 0$ satisfies an estimate of the type $\|R(t) - R_\infty\| \leq C_0 \min(1, \gamma^{2n-1}(1 + t^{n-1})^2e^{-\gamma t})$, valid for all $t \geq 0$, with C_0 independent of t, γ and $R_\infty C^T$ equal to some EK-vector $H(\alpha,\gamma)$. Moreover, there is a quadratic time-varying Lyapunov function for the system $\dot{x} = (A - R(t)C^TC)x + u$ given by $V(x,t) = x^TR^{-1}(t)x$. Therefore, a high-gain observer almost identical to the one defined in [1] can be built as follows: replace their observer gain H (which is the EK-vector of [5]), defined in [1, eq. (9)], by $R(t)C^T$. Then, all their results with exactly the same arguments provided in [1] can be recovered with this new observer. The advantage of our approach lies in the fact that the saturation procedure can be skipped but regarding observation applications the high-gain observer of [1] is definitely more efficient for the reasons already given in Remark 1.

B. Filtering in the Discrete Case

In practical situations, the observation process is monitored only at discrete time. The system from which such observations are taken can either be continuous-time or discrete-time, but it is usual to treat the estimation problem of the output derivative as being a discrete-time problem. In this case, we think about the construction of a discrete plant playing the role of the derivative estimator. Here we return back to the classical theory of the discrete-time Kalman–Bucy filter and show that, for a particular discretization scheme, we can adapt and choose properly the weight matrices appearing in the discrete Riccati equation to force the discrete dynamics of the filter to be an asymptotic differentiator.

We discretize the performance index given by (7) and use a Euler discretization for the constraint $\dot{x} = Ax+v$. We use $\delta > 0$ as the sampling parameter. We are then led to consider the discrete-time Kalman–Bucy filter given by

$$\begin{aligned} x_{k+1} =& A_\delta x_k + K_k(y_k - Cx_k) \\ K_k =& \left(A_\delta P_k C^T\right)\left(CP_kC^T + 1\right)^{-1} \\ P_{k+1} =& A_\delta P_k A_\delta^T - K_k\left(CP_kC^T+1\right)K_k^T + Q_{\delta,\gamma} \end{aligned} \tag{9}$$

where $A_\delta = I + \delta A$, $K_k = A_\delta P_k C^T(CP_kC^T+1)^{-1}$. We know that if $Q_{\delta,\gamma} > 0$, then $\lim_{k\to\infty} P_k = P$, $\lim_{k\to\infty} K_k = K = A_\delta PC^T(CPC^T+1)^{-1}$ such that $A_\delta - KC$ is discrete-Hurwitz and P is the unique definite–positive matrix solution of

$$P = A_\delta P A_\delta^T - K\left(CPC^T+1\right)K^T + Q_{\delta,\gamma}. \tag{10}$$

Note that

$$\begin{aligned} A_\delta - KC =& K(\gamma)^{-1}(A_{\delta\gamma} - \delta\gamma H(\alpha,1)C^T)K(\gamma) \\ =& K(\gamma)^{-1}(Id_n + \delta\gamma A(\alpha))K(\gamma). \end{aligned}$$

where $A(\alpha) \neq A(\alpha,1)$. Then, $A_\delta - KC$ is discrete-Hurwitz if and only if, for every eigenvalue β_i of P_α, $|1-\delta\gamma\beta_i| < 1$, i.e., $\delta\gamma < 2\beta_m/\beta_M$. As in the previous section, we are left with an algebraic problem $(AP)_d$ namely, find a diagonal matrix $Q_{\delta,\gamma} > 0$ such that the solution of (10) verifies $K = A_\delta PC^T(CPC^T+1)^{-1} = \delta H(\alpha,\gamma)$, where $H(\alpha,\gamma)$ is some fixed EK-vector.

We use p_j and p_{lm} resp. to denote the jth column vector and the (l,m)th coefficient of P resp.. We have $(p_1/1+p_{11}) = \delta A_\delta^{-1}(\alpha_i\gamma^i)_{1\leq i\leq n}$. Since $A_\delta^{-1} = \sum_{m\geq 0}(-\delta)^m A^m$, we get, for $1 \leq l \leq n$, that $p_{1l} = (-1)^l \sum_{i=l}^n \alpha_i(-\delta\gamma)^i/\delta^{l-1}\left(1+\sum_{i=1}^n \alpha_i(-\delta\gamma)^i\right)$. We have $1 + p_{11} = (1/1+\sum_{i=1}^n \alpha_i(-\delta\gamma)^i) = (1/\prod_{i=1}^n(1-\delta\gamma\beta_i))$. Since $P > 0$ implies that $p_{11} > 0$, we necessarily have $\prod_{i=1}^n(1-\delta\gamma\beta_i) < 1$. This is the case if δ verifies $\delta\gamma < \beta_m/\beta_M$.

Remark 5: The previous equation implies that the discretization parameter δ and the high-gain γ cannot be chosen independently for this discrete filter to operate. In other words, the better one wants x_k to approximate y_k and its derivatives ($1/\gamma$ small), the finer one has to mesh the interval of observation. Even though natural, this fact appears as a drawback for our filtering scheme in the discrete case. We do not have a general explicit formula for $Q_{\delta,\gamma}$ but we can show that

Theorem 4: Assume that $\alpha = (C_n^i)_{1\leq i\leq n}^T$, i.e., all the roots of P_α are equal to one and that $\delta\gamma < 1$. Then, the matrix $Q_{\delta,\gamma}$ solution of $(AP)_d$ is given by $Q_{\delta,\gamma} = \operatorname{diag}\left(C_n^i\delta^2\gamma^{2i}/(1-\delta\gamma)^i\right)_{1\leq i\leq n}$.

IV. Conclusion

In this note, we first proved that the numerical differentiator introduced in [8] does not exhibit the peaking phenomenon, i.e., there is no unbounded overshoot in the initial times. However, there is no time-invariant cure of the peaking phenomenon in contrast with the use of a saturation (used in [1], [2], and [5]) or the projection method [9]. Our approach has then to be viewed as an alternative to the aforementioned techniques to overcome the peaking phenomenon, better suited for theoretical applications rather than practical ones. This is particularly the case for [11] where the use of our differentiator simplifies the argument. The second part of this note addresses the issue of noise disturbance. The differentiator we propose filters the reference signal while estimating the higher derivatives. In particular, if we assign negative-real roots to the steady-state feedback, the associated quadratic cost to be minimized has a very simple expression. Even though the differentiator attenuates the peaking phenomenon [compared to (1)], it does not eliminate it.

Appendix

A. Proof of Theorem 1

In this paragraph, we assume γ large enough (i.e., larger than a constant only depending on n). Set $A(\alpha,\gamma) \stackrel{def}{=} A - D_\alpha L_\infty^{-1}(\gamma)C^TC$ and $A(\alpha) \stackrel{def}{=} A(\alpha,1)$. We have $K(\gamma)A(\alpha,\gamma)K(\gamma)^{-1} = \gamma A(\alpha)$. Let $\eta(s) \stackrel{def}{=} K(\gamma)\left(x(t) - \widetilde{Y}(t)\right)$, where $s = \gamma t$. Note that $A - D_\alpha L_\infty^{-1}C^TC = A(\alpha)$. We have $(d\eta/ds) = A(\alpha)\eta + g(s)\eta - (y^{(n)}/\gamma^{n+1})c_n$, where $g(s) \stackrel{def}{=} D_\alpha L_\infty^{-1/2}\left(Id_n + G(s)\right)^{-1} G(s) L_\infty^{-1/2}$ with $G(s) > 0$ given by $G(s) := L_\infty^{-1/2}e^{-A_1^Ts}((K(\gamma)^{-2}/\gamma) - L_\infty^{-1})e^{-A_1s}L_\infty^{-1/2}$. For some $C_0 > 0$ independent of γ and $s \geq 0$, $\|G(s)\| \leq C_0\gamma^{2n-1}(1+s^{n-1})^2e^{-s}$. Since $G(s) > 0$, then $\left(Id_n + G(s)\right)^{-1} G(s) > 0$. Therefore, for every $x \in \mathbb{R}^n$, $x^T\left(Id_n + G(s)\right)^{-1} G(s)x \leq \min(x^Tx, x^TG(s)x)$, since, for every $z \in \mathbb{R}^+$, $(z/1+z) \leq \min(1,z)$. Then $\|g(s)\| \leq C_0\min(1,\gamma^{2n-1}(1+s^{n-1})^2e^{-s})$. Consider $V(\eta) := \eta^TP^*\eta$, where P^* is the solution of the Lyapunov equation associated to $A(\alpha)$. Let $\dot{V} = (d/dt)V(\eta(t))$. Then

$$\dot{V} \leq -\left(C_0 - C_1\min(1,\gamma^{2n-1}(1+s^{n-1})^2e^{-s})\right)V + C_2\frac{\|y^{(n)}\|_\infty}{\gamma^{n+1}}V^{1/2}$$

where $C_i > 0$ are independent of γ. Setting $g(s,u) := \int_u^s \left(C_0 - C_1\min(1,\gamma^{2n-1}(1+\xi^{n-1})^2e^{-\xi})\right)d\xi$, we rewrite the previous equation as $\dot{V} \leq -(\partial g/\partial s)V + C_2(\|y^{(n)}\|_\infty/\gamma^{n+1})V^{1/2}$. Using Gronwall's lemma, we deduce that $V(s)^{1/2} \leq C_2(\|y^{(n)}\|_\infty/\gamma^{n+1})\int_0^s(e^{g(\xi,s)/2}d\xi)$, leading to (5).

B. Proof of Theorem 2

Let $z = x - \widetilde{Y}$. Then, the dynamics of z is $\dot{z} = (A - N(\gamma,t)C^TC)z - y^{(n)}c_n$. Define ϕ as the fundamental solution associated to $A - N(\gamma,t)C^TC$, i.e., $(\partial\phi/\partial t) = (A - N(\gamma,t)C^TC)\phi$, $\phi(t,t) = Id_n$. Then, we have $z(t) = \phi(t,0)z(0) + \int_0^t y^{(n)}(s)\phi(t,s)c_n ds$. An explicit expression for ϕ is $\phi(t,s) = N(\gamma,t)e^{-\gamma(t-s)}e^{-(t-s)(\gamma Id_n + A^T)}L(\gamma,s)$. This simply follows from the fact that $\psi(t,s) \stackrel{def}{=} L(\gamma,t)\phi(t,s)$ satisfies $(\partial\psi/\partial t) = -(\gamma Id_n + A^T)\psi$. Then, for $0 \leq s,t$, we have

$$\begin{aligned} \phi(t,s) =& e^{At}\left(Id_n + M(\gamma,t)\right)^{-1}\left(Id_n + M(\gamma,s)\right)e^{-As} \\ \text{with } M(\gamma,t) =& \int_0^t e^{\gamma\tau}e^{A^T\tau}C^TCe^{A\tau}d\tau. \end{aligned} \tag{11}$$

Then, $\phi(t,0) = e^{At}(Id_n + M(\gamma,t))^{-1}$. We have $M(\gamma,t) = \gamma K(\gamma)M(\gamma t)K(\gamma)$, where $M(t) \stackrel{def}{=} M(1,t)$. Rewrite $z(t) = z_1(t) + z_2(t)$, where

$$z_1(t) = \phi(t,0)\left(z(0) + \int_0^t y^{(n)}(s)e^{-As}c_n ds\right)$$
$$z_2(t) = \phi(t,0)\int_0^t y^{(n)}(s)M(\gamma,s)e^{-As}c_n ds. \qquad (12)$$

For $t \geq 0$, we have $S(\gamma,t) = e^{-A_\gamma^T t}M(\gamma,t)e^{-A_\gamma t}$. Then $S(\gamma,t) = \gamma K(\gamma)S(\gamma t)K(\gamma)$, where $S(t) \stackrel{def}{=} S(1,t)$. For $t \geq 0$ and $x \in \mathbb{R}^n$, we have $x^T(Id_n + M(\gamma,t))^{-1}x \leq \min(x^T x, x^T M(\gamma,t)^{-1}x)$. Therefore, there exists $C_0 > 0$ independent of γ such that for every $t \geq 0$ we have

$$\|\phi(t,0)\| \leq C_0 \min(1, \gamma^{2n-1}\exp(-\gamma t)(1+t^{n-1})^2). \qquad (13)$$

From (12) and (13)

$$\|z_1(t)\| \leq C_0 \min(1, \gamma^{2n-1}\exp(-\gamma t)(1+t^{3n-2}))\,(\|z(0)\| + \|y^{(n)}\|_\infty).$$

To estimate z_2, remark that $Id_n + M(\gamma,t) = e^{A_\gamma^T s}(S_1(\gamma,s) + S(\gamma,s) + M(\gamma,t-s))e^{A_\gamma s}$ where $S_1(\gamma,s) \stackrel{def}{=} e^{-A_\gamma^T s}e^{-A_\gamma s}$. Therefore, the first equation shown at the bottom of the page holds true. Changing the integration time s to $\gamma(t-s)$ and setting $X \stackrel{def}{=} \gamma t$, $y_n(s) = y^{(n)}(s/\gamma)$, we have (14), as shown at the bottom of the page, with $W(s) \stackrel{def}{=} 1/\gamma e^{-A_1^T s}K(\gamma)^{-2}e^{-A_1 s}$. Let

$$R(X) \stackrel{def}{=} \int_0^X \|e^{As/\gamma}(W(X-s) + S(X-s) + M(s))^{-1}S(X-s)\|ds.$$

Since $W, S, M > 0$ for $t \geq 0$ and $S(X) < L_\infty$ for $X \geq 0$, we get $\|W(s)^{-1}\| \leq (C_0/\gamma)e^s(1+s^{2n-1})$. For $0 \leq s \leq X$, $(W(X-s) + S(X-s) + M(s))^{-1} \leq W^{-1}(X-s)$. Therefore, $R(X)$ is bounded above independently of γ for $X \leq (1/2)\ln(\gamma)$. For $X \geq (1/2)\ln(\gamma)$, the third equation shown at the bottom of the page holds. For $X \geq (1/2)\ln(\gamma)$

$$\|(W(X-s) + S(X-s) + M(s))^{-1}\| \leq \|M^{-1}(s)\| \leq C_0 e^{-s/2}$$

since $M(s) > C_0 e^{s/2}$. Then, $R(X)$ is bounded above independently of γ. For $1 \leq i \leq n$, $\|(z_2)_i\|_\infty \leq (C_0/\gamma^{n+1-i})\|y^{(n)}\|_\infty$ and Theorem 2 follows.

C. *Proof of Theorem 3*

Proof:

We use p_k and p_{lk}, respectively, to denote the kth column vector and the (l,k)-th coefficient of P_∞, respectively. Note that $Ac_i = c_{i-1}$ and $A^T c_i = c_{i+1}$ with the convention that $c_i = 0$ if $i < 1$ or $i > n$. Multiplying (8) on the right by c_i, we get $p_{i+1} = -Ap_i + (p_1^T p_i) - q_i c_i$, which implies that $p_k = (-A)^{k-1}p_1 + \sum_{j=1}^{k-1}(-A)^{k-1j}(p_{j,1}p_1 - q_j c_j)$. We then have $p_{lk} = (-1)^{k-1}p_{l+k-1,1} + \sum_{j=1}^{k-1}(-1)^{k-1-j}(p_{j1}p_{l+k-1-j,1} - q_j\delta_{l+k-1-j,j})$. Set $p_{lk} = 0$ for $l,k > n$ above and $p_{01} = \alpha_0 = 1$. Since $P_\infty > 0$, by taking $|l-k| = 1$ and $l+k-1 = 2m$, we obtain, for $1 \leq m \leq n$, $q_m = (-1)^m \sum_{j=0}^{2m} p_{2m-j,1}p_{j1}$. Finally, by comparing the expressions listed in the fourth equation shown at the bottom of the page, we conclude that $q_m = \gamma^{2m}\sum \beta_{k_1}^2 \ldots \beta_{k_m}^2 = \gamma^{2m}\tilde{\alpha}_m$. Conversely, if $q = H(\tilde{\alpha},\gamma^2) = K(\gamma)H(\tilde{\alpha},1)K(\gamma)$ has positive coordinates, we must show that $PC^T = H(\alpha,\gamma)$ where $P > 0$ is the solution of the Riccati (8). Let p be the polynomial defined by $p(X) := X^n + \sum_{i=1}^n (PC^T)_i X^{n-i}$. This polynomial is Hurwitz (since $A - PC^T C$ is Hurwitz) and satisfies $p(X)p(-X) = P_{\alpha,\gamma}(X)P_{\alpha,\gamma}(-X)$. Then, $p = P_{\alpha,\gamma}$ which implies

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$$z_2(t) = \int_0^t y^{(n)}(s)e^{A(t-s)}(S_1(\gamma,s) + S(\gamma,s) + M(\gamma,t-s))^{-1}S(\gamma,s)c_n ds.$$

$$z_2(t) = \frac{K(\gamma)^{-1}}{\gamma^{n+1}}\int_0^X y_n(X-s)e^{As/\gamma}(W(X-s) + S(X-s) + M(s))^{-1}S(X-s)c_n ds \qquad (14)$$

$$R(X) = R\left(\frac{1}{2}\ln(\gamma)\right) + \int_{1/2\ln(\gamma)}^X \left\|e^{As/\gamma}(W(X-s) + S(X-s) + M(s))^{-1}S(X-s)\right\| ds.$$

$$P_\alpha(X)P_\alpha(-X) = \begin{cases} (-1)^n \left(\sum_{k=0}^n \alpha_k X^{n-k}\right)\left(\sum_{k=0}^n (-1)^k \alpha_k X^{n-k}\right), \\ (-1)^n \prod_{k=1}^n (X^2 - \beta_k^2), \\ (-1)^n \sum_{k=0}^n (-1)^k \tilde{\alpha}_k X^{2(n-k)} \end{cases}$$

that $PC^T = H(\alpha, \gamma)$. Finally, if α satisfies condition (C), then the q_i's are positive and that finishes the proof.

D. Proof of (F)

We rescale the quadratic cost-minimization problem with $z = K(\gamma)x$ and $s = \gamma t$. The constraint (7) becomes $\min \int u^T \bar{Q}^{-1} u + (\tilde{y} - Cz)^2$ subject to $\dot{z} = Az + u$, where $u = (1/\gamma)K(\gamma)v$, $\bar{Q} = K(\gamma)QK(\gamma)$ with $Q = \mathrm{diag}(H(\tilde{\alpha}, \gamma))$, i.e., $\bar{Q} = \mathrm{diag}(H(\tilde{\alpha}, 1))$ (independent of γ) and $\tilde{y}(s) = (1/\gamma)\bar{y}(t)$. Then, the differentiator $\bar{x}$ is transformed to $(dz/ds) = Az + \bar{P}(s)C^T(\tilde{y} - Cz)$, $z(0) = 0$ and $(d\bar{P}/ds) = A\bar{P} + \bar{P}A^T - \bar{P}C^TC\bar{P} + \bar{Q}$, $\bar{P}(0) = 0$, where $\bar{P}(s) = \gamma K(\gamma)P(t)K(\gamma)$. We also get $A\bar{P}_\infty + \bar{P}_\infty A^T - \bar{P}_\infty C^T C \bar{P}_\infty + \bar{Q} = 0$ and $\bar{P}_\infty C^T = H(\alpha, 1)$. Set $\bar{z}(s) = K(\gamma)\bar{x}(t)$ and $u_y(s) = (1/\gamma)K(\gamma)w(t)$. We have $\|\tilde{y} - Cz\|_2 \leq \|\tilde{y} - C\bar{z}\|_2 + \|C(z - \bar{z})\|_2, \gamma\|\tilde{y} - C\bar{z}\|_2 = \|\bar{y} - C\bar{x}\|_2$ and $\|z - \bar{z}\|_2 \leq K_1\|u_y\|_2$ with K_1 independent of γ. In addition, $\|\bar{z}\|_2 \leq K_1\|\bar{y}\|_2$. Then, ($F$) follows at once.

E. Proof of Theorem 4

We follow the lines of the proof of Theorem 3. Since A_δ^{-1} and A commute, for $0 \leq j \leq n-1$, $p_{j+1} = -A_\delta^{-1}Ap_j - (q_j/\delta)A_\delta^{-1}c_j + \alpha_k \gamma^j p_1$. Then

$$p_k = \sum_{j=0}^{k-1} (-A_\delta)^{k-1-j} A^{k-1-j} \alpha_j \gamma^j p_1 + \sum_{j=1}^{k-1} (-A_\delta)^{k-j} A^{k-1-j} c_j \frac{q_j}{\delta}$$

for $1 \leq k \leq n$ and, for $1 \leq l, k \leq n$

$$p_{lk} = \sum_{j=0}^{k-1} \sum_{s \geq k-2-j} (-1)^{k-1-j} (-\delta)^s C_s^{k-2-j} \alpha_j \gamma^j p_{1,l+k-1+s-j} + \sum_{j=1}^{k-1} (-1)^{k-j} (-\delta)^{2j-l-k+1} C_{2j-l-k+1}^{k-1-j} \frac{q_j}{\delta}.$$

For every $1 \leq m \leq n$, consider $p_{m,m+1}$ and $p_{m+1,m}$. Since $P > 0$, we have

$$\frac{q_m}{\delta} = \sum_{j=0}^{m} \sum_{s \geq j-1} (-1)^j p_{1,m+s+j} (-\delta)^s C_s^{j-1} \alpha_{m-j} \gamma^{m-j} + \sum_{j=1}^{m} \sum_{s \geq j-2} (-1)^j p_{1,m+s+j} (-\delta)^s C_s^{j-2} \alpha_{m-j} \gamma^j. \quad (15)$$

Assume now that α and γ verify the hypothesis of Theorem 4. We get $p_{1l} = ((-1)^l \sum_{i=l}^n C_n^i (-\delta\gamma)^i)/(\delta^{l-1}(1-\delta\gamma)^n)$, and

$$q_m = \frac{\delta^2 \gamma^{2m}}{(1-\delta\gamma)^n} \times \left[\sum_{0 \leq s \leq k \leq n-m} (\delta\gamma)^s (-1)^k C_n^{m+k} C_n^{m+s-k} \left(C_k^s + C_{k-1}^s\right) \right]. \quad (16)$$

We want to show that the expression between brackets in (16) is equal to $C_n^m (1-\delta\gamma)^{n-m}$. Both quantities are polynomials in the variable $\delta\gamma$. To show they are equal, we must establish the equality of their coefficients, i.e., for $0 \leq s \leq n-m$

$$(-1)^s C_n^m C_{n-m}^s = \sum_{s \leq k \leq n-m} (-1)^k C_n^{m+k} C_n^{m+s-k} (C_k^s + C_{k-1}^s)$$

i.e., for $0 \leq s \leq m \leq n$

$$(CC): (-1)^s C_n^m C_m^s = \sum_{0 \leq s \leq k \leq m} (-1)^k C_n^{m-k} C_n^{m-s+k} (C_k^s + C_{k-1}^s.$$

Let $L(n,m,s)$ and $R(n,m,s)$ be the left- and right-hand side, respectively, of (CC). The $L(n,m,s)$'s satisfy the following recurrence relation:

$$L(n+1,m,s) = L(n,m,s) + L(n,m-1,s) - L(n,m-1,s-1). \quad (17)$$

Therefore, using an induction argument, the proof of (CC) reduces in showing that $R(n,m,s)$ satisfies the recurrence relation defined in (17). We first consider $R(n,m-1,s-1)$. Since $C_k^{s-1} + C_{k-1}^{s-1} = C_{k+1}^s + C_k^s - (C_k^s + C_{k-1}^s)$, and

$$\sum_{s-1 \leq k \leq m-1} (-1)^{k+1} \left(C_k^{s-1} + C_{k-1}^{s-1}\right) C_n^{m-k-1} C_n^{m-s+k} = \sum_{s \leq k \leq m} (-1)^k \left(C_k^s + C_{k-1}^s\right) C_n^{m-k} C_n^{m-s+k-1}$$

we rewrite $R(n,m,s) + R(n,m-1,s) - R(n,m-1,s-1)$ as the equation shown at the bottom of the page. The term between brackets is equal to

$$C_n^{m-k}(C_n^{m-s+k} + C_n^{m-s+k-1}) + C_n^{m-k-1}(C_n^{m-s+k-1} + C_n^{m-s+k})$$

then simplified to $C_n^{m-k} C_{n+1}^{m-s+k} + C_n^{m-k-1} C_{n+1}^{m-s+k} = C_{n+1}^{m-k} C_{n+1}^{m-s+k}$. Therefore

$$\begin{aligned} R(n,m,s) &+ R(n,m-1,s) - R(n,m-1,s-1) \\ &= \sum_{s \leq k \leq m} (-1)^k (C_k^s + C_{k-1}^s) C_{n+1}^{m-k} C_{n+1}^{m-s+k} \\ &= R(n+1,m,s). \end{aligned}$$

■

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$$\sum_{s \leq k \leq m} (-1)^k (C_k^s + C_{k-1}^s) \left[C_n^{m-k} C_n^{m-s+k} + C_n^{m-k-1} C_n^{m-s+k-1} + C_n^{m-k} C_n^{m-s+k-1} + C_n^{m-k-1} C_n^{m-s+k} \right].$$

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