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# Note

# Edge separators for graphs of bounded genus with applications

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*Abstract*

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We prove that every $n$-vertex graph of genus $g$ and maximal degree $k$ has an edge separator of size $O(\sqrt{gkn})$. The upper bound is best possible to within a constant factor. This extends the known results on planar graphs and similar results about vertex separators. We apply the edge separator to the isoperimetric number problem, graph embeddings and lower bounds for crossing numbers.

## 1. Introduction

Many divide-and-conquer algorithms on graphs are based on finding a small set of vertices or edges whose removal divides the graph roughly in half. Applications include VLSI layouts [14], Gaussian elimination [15] and graph embeddings [17].

Formally, a class of graphs has $f(n)$ vertex (edge) separator if every $n$-vertex graph in the class has a vertex (edge) cutset of size $f(n)$ that divides the graph into two parts having no more than $2n/3$ vertices. Lipton and Tarjan [16] proved that planar graphs have $O(\sqrt{n})$ vertex separator. The genus of a graph is the minimum number of handles that must be added to a sphere so that the graph can be embedded in the resulting sphere with no crossing edges. Djidjev [7] and Gilbert et al. [9] proposed

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algorithms that find an $O(\sqrt{gn})$ vertex separator for graphs of genus $g$. Further generalization was done in [20, 23] and recently in [19]. Miller [18] and Diks et al. [4] showed that every $n$-vertex planar graph of maximal degree $k$ has an $O(\sqrt{kn})$ edge separator. Extensions of these results can be found in [8].

In this paper we prove that any $n$-vertex graph of positive genus $g$ and maximal degree $k$ has an $O(\sqrt{gkn})$ edge separator. This bound is best possible to within a constant factor. The separator can be found in $O(g+n)$ time provided that we start with an embedding of the graph in its genus surface. We apply the edge separator to the isoperimetric problem, to efficient embeddings of graphs of genus $g$ into various classes of graphs, including trees, meshes and hypercubes, and to showing lower bounds on crossing numbers of $K_n$, $K_{m,n}$ and $Q_n$ drawn on surfaces of genus $g$.

## 2. Separation of graphs of genus $g$

### 2.1. Upper bound

We prove a stronger "weighted" version of the edge separator theorem mentioned in the introduction. Before proving it, we state some notions and an important lemma.

Let $G=(V,E)$ be an $n$-vertex graph of genus $g>0$ and maximal degree $k$ whose vertices have nonnegative weights summing to 1 such that no weight exceeds 2/3. Let us denote the sum of the weights of vertices belonging to a set $X$ as weight($X$). Let $|E|=m$.

**Lemma 2.1** (Djidjev [6]). *If $G$ has a breadth-first search spanning tree of radius $r$ and rooted in a vertex $t$ then there is a partition of $V$ into three sets $A$, $B$, $C$ such that no edge joins a vertex in $A$ with a vertex in $B$, $\text{weight}(A)\leqslant 2/3$, $\text{weight}(B)\leqslant 2/3$ and $|C|\leqslant(4g+2)r+1$, where $t\in C$.*

**Theorem 2.2.** *There is a partition of $V$ into sets $A$, $B$ and a set of edges $D$ such that $\text{weight}(A)\leqslant 2/3$, $\text{weight}(B)\leqslant 2/3$, $|D|\leqslant 5\sqrt{3gkn}$ and every edge between $A$ and $B$ belongs to $D$.*

**Proof.** Let $G$ be connected and $t$ be an arbitrary vertex of $G$. Suppose that $G$ has a breadth-first search spanning tree of radius $r$ rooted in $t$. According to Lemma 2.1, there is a set of vertices $C$ separating $G$ such that $|C|\leqslant(4g+2)r+1$.

Assume that $r<\sqrt{m/((4g+2)k)}$. Let $D$ be the set of all edges incident to the set $C$. Hence,

$$|D|<\left(\sqrt{\frac{m}{(4g+2)k}}(4g+2)+1\right)k<(\sqrt{6}+1)\sqrt{gkm}.$$

By distributing properly the set $C$ between $A$ and $B$, we receive the desired partition.

If $r \geqslant \sqrt{m/((4g+2)k)}$ then we set $s=\sqrt{m/((4g+2)k)}$. We can assume $s>1$; otherwise, $m \leqslant (4g+2)k$ and we have a trivial edge separator of size $m=\sqrt{m}\sqrt{m} \leqslant \sqrt{6gkm}$. Partition the vertices of $G$ into levels $U_0, U_1, \ldots, U_r$ according to their distance from $t$. Define

$$L_j = \{(u,v) \mid u \in U_{i-1},\ v \in U_i,\ i=j \bmod s,\ i=1,2,3,\ldots,r\}.$$

As $|\bigcup_{j=0}^{s-1} L_j| \leqslant m$ and $|L_i \cap L_j|=0$, for $i \neq j$, there exists $s_0$ such that $|L_{s_0}| \leqslant m/s$. By removing the edges of $L_{s_0}$, the graph $G$ is partitioned into connected components $G_i=(V_i,E_i)$, $i=1,2,3,\ldots$

If $\text{weight}(V_i) \leqslant 2/3$ for all $i$ then we easily combine the desired partition $A$, $B$. Set $D=L_{s_0}$. Thus,

$$|D| \leqslant \frac{m}{s} \leqslant \frac{m}{\sqrt{m/((4g+2)k)}} = \sqrt{(4g+2)km} \leqslant \sqrt{6gkm}.$$

Let $\text{weight}(V_i) > 2/3$ for some $i$. Let $h(l)$ be the highest (lowest) level of vertices in $V_i$. Delete the vertices of $G$ at levels $>h$. Shrink the vertices of $G$ at levels $<l$ into the root $t$. The result is a graph $H_i$ of genus $\leqslant g$, $|V_i|+1$ vertices and radius $s$. Apply Lemma 2.1 to the graph $H_i$. We obtain a partition of $H_i$ into sets $A_i$, $B_i$ and $C_i$ such that

$$\text{weight}(A_i) \leqslant \tfrac{2}{3}(\text{weight}(V_i)+\text{weight}(t)),$$
$$\text{weight}(B_i) \leqslant \tfrac{2}{3}(\text{weight}(V_i)+\text{weight}(t))$$

and

$$|C_i| \leqslant (4g+2)s+1.$$

Delete $t$ from $H_i$. Let $D_i$ denote the set of edges incident to $C_i$. Removing $D_i$ from $G_i$, we partition $V_i$ into two sets having weights $\leqslant 2/3$. Thus, we have divided $G$ into components whose weights do not exceed 2/3. One can easily combine the components into the desired partition $A$ and $B$. The total number of deleted edges is

$$|D| = |L_{s_0}| + |D_i| \leqslant \frac{m}{s} + (4g+2)ks = 2\sqrt{(4g+2)km} \leqslant 2\sqrt{6gkm}.$$

Finally, suppose $g \leqslant n/48$. Then

$$|D| \leqslant 2\sqrt{6gkm} \leqslant 2\sqrt{6gk(3n+6g)} \leqslant \sqrt{75gkn}.$$

If $g \geqslant n/48$ then we have a trivial separator of size

$$m=\sqrt{m}\sqrt{m} \leqslant \sqrt{3n+6g}\sqrt{\frac{kn}{2}} \leqslant \sqrt{75gkn}.$$

In the case where $G$ is not connected, we apply the above procedure to the component with the greatest weight. $\square$

Our proof can be directly transformed to an algorithm for finding the edge separator. Provided that we start with an embedding of $G$ in its genus surface, the time complexity is $O(m)=O(g+n)$ because finding both the set $L_{s_0}$ and the vertex cut from Lemma 2.1 requires $O(g+n)$ time [6].

For some applications, it can be useful to have an edge cut that divides the graph into two parts whose numbers of vertices differ at most by 1. Such edge cuts are called bisectors.

**Corollary 2.3.** *Any n-vertex graph of genus g and maximal degree k has a bisector of size* $48\sqrt{gkn}$.

The proof follows the method used in Corollary 3 of [16].

### 2.2. Lower bound

In this section we prove that the bound in Theorem 2.1 is tight to within a constant factor whenever $gk=O(n)$. We show this for the unweighted version of Theorem 2.1, i.e. all vertices have the same weight. We use essentially the following lemma.

**Lemma 2.4** (Gilbert et al. [9]). *There exists a constant* $\alpha$ *such that, for infinitely many* $g$, $n_0$, $g<n_0$, *there is a regular graph* $G_0$ *with* $n_0$ *vertices, genus* $g$ *and of degree 6 whose every vertex cut dividing* $G_0$ *into parts having* $\leqslant 3/4$ *vertices has size* $\geqslant \alpha\sqrt{gn_0}$.

**Corollary 2.5.** *Every edge cut that divides* $G_0$ *into parts having* $\leqslant 3n/4$ *vertices has size at least* $\alpha\sqrt{gn_0}$.

**Proof.** Let the claim be false. Hence, there is an edge cut $D$ of $G_0$ of size $|D|<\alpha\sqrt{gn_0}$ that partitions the vertices of $G_0$ in $A$ and $B$, $|A|,|B|\leqslant 3n/4$. Let $|A|\leqslant|B|$. Delete all vertices that are incident to edges from $D$ and belong to $B$. We have constructed a vertex cut of size $<\alpha\sqrt{gn_0}$ that divides $G_0$ into parts having $\leqslant 3n/4$ vertices. □

**Theorem 2.6.** *For* $k=0 \bmod 6$ *and infinitely many* $g$, $n$, $gk<2n$, *there is a graph* $G$ *of* $n$ *vertices, genus* $g$ *and maximal degree* $k$ *such that every edge separator of* $G$ *has size* $\Omega(\sqrt{gkn})$.

**Proof.** Consider the graph $G_0$ from Lemma 2.2. Replace each edge of $G_0$ by $k/6$ new parallel edges. Put one new vertex on each new edge. We get a graph $G=(V,E)$ of genus $g$, maximal degree $k$ and with $n$ vertices, where $n=n_0k/2+n_0$. Consider a minimal edge separator of $G$, i.e. we have a partition of $V$ in $A$, $B$ such that $|A|\leqslant 2n/3$, $|B|\leqslant 2n/3$ and every edge between $A$ and $B$ belongs to a set $D$. Our aim is to prove that $|D|=\Omega(\sqrt{gkn})$. Let $V_0\subset V$ denote the set of vertices that correspond to the vertices of $G_0$. Denote $A_1=A\cap V_0$, $B_1=B\cap V_0$. It holds that $|A_1|+|B_1|=n_0$. Assume

$|A_1| \leqslant |B_1|$. We distinguish two cases:

(1) $|A_1| \geqslant n_0/4$. Then $|B_1| \leqslant 3n_0/4n$. According to Corollary 2.2, $G$ contains at least $\alpha\sqrt{gn_0}$ tuples $(u, v)$ such that $u \in A$, $v \in B$ and $u$ and $v$ are joined by $k/6$ paths of length 2. From each such path, at least one edge must belong to $D$. Hence,

$$|D| \geqslant \alpha \frac{k}{6}\sqrt{gn_0} = \alpha \frac{k}{6}\sqrt{\frac{2gn}{k+2}} \geqslant \frac{\alpha\sqrt{6}}{12}\sqrt{gkn}.$$

(2) $|A_1| < n_0/4$. We estimate the number of vertices in $A - A_1$ that have a neighbour in $B_1$. Each such vertex together with the neighbour defines an edge that must belong to $D$. Hence,

$$|D| \geqslant |A - A_1| - |\{\text{vertices of } A - A_1 \text{ that have both neighbours in } A_1\}|$$

$$\geqslant \frac{n}{3} - |A_1| - |A_1|\frac{k}{2} > \frac{n}{3} - \frac{n_0}{4}\left(1 + \frac{k}{2}\right) = \frac{n}{12} \geqslant \frac{\sqrt{2}}{24}\sqrt{gkn}. \qquad \square$$

## 3. Applications

In this section we apply the edge separator to the isoperimetric problem, to graph embeddings and to finding lower bounds for crossing numbers of complete, bipartite and hypercube graphs drawn on a surface of genus $g$.

### 3.1. *Isoperimetric number*

The isoperimetric number $i(G)$ of a graph $G = (V, E)$ is defined as

$$i(G) = \min\left\{\frac{|\delta(X)|}{|X|} : X \subset V,\ 1 \leqslant |X| \leqslant \frac{|V|}{2}\right\},$$

where $\delta(X)$ is a set of edges having one edge in $X$ and the other in $V - X$. The quantity $i(G)$ is a discrete analog of the well-known Cheeger [2] isoperimetric constant measuring the minimal possible ratio between the size of the surface and the volume of a geometric figure. Isoperimetric numbers for important graphs are computed in [21]. Boshier [1] proved that if $G$ is an $n$-vertex graph of genus $g$ and maximal degree $k$ then

$$i(G) \leqslant \frac{3(g+2)k}{\sqrt{n/2 - 3(g+2)}}$$

for $n > 18(g+2)^2$. Our edge separator immediately implies the following improvement.

**Theorem 3.1.**

$$i(G) \leqslant 15\sqrt{\frac{3gk}{n}}.$$

**Proof.** Let $A$, $B$ and $D$ be the sets from the unweighted version of Theorem 2.2. Clearly,

$$i(G)=\min_{X\subset V}\frac{|\delta(X)|}{|X|}\leqslant\frac{|D|}{\min\{|A|,|B|\}}\leqslant 15\sqrt{\frac{3gk}{n}}.\quad\square$$

*3.2. Graph embeddings*

Many computational problems can be mathematically formulated as the graph embedding, e.g. representing some kind of data structure by another data structure [17], simulation of interconnection networks of parallel computers [22] and laying out circuits in standard format [14].

Let $G_1=(V_1,E_1)$, $G_2=(V_2,E_2)$ be graphs such that $|V_1|\leqslant|V_2|$. An embedding of $G_1$ into $G_2$ is a couple of mappings $(\phi,\psi)$ satisfying

$\phi: V_1\to V_2$ is an injection,
$\psi: E_1\to\{$set of all simple paths in $G_2\}$,

such that if $(u,v)\in E_1$ then $\psi((u,v))$ is a path between $\phi(u)$ and $\phi(v)$.

We shall study the following measures of the quality of the embedding:

$$\mathrm{adil}(\phi,\psi)=\sum_{e\in E_1}\frac{|\psi(e)|}{|E_1|},$$

where $|\psi(e)|$ denotes the length of the path $\psi(e)$,

$$\exp(\phi,\psi)=\frac{|V_2|}{|V_1|},$$

$$\mathrm{cg}(\phi,\psi)=\max_{e\in E_2}|\{f\in E_1: e \text{ belongs to } \psi(f)\}|.$$

The above measures are called the average dilation, expansion and congestion, respectively.

Lipton and Tarjan [17] pointed out that edge separators can be used to embed graphs with a small average dilation. Applying their method, Diks et al. [4, 5] proved the following results on graph embeddings.

**Theorem 3.2.** *Every n-vertex planar graph of maximal degree k can be embedded in a path, 2-dimensional mesh, d-dimensional mesh* ($d\geqslant 3$), *complete binary tree and a hypercube, with average dilations* $\Theta(\sqrt{kn})$, $\mathrm{O}(\sqrt{k}\log(n/k))$, $\Theta(d\sqrt[d]{k})$, $\Theta(\log k)$, $\mathrm{O}(\log k)$, *and expansion* $\mathrm{O}(1)$.

Using the same method we can extend this result as follows.

**Theorem 3.3.** *Every n-vertex graph of genus g and maximal degree k can be embedded in a path, 2-dimensional mesh, d-dimensional mesh* ($d>2$), *complete binary tree and a hypercube, with average dilations* $O(\sqrt{gkn})$, $O(\sqrt{gk}\log(n/k))$, $O(d\sqrt{g}\sqrt[d]{k})$, $O(\sqrt{g}\log k)$ *and* $O(\sqrt{g}\log k)$, *and expansion* $O(1)$.

For the first average dilation we show an optimal lower bound. We use a method of [3].

**Lemma 3.4.** *Let* $|V_1|=n$ *and* $G_2$ *be an n-vertex path. Let* $f_p(n)$ *be the size of the minimal edge cut that divides* $G_1$ *into two parts having exactly p and* $n-p$ *vertices. Then*

$$\operatorname{adil}(\phi,\psi)\geqslant\frac{1}{|E_1|}\sum_{i=1}^{n/2}f_i(n).$$

**Theorem 3.5.** *Any embedding* $(\phi,\psi)$ *of the graph G constructed in Theorem* 2.6 *into the n-vertex path requires*

$$\operatorname{adil}(\phi,\psi)=\Omega(\sqrt{gkn}).$$

**Proof.** Let $\beta$ be the constant behind the $\Omega$ in Theorem 2.6. Then, setting $G_1=G$ in Lemma 3.4, we get

$$\operatorname{adil}(\phi,\psi)\geqslant\frac{1}{|E|}\sum_{i=1}^{n/2}f_i(n)\geqslant\frac{1}{|E|}\sum_{i=n/3}^{n/2}f_i(n)\geqslant\left(\frac{n}{2}-\frac{n}{3}\right)\beta\sqrt{gkn}\,\frac{k+2}{2kn}=\Omega(\sqrt{gkn}).$$

□

In case $G_2$ is a path, the minimal congestion is usually called cutwidth. This notion has applications in VLSI design. Yannakakis [24] stated an open problem to find a good approximation to the cutwidth of planar graphs. This was partially solved in [4]. We extend the result for graphs of positive geni.

**Theorem 3.6.** *Any n-vertex graph of positive genus g and maximal degree k has a cutwidth of* $O(\sqrt{kgn})$. *This bound cannot be improved in general.*

**Proof.** By breaking the graph recursively into roughly equal parts using the edge separator and embedding the parts into subpaths, one can easily estimate the cutwidth by

$$\sqrt{75kg}\left(\sqrt{n}+\sqrt{\frac{2}{3}n}+\sqrt{\left(\frac{2}{3}\right)^2 n}+\cdots\right)=O(\sqrt{gkn}).$$

Because the cutwidth of a graph is not smaller than the size of its minimal edge separator, the graph from Theorem 2.6 has a cutwidth at least $\Omega(\sqrt{gkn})$. □

### 3.3. Lower bounds on crossing numbers

In this subsection we apply Theorem 2.2 to showing lower bounds for the crossing numbers of complete, bipartite and hypercube graphs drawn in an orientable surface of genus $g$.

The orientable surface $S_g$ of genus $g$ is obtained from a sphere by adding $g$ handles. The crossing number $\mathrm{cr}_g(G)$ of a graph $G$ is defined as the least number of crossings when $G$ is drawn in $S_g$. Very little is known on $\mathrm{cr}_g(G)$. In [10, 11] it is proved that

$$\mathrm{cr}_1(K_n)=\Theta(n^4), \qquad \mathrm{cr}_1(K_{m,n})=\Theta(m^2n^2).$$

Kainen [12] showed that

$$\mathrm{cr}_g(Q_n)=\Theta(\gamma-g)$$

for $g\geqslant\gamma-2^{n-4}$, where $Q_n$ denotes the $n$-dimensional hypercube graph and $\gamma$ its genus.

The following theorem, which describes a lower-bound method for finding crossing numbers, was originally proved for planar graphs [13]. Our extension to graphs of genus $g$ is straightforward.

**Theorem 3.7.** *Let $G=(V,E)$ be a graph. Let* $\mathrm{mec}(G)$ *denote the size of the minimal edge cut that divides G into two parts having $\leqslant 2|V|/3$ vertices. Suppose that the class of graphs of genus $g$ and of maximal degree $k$ has an $f_{g,k}(n)$ edge separator. Then*

$$\mathrm{cr}_g(G)\geqslant f_{g,k}^{-1}(\mathrm{mec}(G))-|V|,$$

*where $f_{g,k}^{-1}$ is the inverse function to $f_{g,k}$.*

**Corollary 3.8.**

$$\mathrm{cr}_g(G)\geqslant\frac{\mathrm{mec}^2(G)}{75gk}-|V|. \tag{1}$$

**Proof.** From Theorem 2.2, we have $f_{g,k}(n)=\sqrt{75gkn}$. □

**Theorem 3.9.**

$$\mathrm{cr}_g(K_n)>\frac{n^4}{6075g}-\frac{n^3}{2}.$$

**Proof.** Define a new graph $H_n$ as follows. Consider a drawing of $K_n$ in $S_g$ with minimal number of crossings. Let $v$ be an arbitrary vertex of $K_n$. Let $u_0, u_1, u_2, \ldots, u_{n-2}$ be its neighbours. In $S_g$, find a region homeomorphic to an open disc that contains $v$ and no crossings. Denote $p=\lfloor n/2\rfloor$. Place new vertices $u_{i1}, u_{i2}, \ldots, u_{ip}$ on the edge $(u_i, v)$ so that they lie in the region for $i=0, 1, 2, \ldots, n-2$. Add edges $(u_{i,j}, u_{(i+1)\bmod (n-1),j})$ for $i=0, 1, 2, \ldots, n-2$ and $j=1, 2, 3, \ldots, p$. Delete the vertex $v$. The resulting graph $H_n$ has

$n(n-1)p$ vertices, genus $g$ and degree 4. Clearly, it holds that

$$\mathrm{cr}_g(K_n) \geqslant \mathrm{cr}_g(H_n). \tag{2}$$

It remains to find $\mathrm{mec}(H_n)$. In what follows, we show that

$$\mathrm{mec}(H_n) \geqslant \frac{2}{9} n^2. \tag{3}$$

Setting $G = H_n$ and substituting (2) and (3) into (1), we obtain the desired result. □

**Lemma 3.10.**

$$\mathrm{mec}(H_n) \geqslant \frac{2}{9} n^2.$$

**Proof.** We use a method of Leighton [13]. Recall the definition of the embedding and the congestion. Leighton proved that

$$\mathrm{mec}(G_2) \geqslant \frac{\mathrm{mec}(G_1)}{\mathrm{cg}(\phi, \psi)}.$$

Let $2K_{n(n-1)p}$ denote the complete graph on $n(n-1)p$ vertices where each edge is replaced by two new parallel edges. Set $G_2 = H_n$, $G_1 = 2K_{n(n-1)p}$. We construct an embedding $(\phi, \psi)$ of $G_1$ in $G_2$ such that

$$\mathrm{cg}(\phi, \psi) \leqslant 2(n-1)^2 p^2.$$

Noting that

$$\mathrm{mec}(2K_{n(n-1)p}) \geqslant \frac{4}{9} n^2 (n-1)^2 p^2,$$

we immediately obtain the lower bound for $\mathrm{mec}(H_n)$.

We shall construct 2 paths between any two vertices of $H_n$ so that the congestion be as small as possible. Let us call the graph induced by the vertices $u_{ij}$, $i = 0, 1, 2, \ldots, n-2$, $j = 1, 2, 3, \ldots, p$, a cobweb.

Let $u_{ij}$ and $u_{rs}$ belong to the same cobweb. Suppose $j \geqslant s$.

If $i = r$ then we join $u_{ij}$ and $u_{rs}$ by two shortest (identical) paths.

If $i \neq r$ then we join $u_{ij}$ with $u_{rs}$ by two paths $u_{ij}, u_{i+1,j}, \ldots, u_{rj}$ and $u_{ij}, u_{i-1,j}, \ldots, u_{rj}$ and prolong these paths to $u_{rs}$ as above.

Let two vertices of $H_n$ belong to different cobwebs. Let $(x, y)$ be an edge joining these cobwebs. Join $x(y)$ to the vertex belonging to the same cobweb as $x(y)$ by two paths as above. Connect the paths by adding twice the edge $(x, y)$. A simple counting analysis shows that the edges between the cobwebs are used by the maximum number of paths, which gives

$$\mathrm{cg}(\phi, \psi) \leqslant 2(n-1)^2 p^2. \qquad \square$$

Using the same approach as above, we can prove the following result.

**Corollary 3.11.**

$$\mathrm{cr}_g(K_{m,n}) > \frac{m^2n^2}{1200g} - \frac{mn(m+n)}{2}, \qquad \mathrm{cr}_g(Q_n) > \frac{4^n}{1500g} - n^2 2^{n-1}.$$

## 4. Conclusions

Our paper leaves several open questions: e.g. improving and completing the upper and lower bounds in Theorems 2.2, 2.6 and 3.3. The most interesting seems to be the problem of drawing of $K_n$, $K_{m,n}$ and $Q_n$ in the surface of orientable genus $g$. We can find drawings that have $O(\sqrt{g})$ times more crossings than the proved lower bounds.

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