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On weak and strong solutions of F-implicit generalized variational inequalities with applications

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Abstract

In this work, we study the F-implicit generalized variational inequalities in a real normed space setting. Weak solutions and strong solutions are introduced. Several existence results are derived. As an application, we study the F-implicit generalized complementarity problems and some existence results are obtained.

Keywords: F-implicit generalized variational inequalities; F-implicit generalized complementarity problems; Strong solution; Weak solution

1. Introduction and Preliminaries

Let X be an arbitrary real normed space with dual space X^*, and $(\cdot, \cdot)$ be the dual pair of X^* and X. Let X and X^* be endowed with their respective norm topologies. Let K be a nonempty closed convex set of X. The mappings $F : K \to \mathbb{R}$ and $g : K \to K$ and $T : K \to 2^{X^*}$ are given. The F-implicit generalized variational inequalities problem (F-IGVIP) is finding an $\bar{x} \in K$ such that

$$\sup_{s \in T(\bar{x})} (s, x - g(\bar{x})) \geq F(g(\bar{x})) - F(x) \tag{1.1}$$

for all $x \in K$.

A solution of (1.1) will be called a *weak solution* of the F-implicit generalized variational inequality problem. The reason for the term *weak solution* can be seen as follows. Suppose that T has compact values, that is, $T(x)$ is compact for all $x \in K$, and $\bar{x}$ is a solution of (1.1). Then since the maximum of the left-hand side of (1.1) is attained, for each $x \in K$, there is an $s \in T(\bar{x})$ such that

$$(s, x - g(\bar{x})) \geq F(g(\bar{x})) - F(x),$$

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and such an s usually depends on x. In contrast, we say that $\bar{x}$ is a *strong solution* of (1.1) if there exists an $\bar{s} \in T(\bar{x})$ such that

$$(\bar{s}, x - g(\bar{x})) \geq F(g(\bar{x})) - F(x), \qquad \text{for all } x \in K.$$

We observe that the above $\bar{s}$ does not depend on $x \in K$.

We now consider some special cases of (F-IGVIP).

(1) If T is a single-valued mapping, then the (F-IGVIP) is equivalent to the F-implicit variational inequality problem (F-IVIP) which is finding an $\bar{x} \in K$ such that

$$(T(\bar{x}), x - g(\bar{x})) \geq F(g(\bar{x})) - F(x) \tag{1.2}$$

for all $x \in K$. This problem was introduced and studied by Huang and Li [5] in a Banach space setting.

(2) If T is a single-valued mapping and g is an identity mapping, then the (F-IGVIP) is equivalent to the (F-VIP) which is to find an $\bar{x} \in K$ such that

$$(T(\bar{x}), x - \bar{x}) \geq F(\bar{x}) - F(x) \tag{1.3}$$

for all $x \in K$. This problem was introduced and studied by Stampacchia [7] in a Hilbert space setting and also investigated in [9].

(3) If $X = \mathbb{R}^n$ and $F \equiv 0$, g is an identity mapping, then the (F-IGVIP) is equivalent to finding an $\bar{x} \in K$ such that

$$\sup_{s \in T(\bar{x})} (s, x - \bar{x}) \geq 0 \tag{1.4}$$

for all $x \in K$. This problem was introduced and studied by Fang and Peterson [3], Yao and Guo [8].

(4) If $X = \mathbb{R}^n$ and $F \equiv 0$, T is a single-valued mapping and g is an identity mapping, then the (F-IGVIP) is equivalent to finding an $\bar{x} \in K$ such that

$$(T(\bar{x}), x - \bar{x}) \geq 0 \tag{1.5}$$

for all $x \in K$. This problem is known as a classical variational inequality. This problem in finite-dimensional spaces has been extensively studied in the literature. For example, see [4].

Summing up the above arguments, it has been shown that for a suitable choice of the mappings F, g, T and the space X, we can obtain a number of known classes of variational inequalities and generalized variational inequalities, implicit generalized variational inequalities. It is also well known that the variational inequality and its variants enable us to study many important problems arising in mathematical, mechanics, operations research and engineering sciences, etc.

In this work we aim to derive some existence results for weak and strong solutions of the F-implicit generalized variational inequality problem. As an application, we will study the F-implicit generalized complementarity problems and derive some existence results for such problems. Let us first recall the following results.

Berge Theorem ([1]). *Let U, V be two topological spaces, the mapping $\phi : U \times V \to \mathbb{R}$ be an upper semicontinuous function, $T : U \to 2^V$ be an upper semicontinuous mapping with nonempty compact values. Then the function $x \to \max_{s \in T(x)} \phi(s, x)$ is upper semicontinuous on U.*

Fan's Lemma ([2]). *Let K be a nonempty subset of Hausdorff topological vector space X. Let $G : K \to 2^X$ be a KKM mapping such that for any $y \in K$, $G(y)$ is closed and $G(y^*)$ is compact for some $y^* \in K$. Then there exists $x^* \in K$ such that $x^* \in G(y)$ for all $y \in K$.*

2. F-implicit generalized variational inequality problems

Now, we shall state and show our main existence results for F-implicit generalized variational inequality problems.

Theorem 2.1. *Let the mappings $F : K \to \mathbb{R}$ be lower semicontinuous, $g : K \to K$ be continuous and $T : K \to 2^{X^*}$ be upper semicontinuous with nonempty compact values, and $\eta : K \times K \to \mathbb{R}$. Suppose that*

(1) $\eta(x, x) \geq 0$ *for all $x \in K$,*

(2) *for each* $x \in K$*, there is an* $s \in T(x)$ *such that for all* $y \in K$*,*

$$\eta(x, y) - (s, y - g(x)) \leq F(y) - F(g(x)),$$

(3) *for each* $x \in K$*, the set* $\{y \in K : \eta(x, y) < 0\}$ *is convex,*

(4) *there is a nonempty compact convex subset* C *of* K*, such that for every* $x \in K \setminus C$*, there is* $y \in C$ *such that for some* $s \in T(x)$*,*

$$(s, y - g(x)) < F(g(x)) - F(y).$$

Then there exists $\bar{x} \in K$ *which is a solution of (*F*-IGVIP). Furthermore, the solution set of (*F*-IGVIP) is compact.*

Proof. Define $\Omega : K \to 2^C$ by

$$\Omega(y) = \left\{x \in C : \max_{s \in T(x)} (s, y - g(x)) \geq F(g(x)) - F(y)\right\}$$

for all $y \in K$. By the Berge Theorem, we know that the function

$$x \to \max_{s \in T(x)} (s, y - g(x)) - F(g(x))$$

is upper semicontinuous on K. Hence the set

$$\left\{x \in K : \max_{s \in T(x)} (s, y - g(x)) \geq F(g(x)) - F(y)\right\}$$

is closed in K and, for each $y \in K$, the set

$$\Omega(y) = \left\{x \in C : \max_{s \in T(x)} (s, y - g(x)) \geq F(g(x)) - F(y)\right\}$$

is compact in C because of the compactness of C.

Next, we shall claim that the family $\{\Omega(y) : y \in K\}$ has the finite intersection property; then the whole intersection $\cap_{y \in K} \Omega(y)$ is nonempty and any element in the intersection $\cap_{y \in K} \Omega(y)$ is a solution of (F-IGVIP). For any given nonempty finite subset N of K, let $C_N = co\{C \cup N\}$, the convex hull of $C \cup N$. Then C_N is a compact convex subset of K. Define the mappings $S, T : C_N \to 2^{C_N}$, respectively, by

$$S(y) = \left\{x \in C_N : \max_{s \in T(x)} (s, y - g(x)) \geq F(g(x)) - F(y)\right\},$$

and

$$T(y) = \{x \in C_N : \eta(x, y) \geq 0\},$$

for each $y \in C_N$. From the conditions (1) and (2), we have

$$\eta(y, y) \geq 0 \qquad \text{for all } y \in C_N, \tag{2.1}$$

and for each $y \in K$, there is an $s \in T(y)$ such that

$$\eta(y, y) - (s, y - g(y)) \leq F(y) - F(g(y)).$$

Hence $y \in S(y)$ for all $y \in C_N$.

We can easily see that T has closed values in C_N. Since, for each $y \in C_N$, $\Omega(y) = S(y) \cap C$, if we prove that the whole intersection of the family $\{S(y) : y \in C_N\}$ is nonempty, we can deduce that the family $\{\Omega(y) : y \in K\}$ has finite intersection property because $N \subset C_N$ and due to the condition (4). In order to deduce the conclusion of our theorem, we can apply Fan's Lemma if we claim that S is a KKM mapping. Indeed, if S is not a KKM mapping, neither is T since $T(y) \subset S(y)$ for each $y \in C_N$ by the condition (2). Then there is a nonempty finite subset M of C_N such that

$$coM \not\subset \bigcup_{u \in M} T(u).$$

Thus there is an element $\bar{u} \in coM \subset C_N$ such that $\bar{u} \notin T(u)$ for all $u \in M$, that is, $\eta(\bar{u}, u) < 0$ for all $u \in M$. By (3), we have

$$\bar{u} \in coM \subset \{u \in K : \eta(\bar{u}, u) < 0\}$$

and hence $\eta(\bar{u}, \bar{u}) < 0$ which contradicts (2.1). Hence T is a KKM mapping, and so is S. Therefore, there exists $\bar{x} \in K$ which is a solution of (F-IGVIP).

Finally, to see that the solution set of (F-IGVIP) is compact, it is sufficient to show that the solution set is closed, due to the coercivity condition (4). To this end, let S denote the solution set of (F-IGVIP). Suppose that $\{x_n\} \subset S$ which converges to some u. Fix any $x \in K$. For each n, there is an $s_n \in T(x_n)$ such that

$$(s_n, x - g(x_n)) \geq F(g(x_n)) - F(x). \tag{2.2}$$

Since T is upper semicontinuous with compact values and the set $\{x_n\} \cup \{u\}$ is compact, it follows that $T(\{x_n\} \cup \{u\})$ is compact [1]. Therefore without loss of generality, we may assume that the sequence $\{s_n\}$ converges to some s. Then $s \in T(u)$ and by taking the liminf in (2.2), we obtain

$$(s, x - g(u)) \geq F(g(u)) - F(x).$$

Hence $u \in S$ and S is closed. This completes the proof. □

When the mapping T is a single-valued mapping and X is a Banach space, Theorem 2.1 reduces to [5, Theorem 3.2].

Theorem 2.2. *Under the assumptions of Theorem 2.1, if, in addition, F is convex and $T(\bar{x})$ is convex, then $\bar{x}$ is a strong solution of F-IGVIP, that is, there exists $\bar{s} \in T(\bar{x})$ such that*

$$(\bar{s}, x - g(\bar{x})) \geq F(g(\bar{x})) - F(x)$$

for all $x \in K$. Furthermore, the set of all strong solutions of (F-IGVIP) is compact.

Proof. From Theorem 2.1, we know that $\bar{x} \in K$ such that (1.1) holds for all $x \in K$. Since $T(\bar{x})$ is compact, the supremum is attained. That is,

$$\max_{s \in T(\bar{x})} (s, x - g(\bar{x})) \geq F(g(\bar{x})) - F(x)$$

for all $x \in K$. Since $T(\bar{x})$ is also convex, by Kneser's minimax theorem [6], we have

$$\begin{aligned} &\max_{s \in T(\bar{x})} \inf_{x \in K} ((s, x - g(\bar{x})) - F(g(\bar{x})) + F(x)) \\ &\quad = \inf_{x \in K} \max_{s \in T(\bar{x})} ((s, x - g(\bar{x})) - F(g(\bar{x})) + F(x)) \geq 0. \end{aligned}$$

Therefore, there exists an $\bar{s} \in T(\bar{x})$ such that

$$(\bar{s}, x - g(\bar{x})) \geq F(g(\bar{x})) - F(x)$$

for all $x \in K$. Hence $\bar{x}$ is a strong solution of F-IGVIP. By an argument similar to that in Theorem 2.1, we can show that the set of all strong solutions of (F-IGVIP) is compact and the corresponding proof will be omitted. □

Theorem 2.3. *Let $F : K \to \mathbb{R}$ be lower semicontinuous on any nonempty compact sets and convex, $g : K \to K$ be continuous and $T : K \to 2^{X^*}$ be upper semicontinuous with nonempty compact values such that*

(1) *for each $x \in K$, there is $s \in T(x)$ such that $(s, x - g(x)) + F(x) - F(g(x)) \geq 0$,*
(2) *there is a nonempty compact convex subset C of K such that for every $x \in K \setminus C$ there is a $y \in C$ such that for some $s \in T(x)$,*

$$(s, y - g(x)) < F(g(x)) - F(y).$$

Then there exists an $\bar{x} \in K$ which is a solution of (F-IGVIP). Furthermore, the solution set of (F-IGVIP) is compact. If, in addition, $T(\bar{x})$ is also convex, then $\bar{x}$ is a strong solution of (F-IGVIP).

Proof. For any given nonempty finite subset N of K, let $C_N = co(C \cup N)$; then C_N is a nonempty compact convex subset of K. Define $S : C_N \to 2^C_N$ as in the proof of Theorem 2.1 and for each $y \in K$, let

$$\Omega(y) = \left\{ x \in C : \max_{s \in T(x)} (s, y - g(x)) + F(y) - F(g(x)) \geq 0 \right\}.$$

We note that for each $x \in K$, $S(x)$ is nonempty since $x \in S(x)$ by condition (1). By the Berge Theorem, we know that for each $x \in C_N$, $S(x)$ is closed in C_N and, for each $y \in K$, $\Omega(y)$ is compact in C. Next, we claim that the mapping S is a KKM mapping. Indeed, if not, there is a nonempty finite subset M of C_N such that $coM \not\subset \cup_{x \in M} S(x)$. Then there is an $x^* \in coM \subset C_N$ such that

$$\max_{s \in T(x^*)} (s, x - g(x^*)) < F(g(x^*)) - F(x)$$

for all $x \in M$. Since F is convex, the mapping

$$x \to \max_{s \in T(x^*)} (s, x - g(x^*)) + F(x)$$

is quasiconvex on C_N. Hence we can deduce that

$$\max_{s \in T(x^*)} (s, x^* - g(x^*)) < F(g(x^*)) - F(x^*)$$

which contradicts condition (1). Therefore, S is a KKM mapping and by Fan's Lemma we have $\cap_{x \in C_N} S(x) \neq \emptyset$. Note that for any $u \in \cap_{x \in C_N} S(x)$, we have $u \in C$ by condition (2). Hence we have

$$\bigcap_{y \in N} \Omega(y) = \bigcap_{y \in N} S(y) \cap C \neq \emptyset,$$

for each nonempty finite subset N of K. Therefore, the whole intersection $\cap_{y \in K} \Omega(y)$ is nonempty. Let $\bar{x} \in \cap_{y \in K} \Omega(y)$. Then $\bar{x}$ is a solution of (F-IGVIP). Since C is compact, the solution set of (F-IGVIP) is compact. Finally, if $T(\bar{x})$ is also convex, then by the same argument as that in Theorem 2.2, we can prove that $\bar{x}$ is a strong solution of (F-IGVIP). □

If T is a single-valued mapping and X is a Banach space, Theorem 2.2 reduces to [5, Theorem 3.4].

3. *F*-implicit generalized complementarity problems

Throughout this section, the set K is assumed to be a closed convex cone of X. We introduce the following F-implicit generalized complementarity problem (F-IGCP): Find $\bar{x} \in K$ and $\bar{s} \in T(\bar{x})$ such that

$$(\bar{s}, g(\bar{x})) + F(g(\bar{x})) = 0 \qquad \text{and} \qquad (\bar{s}, y) + F(y) \geq 0, \qquad \forall y \in K.$$

We remark that the F-implicit generalized complementarity problem covers the classical nonlinear complementarity problem and many of its variants as special cases. See, for example, [4,5,9] and the references therein.

We first establish the following equivalent relation between strong solutions of (F-IGVIP) and solutions of (F-IGCP).

Theorem 3.1. (i) *If $\bar{x}$ solves (F-IGCP), then $\bar{x}$ is a strong solution of (F-IGVIP);* (ii) *if $F : K \to \mathbb{R}$ is a positive homogeneous and convex function and $\bar{x}$ is a strong solution of (F-IGVIP), then $\bar{x}$ solves (F-IGCP).*

Proof. (i) Let $\bar{x}$ solve (F-IGCP). Then, $\bar{x} \in K$ and for some $\bar{s} \in T(\bar{x})$ we have

$$(\bar{s}, g(\bar{x})) + F(g(\bar{x})) = 0 \qquad \text{and} \qquad (\bar{s}, x) + F(x) \geq 0, \qquad \forall x \in K.$$

Hence

$$(\bar{s}, x - g(\bar{x})) \geq F(g(\bar{x})) - F(x)$$

for all $x \in K$. Thus $\bar{x}$ is a strong solution of (F-IGVIP).

(ii) Let $\bar{x}$ be a strong solution of (F-IGVIP). Then there exists $\bar{s} \in T(\bar{x})$ such that

$$(\bar{s}, x - g(\bar{x})) \geq F(g(\bar{x})) - F(x) \tag{$*$}$$

for all $x \in K$. Since $F : K \to \mathbb{R}$ is a positive homogeneous and convex function, and the set K is a closed convex cone of X, substituting $x = 2g(\bar{x})$ and $x = \frac{1}{2}g(\bar{x})$ in ($*$), we obtain

$$(\bar{s}, g(\bar{x})) \geq -F(g(\bar{x})),$$

and

$$(\bar{s}, g(\bar{x})) \leq -F(g(\bar{x})).$$

This implies that $(\bar{s}, g(\bar{x})) + F(g(\bar{x})) = 0$. Combining this result and ($*$), we have

$$(\bar{s}, x) + F(x) \geq 0, \qquad \forall x \in K.$$

Hence $\bar{x}$ is a solution of (F-IGCP). □

When T is a single-valued mapping and X is a Banach space, Theorem 3.1 reduces to [6, Theorem 3.1].

Theorem 3.2. *Let the assumptions of Theorem 2.1 hold. In addition, if $F : K \to \mathbb{R}$ is a positive homogeneous and convex function and T has convex values, then (F-IGCP) has a solution. Furthermore, the solution set is compact.*

Proof. Applying Theorems 2.2 and 3.1, we obtain the conclusion. □

Similarly, combining Theorems 2.3 and 3.1, we have the following result.

Theorem 3.3. *Let the assumptions of Theorem 2.3 hold. In addition, if $F : K \to \mathbb{R}$ is a positive homogeneous function and T has convex values, then (F-IGCP) has a solution. Furthermore, the solution set is compact.*

If T is a single-valued mapping and X is a Banach space, Theorem 3.3 reduces to [5, Theorem 3.5].

We would like to remark that the norm topologies of X and X^* considered in this work are in fact not quite necessary. The consideration of norm topologies is used only to prove that the solution set of (F-IGVIP) and the set of strong solutions are closed. As a matter of fact, we can consider X to be any topological vector space with a dual space X^*. The only assumption that we need to ensure the solution set of (F-IGVIP) and the set of strong solutions to be closed is that the pairing $(\cdot, \cdot)$ between X^* and X is continuous.

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