DIRICHLET SERIES

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The Riemann zeta-function $\zeta(s)$ and Dirichlet L-functions $L(s,\chi)$ are special cases of functions of the form

$$f(s) = \sum_{n\geq 1} \frac{a_n}{n^s} = a_1 + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \cdots,$$

where the a_n are complex numbers and s is a complex variable. Such functions are called *Dirichlet series*. We call a_1 the *constant term*.

A Dirichlet series will often be written as $\sum a_n n^{-s}$, with the index of summation understood to start at $n = 1$. Similarly, $\sum a_p p^{-s}$ runs over the primes, and $\sum a_{p^k} p^{-ks}$ runs over the prime powers *excluding* 1. (Not counting 1 as a prime power in that notation is reasonable in light of the way Dirichlet series that run over prime powers arise in practice, without a constant term.) A Dirichlet series over the prime powers that excludes the primes will be written $\sum_{p,k\geq 2} a_{p^k} p^{-ks}$.

The use of s as the variable in a Dirichlet series goes back to Dirichlet, who took s to be real and positive. Riemann emphasized the importance of letting s be complex. The convention of using σ and t for the real and imaginary parts of s seems to have become common at the beginning of the 20th century,[1] and was universally adopted through the influence of Landau's *Handbuch* [6] (1909).

Example 1. If $a_n = 1$ for all n then $f(s) = \zeta(s)$, which converges for $\sigma > 1$. It does not converge at $s = 1$.

Example 2. If $a_n = \chi(n)$ for a Dirichlet character χ, $f(s)$ is the L-function $L(s,\chi)$ and converges absolutely for $\sigma > 1$. Note $L(s,\chi_4)$ converges for real $s > 0$ since the Dirichlet series is then an alternating series. A general Dirichlet character does not take alternating values ± 1, but we'll see that as long as χ is not a trivial Dirichlet character, $L(s,\chi)$ also converges (though not absolutely) when $0 < \mathrm{Re}(s) \leq 1$.

Example 3. The series $\sum \chi_4(p)p^{-s}$, running over the primes, converges for $\sigma > 1$. Although χ_4 is an alternating function on consecutive odd integers, it is not alternating on consecutive odd primes, so it is not clear whether or not it converges if $0 < \sigma < 1$. Convergence at $s = 1$ is known, but is still unknown for any real $s < 1$, and will probably never be known by a straightforward method since it would lead to a major advance in the Riemann hypothesis for $L(s,\chi_4)$.

Example 4. For a Dirichlet series $\sum a_n n^{-s}$ we can consider $\sum \chi(n)a_n n^{-s}$ for some Dirichlet character χ. We call the latter function a *twist* of the former, by χ. So $L(s,\chi)$ is a twist of the zeta-function.

Example 5. If $a_n = n^k$ for an integer k, then $f(s) = \zeta(s-k)$ converges for $\sigma > k+1$.

[1] This notation is not due to Riemann.

Example 6. If $a_n = 1/n^n$ then $f(s)$ converges for all s. Unlike power series, no Dirichlet series that arises naturally converges on the whole complex plane, so you should not regard examples of this sort as important.

Example 7. A nonconstant polynomial function $c_0 + c_1 s + \cdots + c_m s^m$ $(m > 0)$ is *not* expressible as a Dirichlet series, since it doesn't satisfy the condition of Exercise 10a.

Theorem 8. *If $\sum a_n n^{-s_0}$ converges absolutely at a complex number s_0 then $\sum a_n n^{-s}$ converges absolutely for all s with $\mathrm{Re}(s) \geq \mathrm{Re}(s_0)$.*

Proof. Use the comparison test. □

Our next task is to show Theorem 8 remains true when we weaken absolute convergence to convergence, except the inequality in the conclusion will become strict.

Theorem 9 (Jensen–Cahen)**.** *If $\{a_n\}$ is a sequence such that the partial sums $a_1 + a_2 + \cdots + a_n$ are bounded then the series $\sum a_n n^{-s}$ converges and is an analytic function on the half-plane $\sigma > 0$, with its derivative there computable termwise. Convergence of the series is absolute on the half-plane $\sigma > 1$.*

More generally, if the partial sums of the series $\sum a_n n^{-s}$ at $s_0 = \sigma_0 + it_0$, are bounded then the series $\sum a_n n^{-s}$ converges and is analytic for $\sigma > \sigma_0$, with its derivative there computable termwise. Convergence is absolute on the half-plane $\sigma > \sigma_0 + 1$.

Unless otherwise specified, the undecorated term "half-plane" always refers to a right half-plane of the form $\sigma > \sigma_0$ or $\sigma \geq \sigma_0$.

Proof. Set $A_n = a_1 + \cdots + a_n$ for $n \geq 1$ and $A_0 = 0$, so $\{A_n\}$ is a bounded sequence and $a_n = A_n - A_{n-1}$ for $n \geq 1$. By partial summation

$$\begin{aligned}\sum_{n=1}^{N} \frac{a_n}{n^s} &= \sum_{n=1}^{N} \frac{A_n - A_{n-1}}{n^s} \\ &= \frac{A_N}{N^s} - \sum_{n=1}^{N-1} A_n \left(\frac{1}{(n+1)^s} - \frac{1}{n^s} \right).\end{aligned}$$

For any real numbers a and b,

$$\int_a^b \frac{dx}{x^{s+1}} = -\frac{1}{s}\left(\frac{1}{b^s} - \frac{1}{a^s}\right),$$

so

$$\sum_{n=1}^{N} \frac{a_n}{n^s} = \frac{A_N}{N^s} - \sum_{n=1}^{N-1} A_n (-s) \int_n^{n+1} \frac{dx}{x^{s+1}}.$$

For $x \geq 1$, set $A(x) = A_{[x]}$, *e.g.*,$A(9.7) = A_9$. Then $A(x)$ is a piecewise continuous (step) function and

$$\begin{aligned}\sum_{n=1}^{N} \frac{a_n}{n^s} &= \frac{A_N}{N^s} - \sum_{n=1}^{N-1} (-s) \int_n^{n+1} \frac{A(x)}{x^{s+1}}\, dx \\ &= \frac{A_N}{N^s} + s \sum_{n=1}^{N-1} \int_n^{n+1} \frac{A(x)}{x^{s+1}}\, dx \\ &= \frac{A_N}{N^s} + s \int_1^{N} \frac{A(x)}{x^{s+1}}\, dx. \qquad (1)\end{aligned}$$

Letting $|A_n| \le C$ for all n, when $\sigma > 0$ we have $|A_N/N^s| \le C/N^\sigma \to 0$ as $N \to \infty$ and $\int_1^\infty (A(x)/x^{s+1})dx$ is absolutely convergent. Thus when we let $N \to \infty$ in (1) we get

$$\sum_{n\ge 1} \frac{a_n}{n^s} = s\int_1^\infty \frac{A(x)}{x^{s+1}}dx. \tag{2}$$

Since A_n is a bounded sequence, the numbers $a_n = A_n - A_{n-1}$ are bounded, so the series $\sum |a_n/n^s| = \sum |a_n|/n^\sigma$ converges for all $\sigma > 1$. Thus $\sum a_n n^{-s}$ is absolutely convergent if $\sigma > 1$.

To prove $\sum a_n/n^s$ is analytic for $\sigma > 0$, set

$$f(s) = \sum_{n\ge 1} \frac{a_n}{n^s}, \qquad f_N(s) = \sum_{n=1}^N \frac{a_n}{n^s}.$$

We just showed $f_N \to f$ pointwise when $\sigma > 0$. We will show $f_N \to f$ uniformly on compact subsets of the half-plane $\{s : \sigma > 0\}$, so by basic complex analysis the analyticity of f_N implies both analyticity of f and $f_N' \to f'$ uniformly on compact subsets.

Using (1) and (2), for $\sigma > 0$

$$\begin{aligned} f(s) - f_N(s) &= s\int_1^\infty \frac{A(x)}{x^{s+1}}dx - \left(\frac{A_N}{N^s} + s\int_1^N \frac{A(x)}{x^{s+1}}\,dx\right) \\ &= s\int_N^\infty \frac{A(x)}{x^{s+1}}dx - \frac{A_N}{N^s}, \end{aligned}$$

so

$$|f(s) - f_N(s)| \le |s|\int_N^\infty \frac{C}{x^{\sigma+1}}dx + \frac{C}{N^\sigma} = \frac{|s|C}{\sigma N^\sigma} + \frac{C}{N^\sigma}.$$

Let K be a compact subset of $\{s : \sigma > 0\}$, so for some $b > 0$ and $B > 0$ we have $\sigma \ge b$ and $|s| \le B$ for all $s \in K$. Then

$$s \in K \Longrightarrow |f(s) - f_N(s)| \le \frac{BC}{bN^b} + \frac{C}{N^b},$$

and this upper bound tends to 0 as $N \to \infty$.

Since $f_N' \to f'$ uniformly on compact subsets of $\{s : \sigma > 0\}$, and thus in particular pointwise,

$$f'(s) = \lim_{N\to\infty} f_N'(s) = \lim_{N\to\infty} \sum_{n=1}^N \frac{a_n(-\log n)}{n^s}$$

In the general case where the partial sums of the series $\sum a_n/n^{s_0}$ are bounded, set $b_n = a_n/n^{s_0}$ so the partial sums $b_1 + \cdots + b_n$ are bounded and $\sum b_n/n^s = \sum a_n/n^{s+s_0}$. Conclusions that we have proved for $\sum b_n/n^s$ when $\mathrm{Re}(s) > 0$ transfer to $\sum a_n/n^s$ when $\mathrm{Re}(s) > \mathrm{Re}(s_0)$. □

Example 10. The zeta-function is analytic for $\sigma > 1$, with

$$\zeta'(s) = -\sum_{n\ge 2} \frac{\log n}{n^s}, \quad \zeta''(s) = \sum_{n\ge 2} \frac{(\log n)^2}{n^s}.$$

The contribution of Jensen (1884) to Theorem 9 was a proof that convergence at s_0 implies convergence on the half-plane to the right of s_0. Cahen (1894) weakened the hypothesis to permit bounded partial sums at s_0, and also proved analyticity. (For a discussion of the development of convergence theorems for Dirichlet series, see [2].) Dirichlet did not have the full strength of Theorem 9 back in 1837 when he used L-functions in his proof on primes in arithmetic progression. He looked at L-functions only for real $s > 0$, not complex s, and he proved their convergence and continuity using real analysis.

Corollary 11. *If $\sigma_0 \geq 0$ and $A(x) := \sum_{n \leq x} a_n/n^{s_0}$ satisfies $A(x) = O(x^{\sigma_0})$, or even $A(x) = O(x^{\sigma_0+\varepsilon})$ for all $\varepsilon > 0$, then for $\mathrm{Re}(s) > \sigma_0$*

$$\sum_{n \geq 1} \frac{a_n}{n^s} = s \int_1^\infty \frac{A(x)}{x^{s+1}} dx,$$

where $\int_1^\infty = \lim_{T \to \infty} \int_1^T$.

In the estimate $A(x) = O(x^{\sigma_0+\varepsilon})$, the implicit constant in the estimate may depend on ε.

Proof. Exercise 6. □

Example 12. For real $s > 0$, the alternating series $L(s, \chi_4)$ converges, so it converges and is analytic for complex s with $\mathrm{Re}(s) > 0$. Alternatively, we have convergence on this half-plane since the partial sums of the Dirichlet coefficients are bounded, being only 0 or 1. For $\sigma > 0$,

$$L(s, \chi_4) = s \int_1^\infty \frac{A(x)\,dx}{x^{s+1}},$$

where $A(x) = 0$ or 1.

Although

$$L\left(\frac{1}{2}, \chi_4\right) = \sum_{n \geq 1} \frac{\chi_4(n)}{\sqrt{n}} \text{ and } L'\left(\frac{1}{3}, \chi_4\right) = -\sum_{n \geq 2} \frac{\chi_4(n) \log n}{\sqrt[3]{n}},$$

these sums are *not* absolutely convergent, and this marks a striking difference with power series. A power series is absolutely convergent on the interior of its disc of convergence, but a Dirichlet series can converge nonabsolutely on a vertical strip.

Theorem 13. *Let $\chi\colon (\mathbf{Z}/(m))^\times \to \mathbf{C}$ be a nontrivial Dirichlet character, so $\chi(a) \neq 1$ for some unit a mod m. Set $\chi(n) = 0$ if $(n, m) > 1$. The Dirichlet series*

$$L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}$$

converges for $\sigma > 0$. It converges absolutely for $\sigma > 1$.

Proof. Absolute convergence on $\sigma > 1$ is clear. To get convergence for $\sigma > 0$, we show the partial sums $\chi(1) + \chi(2) + \cdots + \chi(n)$ are all bounded, so Theorem 9 applies.

Since χ is periodic, it suffices to show the sum over a full period vanishes:

$$\sum_{k=N}^{N+m-1} \chi(k) = \sum_{k \in (\mathbf{Z}/(m))^\times} \chi(k) = 0.$$

Let S be this sum. Choosing a unit a mod m so that $\chi(a) \neq 1$,

$$\chi(a)S = \sum_{k \in (\mathbf{Z}/(m))^\times} \chi(ak) = \sum_{k \in (\mathbf{Z}/(m))^\times} \chi(k) = S,$$

so $S = 0$. □

It is instructive to compare properties of functions defined by a power series and by a Dirichlet series. Both such series can be taken as special cases of series of the type $\sum_{n\geq 1} a_n e^{-\lambda_n s}$; Dirichlet series are the case $\lambda_n = \log n$ and power series (in e^{-s}) are the case $\lambda_n = n-1$. We will not pursue such a unification (for that, see [1], [3], [7], or [8]), but instead list a few properties that power series and Dirichlet series share and don't share.

For a power series $\sum c_n(z-a)^n$, convergence at at point z_0 implies convergence on the disc $|z-a| < |z_0 - a|$, a bounded region. But if a Dirichlet series $\sum a_n n^{-s}$ converges at a point s_0 then there is convergence on the half-plane $\sigma > \mathrm{Re}(s_0)$, which is an unbounded region. This illustrates the inherently nonlocal nature of a Dirichlet series, unlike the tool of local power series expansions that pervades complex function theory. For this reason Dirichlet series are not a general purpose tool in complex analysis.

- Without worrying about boundary behavior, a power series has a maximal open disc of convergence $|z-a| < R$. By Theorem 9, a Dirichlet series has a maximal open half-plane of convergence $\sigma > \sigma_c$ for some real number σ_c. The number σ_c is called the *abscissa of convergence* of the Dirichlet series.
- Power series and Dirichlet series are uniform limits of partial sums on compact subsets of the interior of the region of convergence, with derivatives being computable termwise.
- A power series converges absolutely on the interior of its disc of convergence, but a Dirichlet series *need not* converge absolutely on the whole interior of $\sigma > \sigma_c$.

  For every Dirichlet series, there is a maximal open half-plane of absolute convergence $\sigma > \sigma_a$ for some number σ_a, called the *abscissa of absolute convergence*. By Theorem 9, $\sigma_c \leq \sigma_a \leq \sigma_c + 1$. For $\zeta(s)$, $\sigma_a = \sigma_c = 1$, while for $L(s, \chi_4)$, $\sigma_a = 1$ and $\sigma_c = 0$. It can happen that $\sigma_c < \sigma_a < \sigma_c + 1$.
- A Dirichlet series that converges at s_0 converges uniformly on regions larger than compact subsets of $\sigma > \mathrm{Re}(s_0)$, such as sectors

  $$\left\{ s : \sigma \geq \sigma_0,\ |\mathrm{Arg}(s - s_0)| \leq \frac{\pi}{2} - \varepsilon \right\}$$

  for $0 < \varepsilon < \pi/2$. For a proof, see [7, Chap. VI, Prop. 6]. Compact sets of uniform convergence, which we established in Theorem 9, will suffice for us.
- A power series $\sum c_n(z-a)^n$ has radius of convergence given by Hadamard's formula

  $$\frac{1}{R} = \overline{\lim}\, |c_n|^{1/n}.$$

  (Recall that $\overline{\lim}$ simply means "largest limit point.") For a Dirichlet series that does not converge at $s = 0$, the abscissa of convergence σ_c is given by Cahen's formula

  $$\sigma_c = \overline{\lim}\, \frac{\log |a_1 + \cdots + a_n|}{\log n} \geq 0. \tag{3}$$

  A formula for σ_c that is valid with no restrictions on behavior at 0 is due to Knopp [4] (for a generalization, see [5]):

  $$\sigma_c = \overline{\lim}\, \frac{\log |\sum_{n=e^{[x]}}^{e^x} a_n|}{x}$$

  Formulas for σ_a are the same as these, except a_n is replaced by $|a_n|$. We will have no use for such abscissa formulae.

- The coefficients of a power series $f(z) = \sum c_n(z-a)^n$ can be recovered by both Taylor's formula and Cauchy's integral formula:
$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i}\int_{|z-a|=r} \frac{f(z)}{(z-a)^{n+1}}\,dz.$$
Here r is less than the radius of convergence of f at $z = a$.
The coefficients of a Dirichlet series $f(s) = \sum a_n n^{-s}$ can be recovered by integration along vertical lines. For example, if $\sigma > \sigma_c$, then Perron's formula is
$$\sum_{n \le x} a_n = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} f(s)\frac{x^s}{s}\,ds,$$
where x is not an integer. The coefficients a_n are then determined by taking a difference of consecutive partial sums.
An integration formula that recovers the nth coefficient alone is
$$a_n = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(\sigma+it)n^{\sigma+it}\,dt.$$
Here $\sigma > \sigma_c$.
Note that these vertically integrated formulas for Dirichlet coefficients are along *noncompact* contours, while the Cauchy integral formula for Taylor coefficients is along a circle.
The Dirichlet coefficients can also be determined by taking limits to the right, *e.g.*,
$$f(s) = \sum_{n\ge 1}\frac{a_n}{n^s} = a_1 + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \cdots \Longrightarrow \lim_{\sigma\to+\infty} f(s) = a_1.$$
See Exercise 10 for an extension to other coefficients.
- A power series with disc of convergence $|z| < r$ that analytically extends to a point z_0 on the boundary $|z| = r$ is represented by a power series on a small disc around z_0. But a Dirichlet series with half-plane of convergence $\sigma > \sigma_c$ that analytically extends to a point s_c on the boundary $\sigma = \sigma_c$ is *never* represented by a Dirichlet series on a small disc around s_c. Indeed, a Dirichlet series converging on a disc around s_c automatically converges on some half-plane $\sigma > \sigma_c - \varepsilon$, so by uniqueness of coefficients this must be the original Dirichlet series, which only converges on $\sigma > \sigma_c$.
This explains why Dirichlet series are inherently non-local objects, unlike power series. Another way to think about this state of affairs is that a power series can be recentered to a point near the boundary of its region of convergence but a Dirichlet series can't; all Dirichlet series are "centered" at $+\infty$. (Of course, one can always replace $\sum a_n n^{-s}$ with $\sum a_n n^{-s+s_1} = \sum (a_n n^{-s_1})n^{-s}$ to shift the half-plane of convergence, but this is not a recentering since the underlying function has changed.)
- A power series must have an analytic singularity at some point on the boundary of its disc of convergence. (Proof: If we can cover the boundary circle by small balls where the power series admits analytic continuations, finitely many of these balls will cover the boundary circle by compactness. Therefore the function extends analytically to a slightly larger disc, so by complex analysis its series expansion at the center has a slightly larger radius of convergence, which is a contradiction.) If we carry over the proof for power series to the case of Dirichlet series, the proof

fails because (vertical) lines are *noncompact*. And in fact, a Dirichlet series does *not* have to admit any analytic singularities along its abscissa of convergence. For example, the Dirichlet series for $L(s, \chi_4)$ converges only on $\sigma > 0$, but we'll see that $L(s, \chi_4)$ extends to an entire function.

For the rest of this section, we put the background material about infinite products and logarithms of analytic functions to work on the Riemann zeta-function.

Theorem 14. *For* $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} = \exp\left(\sum_{p,k} \frac{1}{kp^{ks}}\right) \neq 0.$$

Proof. To justify expanding

$$\prod_p \frac{1}{1 - p^{-s}} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right)$$

into a Dirichlet series by multiplying out the infinite product of infinite series and rearranging terms any way we want, it is sufficient that the series $\sum_p 1/p^s$ is absolutely convergent, which is true for $\mathrm{Re}(s) > 1$. Therefore when $\mathrm{Re}(s) > 1$ we can expand out $\prod_p 1/(1 - 1/p^s)$ and obtain $\sum 1/n^s = \zeta(s)$ by unique factorization in $\mathbf{Z}$.

For $\mathrm{Re}(s) > 1$ the series $\sum_{p,k} 1/(kp^{ks})$ is absolutely convergent, so continuity of the exponential function implies

$$\exp\left(\sum_p \sum_{k \geq 1} \frac{1}{kp^{ks}}\right) = \prod_p \exp\left(\sum_{k \geq 1} \frac{1}{kp^{ks}}\right) = \prod_p \frac{1}{1 - 1/p^s}.$$

□

Corollary 15. *For* $\mathrm{Re}(s) > 1$,

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n \geq 1} \frac{\mu(n)}{n^s},$$

where μ *is the Möbius function.*

For the reader's convenience, we recall the definition of $\mu(n)$: $\mu(1) = 1$ and $\mu(n) = (-1)^r$ if n is a product of r distinct primes. If n has a multiple prime factor, then $\mu(n) = 0$.

Proof. Since $\zeta(s) \neq 0$ by Theorem 14, we can reciprocate the Euler product:

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right).$$

To write $1/\zeta(s)$ as a Dirichlet series, we give two proofs.

Expanding $\prod(1 - p^{-s})$ into a series, we get the terms $\pm n^{-s}$ where n is squarefree and the $\pm$ sign depends on the parity of the numbers of prime factors of n. That parity is exactly the definition of the Möbius function $\mu(n)$. Since the series $\sum \mu(n)n^{-s}$ converges absolutely on $\sigma > 1$, this computation of the product into a series is justified (Exercise 9).

For a second proof, we begin with the observation that the Dirichlet series $\sum \mu(n)n^{-s}$ converges absolutely on $\sigma > 1$, so by Exercise 7

$$\zeta(s) \cdot \sum_{n \geq 1} \frac{\mu(n)}{n^s} = \sum_{n \geq 1} \frac{c_n}{n^s}$$

where $c_n = \sum_{d|n} \mu(d)$. It is a basic property of the Möbius function that this sum is 1 for $n = 1$ and 0 for $n > 1$. So the product is 1. □

The proof of Theorem 14 brings out a point that is worth making explicit before we continue. It is commonly said that the zeta-function is nonvanishing on $\mathrm{Re}(s) > 1$ because "it has an Euler product." However, we did not prove the nonvanishing as a consequence the Euler product, but rather obtained both nonvanishing and representation as an Euler product simultaneously, as consequences of the representation of the zeta-function as the exponential of another function. So from the viewpoint we are taking, the zeta-function is nonvanishing on $\mathrm{Re}(s) > 1$ not because "it has an Euler product," but because "it is an exponential," and *exponentials are never zero.*

The representation of a function as an infinite product is not by itself an adequate reason for it to be nonvanishing. Consider an example from Corollary 15:

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \lim_{x \to \infty} \prod_{p \leq x} \left(1 - \frac{1}{p^s}\right).$$

It turns out that this formula for $1/\zeta(s)$ for $\mathrm{Re}(s) > 1$ is valid on the line $\mathrm{Re}(s) = 1$. However, while the factors in the product for $1/\zeta(s)$ are all nonvanishing on $\mathrm{Re}(s) \geq 1$, the product vanishes at $s = 1$.

The expression of the zeta-function as an exponential or Euler product as in Theorem 14 should be regarded as a basic structural ingredient, in some sense more fundamental than the usual Dirichlet series definition. Of course the Dirichlet series is important, *e.g.*,it is used in proving the analytic continuation of $\zeta(s)$. However, some similar types of functions that are significant for number theory, such as zeta-functions of varieties over finite fields or Artin L-functions, can only be defined as exponentials or Euler products, not as series. A series expression in these two cases is possible, but not as a starting point.

Definition 16. For $\mathrm{Re}(s) > 1$, let

$$\log \zeta(s) := \sum_{p^k} \frac{1}{kp^{ks}} = \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{2 \cdot 4^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{3 \cdot 8^s} + \frac{1}{2 \cdot 9^s} + \dots.$$

By Theorem 14, the function $\log \zeta(s)$ is indeed a logarithm of the zeta-function, since its exponential is $\zeta(s)$. Although we can reorder the sum in the definition of $\log \zeta(s)$ to take the succinct form $\log \zeta(s) := \sum_p -\mathrm{Log}(1-p^{-s})$, it should be kept in mind that the log notation on the left is a priori only formal; we are not defining $\log \zeta(s)$ as a composite of "log" and "ζ". Still, having checked that $\log \zeta(s)$ is indeed a logarithm of $\zeta(s)$ we can characterize it among all logarithms of $\zeta(s)$ by either of two analytic properties: being real-valued on $(1, \infty)$ (so $\log \zeta(s) = \log(\zeta(s))$ for real $s > 1$) or tending to 0 as $\mathrm{Re}(s) \to \infty$.

Taking the derivative of $\log \zeta(s)$

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{p^k} \frac{\log p}{p^{ks}}. \tag{4}$$

Exercises.

1. Let $h(n)$ be a polynomial in n of degree d. Show $\sum h(n)n^{-s}$ converges absolutely for $\sigma > d+1$.
2. Show the series $\sum a_n n^{-s}$ converges somewhere if and only if a_n has at most polynomial growth, *i.e.*, $a_n = O(n^d)$ for some positive number d.
3. If $\sum_{n\le x} a_n = O(x^\delta)$ for some $\delta \ge 0$, show for $\mathrm{Re}(s) > \delta$ that the Dirichlet series $\sum a_n n^{-s}$ converges. In particular, when $\delta = 0$ we recover Theorem 9, and in fact your solution to this problem should simply involve making the appropriate adjustments to the proof of that theorem.
4. Fix $\sigma_0 > 0$.
   a) If $a_n \ge 0$, show the following are equivalent:
   i) $\sum a_n n^{-s}$ converges for $\sigma > \sigma_0$.
   ii) $\sum_{n\le x} a_n = O(x^{\sigma_0+\varepsilon})$ for all $\varepsilon > 0$, where the O–constant may depend on ε.
   b) Give an example where $a_n \ge 0$, $\sum a_n n^{-s}$ converges for $\sigma > \sigma_0$, and $\sum_{n\le x} a_n \ne O(x^{\sigma_0})$.
5. Let $a_n \ge 0$. Show $f(s) = \sum a_n n^{-s}$ has abscissa of convergence
$$1 + \underline{\lim}\{r : a_n = O(n^r)\}.$$
6. Prove Corollary 11.
7. Let $f(s) = \sum a_n n^{-s}$, $g(s) = \sum b_n n^{-s}$.
   If $\sigma > \sigma_0$ is a common half-plane of absolute convergence for $f(s)$ and $g(s)$, show on this half-plane that formal multiplication is valid:
$$\sum_{n\ge1} \frac{a_n}{n^s} \cdot \sum_{n\ge1} \frac{b_n}{n^s} = \sum_{n\ge1} \frac{c_n}{n^s},$$
   where $c_n = \sum_{d|n} a_d \cdot b_{n/d} = \sum_{dd'=n} a_d b_{d'}$ and $\sum c_n n^{-s}$ converges absolutely on $\sigma > \sigma_0$. This result extends trivially to products of finitely many absolutely convergent Dirichlet series.
8. a) Show the exponential of an absolutely convergent Dirichlet series is an absolutely convergent Dirichlet series. That is, if $\sum b_n n^{-s}$ converges absolutely when $\mathrm{Re}(s) > \sigma_0$, then
$$\exp\left(\sum_{n\ge1} \frac{b_n}{n^s}\right) = \sum_{n\ge1} \frac{a_n}{n^s},$$
   where the right side is absolutely convergent for $\mathrm{Re}(s) > \sigma_0$.
   b) Suppose that all $b_n \ge 0$. Show all $a_n \ge 0$, and in fact $b_n \le a_n$, so $\sum a_n n^{-s}$ and $\sum b_n n^{-s}$ have the same open half-plane of absolute convergence.
9. Let $F\colon \mathbf{Z}^+ \to \mathbf{C}$ be "arithmetically multiplicative": $F(mn) = F(m)F(n)$ for relatively prime m and n in $\mathbf{Z}^+$, but possibly $F(mn) \ne F(m)F(n)$ if m and n have a common factor. Examples include $d(n)$ and $\varphi(n)$. If $F(n)$ grows at most polynomially in n, prove the product decomposition
$$\sum_{n\ge1} \frac{F(n)}{n^s} = \prod_p \left(1 + \frac{F(p)}{p^s} + \frac{F(p^2)}{p^{2s}} + \frac{F(p^3)}{p^{3s}} + \dots\right)$$
   on a half-plane where the Dirichlet series on the left converges absolutely.

10. a) Show that
$$f(s) = \sum_{n\geq 1} \frac{a_n}{n^s} = a_1 + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \cdots \Longrightarrow \lim_{\sigma\to+\infty} f(s) = a_1.$$

b) If in part a) the coefficients $a_1, \ldots, a_{k-1}$ have been determined, bring the first $k-1$ terms to the other side and consider a Dirichlet series starting with the kth term:
$$g(s) = \frac{a_k}{k^s} + \frac{a_{k+1}}{(k+1)^s} + \frac{a_{k+2}}{(k+2)^s} + \ldots,$$
so
$$k^s g(s) = a_k + a_{k+1}\left(\frac{k}{k+1}\right)^s + a_{k+2}\left(\frac{k}{k+2}\right)^s + \cdots = \sum_{n\geq k} a_n \left(\frac{k}{n}\right)^s.$$
Show that $\lim_{\sigma\to+\infty} k^s g(s) = a_k$, so all the coefficients are inductively determined by taking various limits to ∞.

c) Suppose $h(s) = \sum_{n\geq k} b_n n^{-s}$, where $b_k \neq 0$. Show $|h(s)| \geq (1/2)|b_k|/k^\sigma$ for all sufficiently large σ. Therefore a Dirichlet series is bounded away from 0 and ∞ on all vertical lines sufficiently far to the right.

d) If $f(s_k) = 0$ for a sequence s_k with $\mathrm{Re}(s_k) \to \infty$, show all $a_n = 0$, so f is identically zero. (This is due to Dirichlet for real s_k and Perron for the general case.)

e) If $a_n \neq 0$ for some $n > 1$, show for every $c \in \mathbf{C}$ that $f(s) \neq c$ if $\mathrm{Re}(s)$ is sufficiently large.

11. a) If $\sum_{n\geq 1} a_n n^{-s}$ converges on $\sigma > \sigma_0$, show $-\sum_{n\geq 2}(a_n/\log n)n^{-s}$ does as well, so a function on an open half-plane that is represented by a Dirichlet series has its antiderivative on that half-plane also represented by a Dirichlet series provided $a_1 = 0$.

b) By part a), the series $h(s) = \sum_{n\geq 2} 1/(n^s \log n)$ converges on $\sigma > 1$ and $h'(s) = 1 - \zeta(s)$. Since $\sum 1/(n\log n)$ diverges (the companion integral $\int_2^\infty dx/x\log x$ diverges), we expect that $h(s) \to \infty$ as $s \to 1^+$. Prove this.

12. Let $f(s) = \sum a_n n^{-s}$ with $a_1 = 1$ and a_n real for all n. If for some real σ_0, $f(s)$ is nonvanishing for $\sigma > \sigma_0$, show $f(s)$ is positive for real $s > \sigma_0$.

13. Show the only rational functions that are represented on some (nonempty) half-plane by a Dirichlet series are constants.

14. (Dirichlet integral) Let $a(x)$ be an integrable function on each interval $[1, T]$. Let
$$f(s) := \int_1^\infty \frac{a(x)}{x^s}\,dx = \lim_{T\to\infty} \int_1^T \frac{a(x)}{x^s}\,dx$$
when this integral converges. Think of $\int_1^T a(x)x^{-s}\,dx$ as analogous to $\sum_{n\leq x} a_n n^{-s}$.

a) Explain why, for fixed T, the integral $\int_1^T a(x)x^{-s}\,dx$ is analytic in s and can be differentiated under the integral sign.

b) If for some s_0 the integrals $A(T) = \int_1^T a(x)x^{-s_0}\,dx$ are bounded uniformly in $T \geq 1$, show $f(s)$ converges and is analytic for $\mathrm{Re}(s) > \mathrm{Re}(s_0)$, with
$$f'(s) = \int_1^\infty \frac{-a(x)\log x}{x^s}\,dx.$$
(Hint: Reduce to the case $\mathrm{Re}(s_0) = 0$ and use integration by parts.)

c) Show $f(s)$ converges absolutely for $\mathrm{Re}(s) > \sigma_0 + 1$.

15. Let $\sum a_n n^{-s}$ have abscissa of convergence σ_c. Without using formulas for σ_c, show $\sum \overline{a}_n n^{-s}$ also has abscissa of convergence σ_c, so by addition the series $\sum \mathrm{Re}(a_n) n^{-s}$ and $\sum \mathrm{Im}(a_n) n^{-s}$ converge on $\sigma > \sigma_c$.

    Also show the functions $\sum a_n n^{-s}$ and $\sum \overline{a}_n n^{-s}$ have analytic continuations to the same half-planes, although these continuations need not be everywhere expressible as a Dirichlet series.

16. For $\varepsilon > 0$, let

$$f_\varepsilon(s) = \sum_{n \geq 2} \left(\frac{1}{\log n} \right)^{1+\varepsilon} \frac{1}{n^s}.$$

    Show $f_\varepsilon(s)$ has $\sigma_a = \sigma_c = 1$ and the series converges absolutely on $\sigma = 1$.

17. Let the partial sums $A_n = a_1 + \cdots + a_n$ be bounded, say $|A_n| \leq C$ for all n.

    a) Prove the following tail end estimate for $\sigma > 0$:

$$\left| \sum_{n \geq M} \frac{a_n}{n^s} \right| \leq C \left(\frac{|s|}{\sigma} + 1 \right) \frac{1}{M^\sigma}.$$

    In particular, if $\sum_{n \leq x} a_n = O(1)$ then $\sum_{n \geq x} a_n / n = O(1/x)$.

    b) For real $t \neq 0$, show that as $N \to \infty$,

$$\left| \sum_{n=1}^{N} \frac{a_n}{n^{it}} \right| \leq C|t| \log(N+1) + C \sim C|t| \log N.$$

18. Suppose $\sum |a_n|$ converges. Show the series $\sum a_n n^{-s}$ is uniformly continuous on compact subsets of the closed half-plane $\sigma \geq 0$. (The main point is to check compact sets containing a piece of the boundary $\sigma = 0$.)

19. Let a_n, b_n be sequences of positive numbers, with $a_n \sim b_n$ as $n \to \infty$.

    a) For $s > 1$, show $\sum a_n n^{-s}$ converges if and only if $\sum b_n n^{-s}$ converges.

    b) As $s \to 1^+$, show $\sum a_n n^{-s}$ tends to ∞ if and only if $\sum b_n n^{-s}$ does, in which case the two Dirichlet series are asymptotic as $s \to 1^+$.

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