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On multiply-connected Fatou components in iteration of meromorphic functions ☆

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Abstract

Let $f:\mathbf{C}\mapsto\hat{\mathbf{C}}$ be a transcendental meromorphic function with at most finitely many poles. We mainly investigated the existence of the Baker wandering domains of $f(z)$ and proved, among others, that if $f(z)$ has a Baker wandering domain U, then for all sufficiently large n, $f^n(U)$ contains a round annulus whose module tends to infinity as $n\to\infty$ and so for some $0<d<1$,

$$M_c(r,a,f)^d\leqslant m_c(r,a,f),\quad r\in G,$$

where G is a set of positive numbers with infinite logarithmic measure. Therefore, we give out several criterion conditions for non-existence of the Baker wandering domains.

Keywords: Iteration of meromorphic functions; Fatou components; Baker wandering domain

1. Introduction and results

Let $f:\mathbf{C}\mapsto\hat{\mathbf{C}}$ be a transcendental meromorphic function, and f^n, $n\in\mathbf{N}$, denote the nth iterate of f. Then $f^n(z)$ is defined for all $z\in\mathbf{C}$ except for a countable set of the poles of $f, f^2,\dots,f^{n-1}$. Define the Fatou set of f by

$$F_f=\{z\in\hat{\mathbf{C}}\colon\ \{f^n\}\text{ is defined and normal in some neighbourhood of }z\}$$

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and the Julia set of f by $J_f = \hat{\mathbf{C}} \setminus F_f$. It is well known that F_f is open and completely invariant under f, i.e., $z \in F_f$ if and only if $f(z) \in F_f$. Let U be a connected component of F_f, then $f^n(U)$ is contained in a component of F_f, denoted by U_n. If for some integer $p \geqslant 1$, $U_p = U$, that is, $f^p(U) \subseteq U$, then U is called a periodic domain and the smallest integer p such that $U_p = U$ is the period of U; if for some pair of $n \neq m$, $U_n = U_m$, but U is not periodic, then U is called preperiodic; if for $n \neq m$, $U_n \neq U_m$, then U is called a wandering domain of f. And U is called the Baker wandering domain if U is wandering and all U_n are multiply-connected and surround 0 and the Euclidean distance $\mathrm{dist}(0, U_n) \to +\infty$ as $n \to +\infty$.

It was proved in [2] that for a transcendental entire function $f(z)$, every multiply-connected component of its Fatou set must be Baker wandering and in this case, F_f and J_f both have only bounded components. And $f(z)$ has only simply connected Fatou components if it has a finite asymptotic value. The same result was proved in [18] if it has a finite Nevanlinna deficient value (for the definition of Nevanlinna deficient value, please see the statement before Corollary 2). This result is interesting, because we can distinguish non-existence of multiply-connected Fatou components by the quantity, Nevanlinna deficiency, in theory of value distribution, and thus we establish some relationship between theory of complex dynamics and value distribution of meromorphic functions. In this paper, we investigate the necessary condition under which a transcendental meromorphic function has multiply-connected Fatou components.

Theorem. *Let $f(z)$ be a transcendental meromorphic function with at most finitely many poles. If $f(z)$ has a Baker wandering domain U, then for a multiply-connected domain A in U such that $f^n(A)$ contains a closed curve which is not null-homotopic in U_n, there exists a positive number n_0 such that for each $n > n_0$, we have a round annulus $D_n = \{r_n < |z| < R_n\}$ in $f^n(A)$ and* $\mathrm{dist}(0, D_n) \to \infty$ *and* $\mathrm{mod}(D_n) \to \infty$ *as $n \to \infty$.*

Remark.

(A) It was proved in Theorem 1 of Zheng [18] that the Julia set of a transcendental meromorphic function with at most finitely many poles has only bounded components if and only if it has a Baker wandering domain.

(B) From the proof of Theorem (see also [9] and result (II) in Theorem 1 of Zheng [18]), there exists a $R > 0$ such that for any closed curve γ in $\{|z| > R\}$ surrounding 0, $f(\gamma)$ and so $f^p(\gamma)$ contains a closed curve Γ_p with $n(\Gamma_p, 0) = 1$, where $n(\Gamma_p, 0)$ is the winding number of Γ_p with respect to 0. Therefore, a Baker wandering domain U in Theorem definitely contains a multiply-connected domain A satisfying the condition stated in Theorem. Furthermore, we deduce that for each sufficiently large n, U_n contains a round annulus D_n such that $\mathrm{dist}(0, D_n) \to \infty$ and $\mathrm{mod}(D_n) \to \infty$ as $n \to \infty$. This is an improvement of result (III) in Theorem 1 of Zheng [18] which only asserts that $\bigcup_{n=1}^{\infty} U_n$ contains a sequence of round annuli D_n such that $\mathrm{dist}(0, D_n) \to \infty$ and $\mathrm{mod}(D_n) \to \infty$ as $n \to \infty$.

(C) We can prove by using the same method as in Lemma 3.3 in [15] that a multiply-connected wandering domain U is Baker wandering, if U contains a closed curve γ such that there exists a sequence $f^{n_k}(\gamma)$ not to be null-homotopic in F_f. Then we raise a question: Is any multiply-connected wandering domain Baker wandering? If f has infinitely many poles, then the answer to this question is negative. In fact, we can construct an example of meromorphic function which has a multiply-connected wandering domain U such that $f^{np}|_U \to a$ as $n \to \infty$ and $f^{p-1}(a) = \infty$. Therefore, we should put our attention on a meromorphic function with finitely many poles about this question.

Next we discuss some consequences of the theorem. Set

$$M_c(r, a, f) = \max\{|f(z)|: |z-a| = r\},$$
$$m_c(r, a, f) = \min\{|f(z)|: |z-a| = r\},$$
$$M_s(r, a, f) = \max\{|f(z)|: |\operatorname{Re} z - \operatorname{Re} a| = r,\ |\operatorname{Im} z - \operatorname{Im} a| = r\}, \quad \text{and}$$
$$m_s(r, a, f) = \min\{|f(z)|: |\operatorname{Re} z - \operatorname{Re} a| = r,\ |\operatorname{Im} z - \operatorname{Im} a| = r\}.$$

When $a = 0$, we simply write $M(r, f)$ for $M_c(r, 0, f)$. As an application of Theorem, we can deduce the following.

Corollary 1. *Let $f(z)$ be a transcendental meromorphic function with at most finitely many poles. If J_f has only bounded components, then for any complex number a, there exists a constant $0 < d < 1$ and two sequences $\{r_n\}$ and $\{R_n\}$ of positive numbers with $r_n \to \infty$ and $R_n/r_n \to \infty$ $(n \to \infty)$ such that*

$$M_c(r, a, f)^d \leqslant m_c(r, a, f), \quad r \in G, \tag{1}$$

and

$$M_s(r, a, f)^d \leqslant m_s(r, a, f), \quad r \in G, \tag{2}$$

where

$$G = \bigcup_{n=1}^{\infty} \{r: r_n < r < R_n\}.$$

It is obvious that the set G in Corollary 1 has infinite logarithmic measure, that is,

$$\operatorname{lm}(G) := \int_G \frac{dt}{t} = \infty.$$

From Corollary 1, we can immediately obtain some criterion conditions of non-existence of the Baker wandering domains of such a transcendental meromorphic function. In order to state one of such consequences, below we introduce the basic notations of Nevanlinna theory (see [12]). Let $f(z)$ be a meromorphic function. Define

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta \tag{3}$$

and

$$N(r,f)=\int_0^r \frac{n(t,f)-n(0,f)}{t}\,dt+n(0,f)\log r,$$

where $\log^+ x=\max(\log x,0)$ and $n(r,f)$ denotes the number of poles of f in $\{|z|<r\}$, and the Nevanlinna characteristic is

$$T(r,f)=m(r,f)+N(r,f).$$

The lower order μ and the order λ of $f(z)$ are in turn defined as follows:

$$\mu=\mu(f)=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r},\qquad \lambda=\lambda(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

For $a\in\mathbf{C}$, the quantity

$$\delta(a,f)=\liminf_{r\to\infty}\frac{m(r,\frac{1}{f-a})}{T(r,f)}=1-\limsup_{r\to\infty}\frac{N(r,\frac{1}{f-a})}{T(r,f)}$$

is called deficiency of Nevanlinna of the value a. If $\delta(a,f)>0$, then a is called a deficient value of Nevanlinna. By E we denote a set of positive numbers with finite logarithmic measure which may not be the same for each occurrence. From (1) with $a=0$, we immediately deduce the following consequence.

Corollary 2. *Let $f(z)$ be a transcendental meromorphic function with at most finitely many poles. Then each of the following statements implies that J_f has an unbounded component, so that $f(z)$ has no Baker wandering domains*:

(I) *for any $\varepsilon>0$, there exists a curve Γ tending to ∞ such that*

$$\log\bigl|f(z)\bigr|<\varepsilon\log M\bigl(|z|,f\bigr),\quad z\in\Gamma;\tag{4}$$

(II) *for some meromorphic function $h(z)$, which may be a complex number, satisfying*

$$\log M(r,h)=o\bigl(\log M(r,f)\bigr),\quad r\notin E,\tag{5}$$

we have $\delta(0,f-h)>0$.

The result in Corollary 2 corresponding to (I) follows immediately from (1) with $a=0$. Indeed, if J_f has no unbounded components, then we have (1) for some $d>0$. On the other hand, for $0<\varepsilon<d$ we have a curve Γ tending to ∞ such that (4) holds on Γ. For sufficiently large $r\in G$, Γ intersects the circle $\{|z|=r\}$. By z_0 we denote the intersecting point. Then

$$d\log M(r,f)\leqslant\log m_c(r,f)\leqslant\log\bigl|f(z_0)\bigr|<\varepsilon\log M(r,f),$$

so that $d<\varepsilon$, which derives a contradiction.

By the Hadamard three circle theorem, for all sufficiently large $r>0$, we have

$$\log M\bigl(r^{\varepsilon},f\bigr)\leqslant(1-\varepsilon)\log M(1,f)+\varepsilon\log M(r,f).$$

This implies that our result is an extension of [5, Theorem 10] which considers that when f is a transcendental entire function, for all $\varepsilon > 0$, there exists a curve Γ tending to ∞ such that on Γ, $|f(z)| \leqslant M(|z|^{\varepsilon}, f)$.

In order to prove the result in Corollary 2 corresponding to (II), suppose that J_f has only bounded components, then we have (1) with $a = 0$. Under the assumption of (5), for sufficiently large $r \in G \setminus E$, we have

$$\bigl|f(z) - h(z)\bigr| \geqslant \bigl|f(z)\bigr| - \bigl|h(z)\bigr| \geqslant m_c(r, 0, f) - M(r, h) > 1 \quad \text{for } |z| = r,$$

$\log^{+}(1/|f(z) - h(z)|) = 0$ and hence $m(r, 1/(f - h)) = 0$, $r \in G \setminus E$. This implies that $\delta(0, f - h) = 0$. We derive a contradiction, and thus the result in Corollary 2 under (II) follows.

Furthermore, we can deduce the following result.

Corollary 3. *Let $f(z)$ be an entire function with order ρ and all but finitely many zeros of f be real. If $2 < \rho \leqslant \infty$, then $f(z)$ has only simply connected Fatou components.*

When $2 < \rho < \infty$, a result of Edrei et al. [10, Corollary 1.2] asserts that $\delta(0, f) > 0$, and therefore Corollary 3 follows from result (II) in Corollary 2; when $\rho = \infty$, a result of Miles [13, Theorem 1] asserts that

$$N\left(r, \frac{1}{f}\right) = o\bigl(\log M(r, f)\bigr).$$

Suppose that $f(z)$ has a multiply-connected Fatou component. Then (1) holds with $a = 0$. Therefore on the circle $|z| = r \in G$, $|f(z)| > 1$, that is, $m(r, 1/f) = 0$. By the first fundamental theorem of Nevanlinna (see [12, Theorem 1.2]), for $r \in G$, we have

$$T(r, f) = T\left(r, \frac{1}{f}\right) + O(1) = N\left(r, \frac{1}{f}\right) + O(1) = o\bigl(\log M(r, f)\bigr).$$

On the other hand, from (1) we have

$$T(r, f) = m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^{+}\bigl|f\bigl(re^{i\theta}\bigr)\bigr|\, d\theta \geqslant d \log M(r, f), \quad r \in G.$$

Thus we derive a contradiction from which Corollary 3 follows.

Below as another application of Corollary 1, we exhibit non-existence of the Baker wandering domains of a transcendental meromorphic function $f(z)$ with at most finitely many poles in terms of the existence of two values which $f(z)$ assumes few times in an angular domain. By this result, we relate theory of complex dynamics to that of value distribution again. By $\Omega(\alpha, \beta)$ we denote the angular domain $\{z\colon \alpha < \arg z < \beta\}$, $0 \leqslant \alpha < \beta < 2\pi$ and by $\bar{n}(r, \Omega(\alpha, \beta), f = a)$ the number of the distinct roots of $f(z) = a$ in $\{|z| < r\} \cap \Omega(\alpha, \beta)$.

Corollary 4. *Let $f(z)$ be a transcendental meromorphic function with at most finitely many poles. If we have an angular domain $\Omega = \Omega(\alpha, \beta)$, $0 \leqslant \alpha < \beta < 2\pi$ and $\mu(f) > \frac{\pi}{\beta - \alpha}$ such that for some integer $k \geqslant 0$ and $0 \leqslant \rho < \mu = \mu(f)$, we have*

$$\bar{n}(r, \Omega, f = 0) + \bar{n}\bigl(r, \Omega, f^{(k)} = 1\bigr) < r^{\rho} + O(\log r), \quad r \notin E, \tag{6}$$

then J_f has an unbounded component and $f(z)$ has no Baker wandering domains.

Remark.

(D) A natural problem is whether Corollary 4 still holds for the order replacing the lower order.

(E) If $\mu(f) = \infty$, then the result in Corollary 4 holds, as long as we assume that there exists an angular domain $\Omega = \Omega(\alpha, \beta)$ such that

$$\limsup_{r \notin E \to \infty} \frac{\bar{n}(r, \Omega, f = 0) + \bar{n}(r, \Omega, f^{(k)} = 1)}{\log r} < \infty.$$

Corollary 5. *Let $f(z)$ be a transcendental meromorphic function with at most finitely many poles. If for all sufficiently large $r > 0$ and $d > 1$, we have*

$$\log M(2r, f) > d \log M(r, f), \tag{7}$$

then J_f has an unbounded component and $f(z)$ has no Baker wandering domains.

Remark.

(F) Corollary 5 is still true if (7) is replaced by $T(2r, f) > dT(r, f)$. From $T(2r, f) > dT(r, f)$, by using Chuang's inequality (see [7]), we can deduce that $T(2r, f^{(k)}) > d_k T(r, f^{(k)})$, $d_k > 1$ and so $f^{(k)}(z)$ has no Baker wandering domains.

(G) It is obvious that if

$$\log M(r, f) \sim \frac{r}{\log r} \quad (r \to \infty), \tag{8}$$

then (7) holds. Therefore such a transcendental entire function has only simply connected Fatou components. $\prod_{n=2}^{\infty}(1 + \frac{z}{n(\log n)^2})$ is an entire function to satisfy (8).

(H) Let $\Gamma(z)$ be the gamma function. Then $1/\Gamma(z)$ is an entire function. Since

$$\log \frac{1}{\Gamma(z)} = z \log z + O(z)$$

uniformly as $z \to \infty$ for $|\arg z| < \pi - \delta$,

$$T(r, 1/\Gamma) = \bigl(1 + o(1)\bigr)\frac{1}{\pi} r \log r$$

so that $T(2r, 1/\Gamma) > dT(r, 1/\Gamma)$, $d > 1$. On the other hand, by noting that $N(r, 1/\Gamma) \sim r$, we have $\delta(0, 1/\Gamma) = 1$. Therefore, $1/\Gamma(z)$ has not multiply-connected components of the Fatou set.

(I) Let $f(z)$ be an entire function satisfying (7) and $g(z)$ a transcendental entire function. Since $\log M(r, g)$ is convex in $\log r$, $\frac{M(3r/2, g)}{M(r, g)} \to \infty$ $(r \to \infty)$. Then from a theorem of Pólya [14] (also see [8]), for some $0 < \rho < 1$ and all sufficiently large r, we have

$$\begin{aligned} \log M(2r, f \circ g) &\geqslant \log M\bigl(\rho M(3r/2, g), f\bigr) \geqslant \log M\bigl(2M(r, g), f\bigr) \\ &> d \log M\bigl(M(r, g), f\bigr) \geqslant d \log M(r, f \circ g), \end{aligned}$$

so that from Corollary 5 it follows that $f \circ g$ has no multiply-connected components of the Fatou set.

Corollary 6. *Every transcendental meromorphic function satisfying linear differential equation with rational coefficients must have no the Baker wandering domains and its Julia set has an unbounded component.*

Remark.

(J) $\sum_{j=1}^{m} Q_j(z)e^{P_j(z)}$ satisfies a linear differential equation with rational coefficients, where $Q_j(z)$ is rational and $P_j(z)$ a polynomial.

(K) Let $J_v(z)$ be v order Bessel function of the first kind which comes from the Bessel differential equation which is a linear differential equation with rational coefficients. Entire function $J_v(z)(z/2)^{-v}$ has no multiply-connected components of the Fatou set from Corollary 6.

The main results in this paper were addressed in the course of Complex Dynamics which was taught by the present author in Tsing Hua University from February to June, 2002.

2. Proof of Theorem and its corollaries

By using the same method as in the proof of Theorem 1 of Zheng [18], we can prove the following result which shall be used to the proof of Theorem and for completeness whose proof we shall give.

Lemma 1. *Let $f(z)$ and U be given as in Theorem. Then for A in Theorem, there exists in $\bigcup_{m=1}^{\infty} f^m(A)$ a sequence of round annuli D_n with 0 as center and $\operatorname{mod}(D_n) \to \infty$ and $\operatorname{dist}(0, D_n) \to \infty$ as $n \to \infty$.*

The hyperbolic metric plays a key role in the proof of Lemma 1. On a hyperbolic open set Ω, that is, $\hat{\mathbf{C}} \setminus \Omega$ contains at least three points, we have the hyperbolic density $\lambda_\Omega(z)$ defined as follows: $\lambda_\Omega(z)$ is the hyperbolic density on every component of Ω. We define the hyperbolic length of a curve α in Ω by

$$L_\Omega(\alpha) = \int_\alpha \lambda_\Omega(z)|dz|.$$

For $a \notin \Omega$ define

$$C_\Omega(a) := \inf\{\lambda_\Omega(z)|z-a|:\ z \in \Omega\}.$$

In the proof of Lemma 1, we need the following result, which is essentially due to Beardon and Pommerenke [4] and see [17].

Lemma 2. *Let Ω be a hyperbolic open set. Then for $a \notin \Omega$, we have*

$$\frac{1}{2(\beta_\Omega(z;a)+\kappa)} \leqslant \lambda_\Omega(z)|z-a| \leqslant \frac{\pi}{4\beta_\Omega(z;a)}, \tag{9}$$

where $\kappa = \Gamma(1/4)^4/(4\pi^2)$ *and*

$$\beta_\Omega(z;a) = \inf\left\{\left|\log\frac{|z-a|}{|b-a|}\right|:\ b \in \partial\Omega\right\}.$$

Proof of Lemma 1. Set $H = \bigcup_{m=0}^{\infty} f^m(A)$. Take a point b in a non-degenerate boundary of A such that $f(z)$ assumes value b at infinitely many points and obviously $b \in \partial H$. We want to prove that $C_H(b) = 0$. To the end, suppose that $C_H(b) > 0$.

We take a Jordan curve γ in A which separates the boundaries of A and choose a sufficiently large $R_0 > 0$ such that

$$n\left(R_0, \frac{1}{f-b}\right) - n(R_0, f) > \frac{1}{2\pi C_H(b)} L_A(\gamma). \tag{10}$$

Since $f^n(A) \to \infty$ $(n \to \infty)$, for a positive integer p we have a closed continuum $\Gamma_p \subset f^p(\gamma) \subset f^p(A)$ such that $\Gamma_p \subset \{|z| > R_0\}$ and $n(\Gamma_p, 0) = 1$.

Since $f : f^p(A) \to f^{p+1}(A)$, by the principle of the hyperbolic metric [1], we have

$$\lambda_{f^{p+1}(A)}\big(f(z)\big)\big|f'(z)\big| \leqslant \lambda_{f^p(A)}(z), \quad z \in f^p(A).$$

From the definition of $C_H(b)$, we have

$$\begin{aligned} &C_H(b)\big|f'(z)\big| \leqslant \lambda_{f^{p+1}(A)}\big(f(z)\big)\big|f(z)-b\big|\big|f'(z)\big| \leqslant \lambda_{f^p(A)}(z)\big|f(z)-b\big|, \\ &\quad z \in f^p(A). \end{aligned} \tag{11}$$

From (11), (10) and the argument principle, we have

$$\begin{aligned} L_A(\gamma) &\geqslant L_{f^p(A)}\big(f^p(\gamma)\big) \geqslant L_{f^p(A)}(\Gamma_p) = \int_{\Gamma_p} \lambda_{f^p(A)}(z)|dz| \\ &\geqslant C_H(b)\int_{\Gamma_p} \frac{|f'(z)|}{|f(z)-b|}|dz| \geqslant 2\pi C_H(b)\left|\frac{1}{2\pi i}\int_{\Gamma_p}\frac{f'(z)}{f(z)-b}\,dz\right| \\ &= 2\pi C_H(b)\left(n\left(\Gamma_p, \frac{1}{f-b}\right) - n(\Gamma_p, f)\right) \\ &> 2\pi C_H(b)\left(n\left(R_0, \frac{1}{f-b}\right) - n(R_0, f)\right) > L_A(\gamma). \end{aligned}$$

This is impossible. Therefore, we have proved that $C_H(b) = 0$.

Then there exists a sequence $\{z_n\}$ in H such that $\lambda_H(z_n)|z_n - b| \to 0$ as $n \to \infty$. Put $\delta_n = |z_n - b|$ and $\beta_n = \beta_H(z_n; b)$. From (9) it follows that $\beta_n \to +\infty$ as $n \to \infty$. The definition of $\beta_H(z; b)$ implies that for the annuli

$$B_n = \left\{z:\ \delta_n e^{-\beta_n} < |z-b| < \delta_n e^{\beta_n}\right\},$$

we have $B_n \subset H$ and $\operatorname{mod}(B_n) = 2\beta_n \to \infty$. Since b lies in a continuum of ∂H, $\delta_n \nrightarrow 0$, otherwise $\beta_n \to 0$. From the formula of $\beta_H(z; b)$, we deduce that $\delta_n \to +\infty$ as $n \to \infty$. Since F_f has only bounded components, each $f^m(A)$ is bounded, and so contains at most finitely many B_n. Thus $\operatorname{dist}(0, B_n) \to \infty$ $(n \to \infty)$.

Then there exists in H a sequence of annuli $\{D_n\}$ with center 0 such that $\mathrm{mod}(D_n) = \mathrm{mod}(B_n) + O(1)$ and $\mathrm{dist}(0, D_n) \to \infty$ as $n \to \infty$. Lemma 1 follows. □

Now we start to prove Theorem.

Proof of Theorem. Let $S(z)$ be the rational function consisting of sum of the singular parts of Laurant series of $f(z)$ at its poles. Then we can write

$$f(z) = g(z) + S(z),$$

where $g(z)$ is an entire function. We take a positive number R such that $f(z)$ is analytic and $|S(z)| < 1$ in $\{|z| > R\}$.

From Lemma 1 it follows that for some sufficiently large p, $f^p(A) \supseteq \{s/2 < |z| < 8s\}$ and $s > 2(M(R, g) + 1 + |g(0)|)$. Put

$$h(z) = \frac{2(g(8sz) - g(0))}{M(4s, g)}, \quad |z| < 1.$$

By the theorem of Bohr (see [8, Theorem 6.9]), $h(\{|z| < 1\}) \supset \{|z| = \tilde{r}\}$, $\tilde{r} > c$, c is a universal constant and $0 < c < 1$. Therefore, $g(\{|z| < 8s\}) \supset \{|z + g(0)| = r_s\}$, $r_s > \frac{c}{2} M(4s, g)$. By noting that $\log M(r, g)$ is convex in $\log r$, we have

$$\frac{M(2r, g)}{M(r, g)} \to \infty \quad (r \to \infty).$$

We can assume that for $r > R$, $M(2r, g) > \frac{16}{c} M(r, g)$. Thus $r_s > 8M(2s, g) > M(s, g)$ and since $g(\{|z| \leqslant s\}) \subset \{|z| \leqslant M(s, g)\}$, we have that

$$g\bigl(\{s < |z| < 8s\}\bigr) \supset \bigl\{\bigl|z + g(0)\bigr| = r_s\bigr\}$$

so that $f(\{s/2 < |z| < 8s\}) \supset \{|z + g(0)| = r_s\}$. It follows from $f(\{|z| = s/2\}) \subset \{|z| < M(s, g)/2\}$ that $\partial f(\{s/2 < |z| < 8s\}) \cap f(\{|z| = 8s\})$ lies in $\{|z| > 8M(2s, g)\}$ and surrounds the origin. It is easy to see that

$$f^{p+1}(A) \supset f\bigl(\{s/2 < |z| < 8s\}\bigr) \supset \bigl\{M(s, g)/2 < |z| < 8M(2s, g)\bigr\}.$$

By induction with $M(2s, g)$ replacing s, we have

$$f^{p+n}(A) \supset \bigl\{M_n(s, g) < |z| < 8M_n(2s, g)\bigr\}, \tag{12}$$

where $M_n(r, g) = M_{n-1}(M(r, g), g)$. We want to show the round annuli $\{M_n(s, g) < |z| < M_n(2s, g)\}$ satisfies the requirement of theorem, indeed, we have

$$M_n(2s, g) \geqslant M_{n-1}\bigl(2^2 M(s, g), g\bigr) \geqslant M_{n-2}\bigl(2^4 M(s, g), g\bigr) \geqslant \cdots \geqslant 2^{2^n} M_n(s, g).$$

We complete the proof of Theorem. □

When $f(z)$ is entire, we give another proof of Theorem as follows.

Suppose that the result of Theorem does not hold. Then there exists a sequence $\{n_k\}$ such that $\bigcup_{k=1}^{\infty} f^{n_k}(A)$ does not contain any sequence of round annuli B_m with the origin center and $\mathrm{mod}(B_m) \to \infty$ and $\mathrm{dist}(0, B_m) \to \infty$ as $m \to \infty$. Consider $H = \bigcup_{k=1}^{\infty} f^{n_k}(A)$. From Corollary 4 of Zheng [18], $\delta(0, f - z) = 0$, and so $f(z)$ has a fixed-point b, and it is obvious that b is not in the closure of H, for any wandering domain of $f(z)$ cannot contain periodic points of $f(z)$. We can choose b not to be a Nevanlinna deficient value of $f(z)$, and hence the equation $f(z)=b$ has infinitely many roots. Since all the roots of $f(z)=b$ are the roots of $f^n(z)=b$, $n\geqslant 1$, for each n and $r>0$ we have

$$n\left(r,\frac{1}{f^n-b}\right)\geqslant n\left(r,\frac{1}{f-b}\right).$$

Thus we have a sufficiently large $R_0>0$ such that (10) holds for $f^{n_{k+1}-n_k}(z): f^{n_k}(A)\to f^{n_{k+1}}(A)$, provided that $C_H(b)>0$. Then by the same argument as in the proof of Lemma 1, we can prove $C_H(b)=0$ and further we obtain a sequence of round annuli B_m with the origin center and $\operatorname{mod}(B_m)\to\infty$ and $\operatorname{dist}(0,B_m)\to\infty$ as $m\to\infty$. Thus we derive a contradiction.

Theorem follows.

In the proof of Corollary 1, we need the following, which is essentially due to Baker [3] (see [5, Lemma 7] and [19, Lemma 4]).

Lemma 3. *Let U be a domain in the complex plane and $f(z)$ be defined and analytic in $f^n(U)$ $(n=0,1,\ldots)$ inductively such that $H=\bigcup_{n=0}^{\infty}f^n(U)$ has at least two finite boundary points in the complex plane. If $f^{n_k}|_U\to\infty$ $(k\to\infty)$, then for any compact subset K of H, there exists a positive constant M such that for all sufficiently large k, we have*

$$\left|f^{n_k}(z)\right|\leqslant\left|f^{n_k}(z')\right|^M,\quad \text{for all } z,z'\in K. \tag{13}$$

Proof of Corollary 1. Under the assumption of Corollary 1, there exists a Baker wandering domain U containing an A in Theorem. From Theorem, we have in $\bigcup_{n=0}^{\infty}f^n(A)$ a sequence $D_n=\{z\colon r_n\leqslant|z-a|\leqslant R_n\}$ with $r_n\to\infty$ and $R_n/r_n\to\infty$ $(n\to\infty)$. Set

$$G=\bigcup_{n=1}^{\infty}\{r\colon r_n\leqslant r\leqslant R_n/\sqrt{2}\}\quad\text{and}\quad C(r,a)=\bigl\{z\colon |z-a|=r\bigr\}.$$

Obviously, G has the infinite logarithmic measure. For every $r\in G$, we have $C(r,a)\subset D_n$ for some n and we have a curve $\gamma(r)$ in A and a positive integer $m(n)$ such that $C(r,a)=f^{m(n)}(\gamma(r))$. Thus we have two points w_0 and w_1 in $\gamma(r)\subset A$ such that

$$M_c(r,a,f)=\max_{z\in C(r,a)}\left|f(z)\right|=\left|f\bigl(f^{m(n)}(w_0)\bigr)\right| \tag{14}$$

and

$$m_c(r,a,f)=\min_{z\in C(r,a)}\left|f(z)\right|=\left|f\bigl(f^{m(n)}(w_1)\bigr)\right|. \tag{15}$$

By noting that $\overline{A}$ is a compact subset of U and $f^n(U)\to\infty$ $(n\to\infty)$, we can apply Lemma 3 to show that for some constant $M>1$, we have

$$\left|f^{m(n)+1}(w_0)\right|\leqslant\left|f^{m(n)+1}(w_1)\right|^M. \tag{16}$$

Combining (14)–(16), we obtain (1) with $d=1/M$.

Set $S(r,a)=\{z\colon |\operatorname{Re} z-\operatorname{Re} a|=r, |\operatorname{Im} z-\operatorname{Im} a|=r\}$. $S(r,a)$ is a square with center a. It is easy to see that for every $r\in G$, $S(r,a)\subset D_n$ for some n. Then by the same argument as in above, we can imply (2).

The proof of Corollary 1 is completed. □

In order to prove Corollary 4, we need the Nevanlinna theory on angular domain. Let $f(z)$ be a meromorphic function on the angular domain $\overline{\Omega}(\alpha,\beta)=\{z\colon \alpha\leqslant \arg z\leqslant\beta\}$, where $0<\beta-\alpha\leqslant 2\pi$. Following Nevanlinna (see [11]) define

$$A_{\alpha,\beta}(r,f)=\frac{\omega}{\pi}\int_1^r\left(\frac{1}{t^\omega}-\frac{t^\omega}{r^{2\omega}}\right)\left\{\log^+\left|f\left(te^{i\alpha}\right)\right|+\log^+\left|f\left(te^{i\beta}\right)\right|\right\}\frac{dt}{t},$$

$$B_{\alpha,\beta}(r,f)=\frac{2\omega}{\pi r^\omega}\int_\alpha^\beta\log^+\left|f\left(re^{i\theta}\right)\right|\sin\omega(\theta-\alpha)\,d\theta,$$

$$C_{\alpha,\beta}(r,f)=2\sum_{1<|b_m|<r}\left(\frac{1}{|b_m|^\omega}-\frac{|b_m|^\omega}{r^{2\omega}}\right)\sin\omega(\theta_m-\alpha), \tag{17}$$

where $\omega=\frac{\pi}{\beta-\alpha}$ and $b_m=|b_m|e^{i\theta_m}$ are the poles of $f(z)$ on $\overline{\Omega}(\alpha,\beta)$ appeared according to their multiplicities. $\overline{C}_{\alpha,\beta}(r,f)$ has the same formula as in $C_{\alpha,\beta}(r,f)$ for the distinct poles of $f(z)$ on $\overline{\Omega}(\alpha,\beta)$ and the Nevanlinna's angular characteristic is defined as follows:

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f).$$

Lemma 4 [20]. *Let $f(z)$ be transcendental and meromorphic in* **C**. *If for $\rho>0$,*

$$\limsup_{r\to\infty}\frac{\log\bar{n}(r,\Omega(\alpha,\beta),f=a)}{\log r}\leqslant\rho, \tag{18}$$

then given arbitrary small $\epsilon>0$, for sufficiently large $r>0$, we have

$$\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)<K\left(r^{\rho-\omega+\epsilon}+\log r\right), \tag{19}$$

where $\omega=\frac{\pi}{\beta-\alpha}$ and K is a positive constant depending on ϵ.

The following is the second Nevanlinna fundamental theorem on angular domain.

Lemma 5 [11]. *Let $f(z)$ be meromorphic in the complex plane and consider angular domain $\overline{\Omega}(\alpha,\beta)$. Then for arbitrary integer $k\geqslant 0$ we have*

$$\begin{aligned}S_{\alpha,\beta}(r,f)<{}&\overline{C}_{\alpha,\beta}(r,f)+C_{\alpha,\beta}\left(r,\frac{1}{f}\right)+C_{\alpha,\beta}\left(r,\frac{1}{f^{(k)}-1}\right)-C_{\alpha,\beta}\left(r,\frac{1}{f^{(k+1)}}\right)\\&+R_{\alpha,\beta}(r,f),\end{aligned} \tag{20}$$

where $R_{\alpha,\beta}(r,f)=O(\log rT(r,f))$, $r\notin E$.

The proof of Lemma 5 is omitted, since it can be proved by the similar arguments to those in the proof of Milloux inequality in the complex plane (see Hayman [12]).

Now we can prove Corollary 4.

Proof of Corollary 4. Take a positive number ε such that $\rho + \varepsilon < \mu$. Under (6), applying Lemma 4 gives that

$$\overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) + \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f^{(k)} - 1}\right) < K\left(r^{\rho-\omega+\epsilon} + \log r\right), \quad r \notin E. \tag{21}$$

Now it follows from Lemma 5 and (21) that

$$\begin{aligned} B_{\alpha,\beta}(r, f) &\leqslant S_{\alpha,\beta}(r, f) - C_{\alpha,\beta}(r, f) \\ &< (k+1)\overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) + \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f^{(k)} - 1}\right) + R_{\alpha,\beta}(r, f) \\ &< \tilde{K}\left(r^{\rho-\omega+\epsilon} + \log rT(r, f)\right), \quad r \notin E. \end{aligned}$$

Thus

$$\int_{\alpha-\varepsilon}^{\beta-\varepsilon} \log^+\left|f\left(re^{i\theta}\right)\right| d\theta \leqslant \frac{\pi}{2\omega}\sin(\varepsilon\omega)\tilde{K}\left(r^{\rho+\epsilon} + r^{\omega}\log rT(r, f)\right), \quad r \notin E. \tag{22}$$

If J_f has no unbounded components, then $f(z)$ has the Baker wandering domains and we have (1) for some positive number d and a set G such that $G \setminus E$ has infinite logarithmic measure. Thus applying (1) to (22) yields that for $r \in G \setminus E$,

$$d(\beta - \alpha - 2\varepsilon)\log M(r, f) < \frac{\pi}{2\omega}\sin(\varepsilon\omega)\tilde{K}\left(r^{\rho+\epsilon} + r^{\omega}\log rT(r, f)\right),$$

so that $\mu \leqslant \max\{\rho + \varepsilon, \omega\}$. We derive a contradiction. Corollary 4 follows. □

Proof of Corollary 5. Suppose on the contrary that J_f has no unbounded components, then we have a sequence of round annulus $D_n = \{z\colon r_n \leqslant |z| \leqslant R_n\}$ such that $R_n/r_n \to \infty$ and $r_n \to \infty$ and $f(D_n) \subset \{|z| > R_n\}$ so that $|f(z)| > 1$ in D_n and $n(R_n, 1/f) = n(r_n, 1/f)$. From (12) and (7), we can require that $R_n = r_n^d$.

Thus by using the first Nevanlinna fundamental theorem, we have

$$\begin{aligned} &T(R_n, f) + O(1) \\ &\quad = T(R_n, 1/f) = m(R_n, 1/f) + N(R_n, 1/f) = N(R_n, 1/f) \\ &\quad = N(r_n, 1/f) + \int_{r_n}^{R_n} \frac{n(t, 1/f) - n(0, 1/f)}{t}\, dt + n(0, 1/f)\log\frac{R_n}{r_n} \\ &\quad = N(r_n, 1/f) + \left(n(r_n, 1/f) - n(0, 1/f)\right)\log\frac{R_n}{r_n} + n(0, 1/f)\log\frac{R_n}{r_n} \\ &\quad < N(r_n, 1/f) + \log\frac{R_n}{r_n} N(er_n, 1/f) < \left(2 + (d-1)\log r_n\right)T(er_n, f). \end{aligned}$$

On the other hand, by x_n we denote the maximum integer part of $[\log 2]^{-1}\log(r_n^{d-1}/2e)$. Then by using (7), we have

$$\begin{aligned} T(R_n, f) = T\big(r_n^d, f\big) &\geqslant \frac{1}{3}\log M\big(r_n^d/2, f\big) \geqslant \frac{1}{3}\log M\big(2^{x_n} e r_n, f\big) \\ &\geqslant \frac{1}{3} d^{x_n} \log M(e r_n, f) \geqslant K r_n^c \log M(e r_n, f), \end{aligned}$$

where $c = (d-1)\frac{\log d}{\log 2}$ and $K = (3d)^{-1}(2e)^{-c}$.

Thus we obtain the inequality $2 + (d-1)\log r_n \geqslant K r_n^c$. This is impossible, since $r_n \to \infty$ as $n \to \infty$.

Corollary 5 follows. □

Finally, we give a proof of Corollary 6 by using asymptotic integration theory (see Brüggemann [6], Steinmetz [16] and Zheng [21]).

Proof of Corollary 6. Let $f(z)$ be a function in Corollary 6. From the theory of asymptotic integration, a nth order linear differential equation with rational coefficients has n linearly independent formal solutions

$$W_j = \exp\big(P_j(z)\big) z^{d_j} \big[\log z^{1/p}\big]^{m_j} Q_j(z, \log z), \quad 1 \leqslant j \leqslant n, \tag{23}$$

with $P_j(z)$ being a polynomial in $z^{1/p}$, $d_j \in C$, $m_j \in N_0$ and $Q_j(z, \log z) = 1 + O(1/\log z)$, as $|z| \to +\infty$. There exist a finitely many rays $\arg z = \theta_i$, $1 \leqslant i \leqslant m$, such that $\theta_1 < \theta_2 < \cdots < \theta_m$ and in $S_i = \{z \in C\colon \theta_i < \arg z < \theta_{j+1}\}$, the differential equation has a fundamental solution system with form in (23). Then in S_i, we have

$$f(z) = c_1 W_1 + c_2 W_2 + \cdots + c_n W_n,$$

where c_j is a complex number. We can assume without any loss of generalities that $P := P_1 = \cdots = P_k$ and for $k < s \leqslant n$ and $\theta_i < \theta < \theta_{i+1}$,

$$\mathrm{Re}\big(P\big(re^{i\theta}\big) - P_s\big(re^{i\theta}\big)\big) \to +\infty,$$

as $r \to +\infty$. Thus

$$\log\big|f\big(re^{i\theta}\big)\big| = \mathrm{Re}\big(P\big(re^{i\theta}\big)\big) + O\big(\log|z|\big). \tag{24}$$

Since any meromorphic solution of linear differential equation with rational coefficients has only finitely many poles and the finite positive order of growth, we define the indicator function $h_f(\theta)$ of $f(z)$ by

$$h_f(\theta) := \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r^\rho},$$

where ρ is the order of $f(z)$. Therefore, from (24), it follows that

$$T(r, f) = m(r, f) + O(\log r) = \frac{r^\rho}{2\pi}\int_0^{2\pi} \max\big\{0, h_f(\theta)\big\}\, d\theta + O\big(r^{\rho-\varepsilon}\big)$$

(see [16, Theorem 2]). It is easy to see that $T(2r, f) > dT(r, f)$, $d > 1$. Since $f(z)$ has only finitely many poles, Corollary 6 follows from Corollary 5. □

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