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Elimination of Composite Superpositions May Cause Abortion

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We will show by examples that the elimination of composite superpositions during the completion of a term rewriting system may lead to an unnecessary stop with failure that cannot be prevented by postponing critical pairs.

We would like to make a remark on the article "Only Prime Superpositions Need be Considered in the Knuth-Bendix Completion Procedure" by D. Kapur, D. R. Musser and P. Narendran (1988). In that article the authors presented a criterion that can be used to eliminate critical pairs during the Knuth-Bendix completion of a term rewriting system. They mentioned that the completion procedure may encounter an unorientable rule if their criterion is incorporated whereas this unorientable rule is not encountered if their criterion is not used. Furthermore, they made the conjecture that such a case does not pose any problems in generating complete systems using the RRL or the REVE system because of the option to postpone the consideration of a critical pair when such a situation arises (Kapur *et al.*, 1988, pp. 33–34).

The following example shows that this conjecture is not true. Even if postponing of critical pairs is allowed, it may happen that the usage of the criterion of Kapur *et al.* (1988) leads to a stop with failure, whereas the Knuth-Bendix completion procedure stops with success if this criterion is not used.

EXAMPLE

Let $\mathscr{R}_1 = \{l_i \to r_i / i = 1 \ldots 4\}$ where:

$$l_1 = f(g(h(x_1, y_1))) \qquad r_1 = f'(h'(x_1))$$

$$l_2 = g(h(x_2, y_2)) \qquad r_2 = g'(x_2)$$

$$l_3 = h(x_3, y_3) \qquad r_3 = f'(h'(a))$$

$$l_4 = g(f(g'(a))) \qquad r_4 = a$$

These rules are oriented according to the Knuth-Bendix ordering $<$ with the following weight function ϕ:

$$\phi(h) = 4$$

$$\phi(f') = 2$$

$$\phi(f) = \phi(g) = \phi(h') = \phi(g') = \phi(a) = 1$$

$$\phi(y) = 1 \quad \forall y \in \mathscr{V} \ (\mathscr{V} \text{ denotes the set of the variables})$$

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and any precedence.
Then there exist the following superpositions (according to the definition in (Kapur *et al.*, 1988)):

$$o_1 = \langle f(g(h(x_1, y_1))), \lambda, l_1 \to r_1, 1, l_2 \to r_2, \{x_2 \leftarrow x_1, y_2 \leftarrow y_1\}\rangle$$

$$o_2 = \langle g(h(x_2, y_2)), \lambda, l_2 \to r_2, 1, l_3 \to r_3, \{x_3 \leftarrow x_2, y_3 \leftarrow y_2\}\rangle$$

$$o_3 = \langle f(g(h(x_1, y_1))), \lambda, l_1 \to r_1, 1.1, l_3 \to r_3, \{x_3 \leftarrow x_1, y_3 \leftarrow y_1\}\rangle$$

(i) If the completion procedure with the criterion of Kapur *et al.* (1988) is started with input $\mathcal{R}_1$ and $<$, then the superposition o_1 will be eliminated (o_1 is composite), independently of the order in which these superpositions are treated. The consideration of the remaining superpositions will lead to:

o_2:

$$\begin{array}{ccc} & g(h(x_2, y_2)) & \\ {\scriptstyle l_2 \to r_2} \swarrow & & \searrow {\scriptstyle l_3 \to r_3} \\ g'(x_2) & & g(f'(h'(a))) \end{array}$$

$g'(x_2)$ and $g(f'(h'(a)))$ are irreducible and incomparable;
$(\phi(g'(x_2)) = 2;\ \phi(g(f'(h'(a)))) = 5$ and $|g'(x_2)|_{x_2} > |g(f'(h'(a)))|_{x_2})$

o_3:

$$\begin{array}{ccc} & f(g(h(x_1, y_1))) & \\ {\scriptstyle l_1 \to r_1} \swarrow & & \searrow {\scriptstyle l_3 \to r_3} \\ f'(h'(x_1)) & & f(g(f'(h'(a)))) \end{array}$$

$f'(h'(x_1))$ and $f(g(f'(h'(a))))$ are irreducible and incomparable;
$(\phi(f'(h'(x_1))) = 4;\ \phi(f(g(f'(h'(a))))) = 6$ and $|f'(h'(x_1))|_{x_1} > |f(g(f'(h'(a))))|_{x_1})$

There are no further superpositions.
Thus, the Knuth-Bendix completion algorithm stops with failure. It is obvious that this stop cannot be prevented by postponing critical pairs, by changing the reduction strategy and even not by considering the superpositions in a different order.

(ii) Otherwise, if the criterion of Kapur *et al.* (1988) is not used, then the Knuth-Bendix algorithm will stop with success:

o_1:

$$f(g(h(x_1, y_1))) \begin{cases} \xrightarrow{l_1 \to r_1} f'(h'(x_1)) \\ \xrightarrow[l_2 \to r_2]{} f(g'(x_1)) \end{cases}$$

new rule: $f'(h'(x_5)) \to f(g'(x_5))$ $\quad (l_5 := f'(h'(x_5));\ r_5 := f(g'(x_5)))$

o_2:

$$\begin{tikzcd}[column sep=small]
 & g'(x_2) & & \\
g(h(x_2, y_2)) \arrow[ur, "l_2 \to r_2"] \arrow[dr, "l_3 \to r_3"'] & & & \\
 & g(f'(h'(a))) \arrow[r, "l_5 \to r_5"'] & g(f(g'(a))) \arrow[r, "l_4 \to r_4"'] & a
\end{tikzcd}$$

new rule: $g'(x_6)) \to a \qquad (l_6 := g'(x_6);\ r_6 := (a)$

o_3:

$$\begin{tikzcd}[column sep=small]
 & f'(h'(x_1)) \arrow[r, "l_5 \to r_5"] & f(g'(x_1)) \arrow[dr, "l_6 \to r_6"] & \\
f(g(h(x_1, y_1))) \arrow[ur, "l_1 \to r_1"] \arrow[dr, "l_3 \to r_3"'] & & & f(a) \\
 & f(g(f'(h'(a)))) \arrow[r, "l_5 \to r_5"'] & f(g(f(g'(a)))) \arrow[ur, "l_4 \to r_4"'] &
\end{tikzcd}$$

Superposition between $g'(x_6) \to a$ and $g(f(g'(a))) \to a$:
$o_4 = \langle g(f(g'(a))),\ \lambda,\ l_4 \to r_4,\ 1.1,\ l_6 \to r_6,\ \{x_6 \leftarrow a\}\rangle$

o_4:

$$\begin{tikzcd}
 & a. \\
g(f(g'(a))) \arrow[ur, "l_4 \to r_4"] \arrow[dr, "l_6 \to r_6"'] & \\
 & g(f(a))
\end{tikzcd}$$

new rule: $g(f(a)) \to a \qquad (l_7 := g(f(a));\ r_7 := a)$

There are no further superpositions:
Thus, the Knuth-Bendix completion algorithm stops with

$$\begin{aligned}
\mathcal{R}_1' = \mathcal{R}_1 \cup \{f'(h'(x_5)) &\to f(g'(x_5));\\
g'(x_6) \quad &\to a;\\
g(f(a)) \ &\to a\}
\end{aligned}$$

as the completed system.
Note that the system $\mathcal{R}_1'$ or $\mathcal{R}_1' \cup \{f(g(f(a))) \to f(a)\}$ will always be generated if the technique of postponing critical pairs is used. In that case the success of the algorithm depends neither on the order in which the superpositions are treated nor on the reduction strategy used.

Hence, the use of the criterion developed by Kapur *et al.* (1988) in the completion procedure may lead to an unnecessary stop with failure. A lot of similar examples can be constructed. For example, if o_3 does not exist or its corresponding critical pair is confluent, but o_2 leads to an incomparable pair, then o_1 may be necessary to solve this incomparable pair. The same problem may also arise if o_2 is eliminated too. In this case the connectedness of the corresponding critical pair below the term from which it is derived is established by other critical pairs if the completion procedure terminates with success. But, if one of these critical pairs is incomparable, o_1 may be necessary to solve this critical pair.

One may argue that the term rewriting system $\mathcal{R}_1$ is not interreduced. If $\mathcal{R}_1$ gets interreduced before any superposition is computed, then it depends on the strategy used for interreduction whether the Knuth-Bendix completion algorithm stops with success or with failure. In this example interreduction of the first rule by the second one is similar to overlapping these two rules. Thus, this interreduction leads to the key-rule $f'(h'(x_5)) \rightarrow f(g'(x_5))$ and a complete system can be generated as mentioned. But, if the first rule is interreduced by the third one, then the superposition o_1 is omitted and a stop with failure cannot be prevented.

However, the use of interreduction may not always solve these problems:

EXAMPLE

Let $\mathcal{R}_2 = \{l_i \rightarrow r_i / i = 1 \ldots 4\}$ where:

$$\begin{array}{ll} l_1 = f(g(x_1, y_1, y_1, z_1)) & r_1 = g'(x_1, y_1) \\ l_2 = g(x_2, y_2, a, h(y_2)) & r_2 = g'(y_2, y_2) \\ l_3 = h(a) & r_3 = h'(a) \\ l_4 = f(g'(a, a)) & r_4 = a \end{array}$$

These rules are oriented according to the Knuth-Bendix ordering $<$ with the following weight function ϕ:

$$\phi(g') = 6$$
$$\phi(f) = \phi(h) = 4$$
$$\phi(g) = \phi(h') = \phi(a) = 1$$
$$\phi(y) = 1 \qquad \forall y \in \mathcal{V}$$

and any precedence.

Then there exist the following superpositions:

$$o_1 = \langle f(g(x_1, a, a, h(a))), \lambda, l_1 \rightarrow r_1, 1, l_2 \rightarrow r_2, \{y_1 \leftarrow a, z_1 \leftarrow h(a), x_2 \leftarrow x_1, y_2 \leftarrow a\}\rangle$$
$$o_2 = \langle g(x_2, a, a, h(a)), \lambda, l_2 \rightarrow r_2, 4, l_3 \rightarrow r_3, \{y_2 \leftarrow a\}\rangle$$

(i) In this example the set of rules is interreduced. Thus, superpositions have to be computed in order to complete $\mathcal{R}_2$.
Since o_1 is composite it will be eliminated if the criterion of Kapur *et al.* (1988) is used. There remains only the consideration of the superposition o_2. But, since the corresponding critical pair is irreducible and incomparable the completion algorithm will stop with failure:

$$o_2: \quad \begin{array}{ccc} & g(x_2, a, a, h(a)) & \\ {\scriptstyle l_2 \rightarrow r_2} \swarrow & & \searrow {\scriptstyle l_3 \rightarrow r_3} \\ g'(a, a) & & g(x_2, a, a, h'(a)) \end{array}$$

$g'(a, a)$ and $g(x_2, a, a, h'(a))$ are irreducible and incomparable;
$(\phi(g'(a, a)) = 8;\ \phi(g(x_2, a, a, h'(a))) = 6$ and $|g(x_2, a, a, h'(a))|_{x_2} > |g'(a, a)|_{x_2})$

(ii) As in the previous example such a failure will not occur, if the criterion of Kapur *et al.* (1988) is not applied:

o_1:

$$\begin{tikzcd}
 & g'(x_1, a) & \\
f(g(x_1, a, a, h(a))) \arrow[ur, "l_1\to r_1"] \arrow[dr, "l_2\to r_2"'] & & \\
 & f(g'(a, a)) \arrow[r, "l_4\to r_4"'] & a
\end{tikzcd}$$

new rule: $g'(x_5,a)\to a$ $\quad (l_5 := g'(x_5, a);\ r_5 := a)$

o_2:

$$\begin{tikzcd}
 & g'(a, a) \arrow[r, "l_5\to r_5"] & a \\
g(x_2, a, a, h(a)) \arrow[ur, "l_2\to r_2"] \arrow[dr, "l_3\to r_3"'] & & \\
 & g(x_2, a, a, h'(a)) &
\end{tikzcd}$$

new rule: $g(x_6, a, a, h'(a))\to Ma$ $\quad (l_6 := g(x_6, a, a, h'(a));\ r_6 := a)$

Superposition between $g'(x_5, a)\to a$ and $f(g'(a, a))\to a$:
$o_3 = \langle f(g'(a, a)), \lambda, l_4\to r_4, 1, l_5\to r_5, \{x_5\leftarrow a\}\rangle$

o_3:

$$\begin{tikzcd}
 & a \\
f(g'(a, a)) \arrow[ur, "l_4\to r_4"] \arrow[dr, "l_5\to r_5"'] & \\
 & f(a)
\end{tikzcd}$$

new rule: $f(a)\to a$ $\quad (l_7 := f(a);\ r_7 := a)$

Superposition between $g(x_6, a, a, h'(a))\to a$ and $f(g(x_1, y_1, z_1))\to g'(x_1, y_1)$:
$o_4\times\langle f(g(x_1, a, a, h'(a))), \lambda, l_1\to r_1, 1, l_6\to r_6, \{y_1\leftarrow a, z_1\leftarrow h'(a), x_6\leftarrow x_1\}\rangle$

o_4:

$$\begin{tikzcd}
 & g'(x_1, a) \arrow[dr, "l_5\to r_5"] & \\
f(g(x_1, a, a, h'(a))) \arrow[ur, "l_1\to r_1"] \arrow[dr, "l_6\to r_6"'] & & a \\
 & f(a) \arrow[ur, "l_7\to r_7"'] &
\end{tikzcd}$$

There are no further superpositions.
Thus, the Knuth-Bendix completion algorithm stops with

$$\begin{aligned}\mathcal{R}_2' = \mathcal{R}_2\cup\{g'(x_5, a) &\to a;\\ g(x_6, a, a, h'(a)) &\to a;\\ f(a) &\to a\}\end{aligned}$$

as the completed system.

Our first example has shown that there is a strong connection between critical pair criteria and interreduction (see also Küchlin, 1985). Therefore one may suppose that even interreduction may lead to an unnecessary stop with failure. Indeed such situations may arise:

EXAMPLE

Let $\mathcal{R}_3 = \{l_i \to r_i / i = 1 \ldots 5\}$ where:

$$
\begin{array}{ll}
l_1 = f(x_1, g(a, y_1, y_1)) & r_1 = f'(y_1, y_1) \\
l_2 = g(x_2, y_2, h(x_2)) & r_2 = g'(x_2, y_2) \\
l_3 = h(x_3) & r_3 = h'(x_3) \\
l_4 = f(x_4, g'(a, h'(a))) & r_4 = h'(x_4) \\
l_5 = f'(h'(a), h'(a)) & r_5 = a
\end{array}
$$

These rules are oriented according to the Knuth-Bendix ordering $<$ with the following weight function ϕ:

$$\phi(h) = \phi(g') = 4$$

$$\phi(f) = \phi(g) = \phi(h') = \phi(f') = \phi(a) = 1$$

$$\phi(y) = 1 \qquad \forall y \in \mathcal{V}$$

and any precedence.
Then there exist the following superpositions:

$$o_1 = \langle f(x_1, g(a, h(a), h(a))), \lambda, l_1 \to r_1, 2, l_2 \to r_2, \{y_1 \leftarrow h(a), x_2 \leftarrow a, y_2 \leftarrow h(a)\}\rangle$$

$$o_2 = \langle g(x_2, y_2, h(x_2)), \lambda, l_2 \to r_2, 3, l_3 \to r_3, \{x_3 \leftarrow x_2\}\rangle$$

(i) Aside from the second rule all rules of $\mathcal{R}_3$ are in reduced form. The right-hand side of the second rule is also irreducible, while the left-hand side can be reduced to $g(x_2, y_2, h'(x_2))$ by the third rule. But, $g'(x_2, y_2)$ and $g(x_2, y_2, h'(x_2))$ are incomparable ($\phi(g'(x_2, y_2)) = 6$; $\phi(g(x_2, y_2, h'(x_2))) = 5$ and $|g(x_2, y_2, h'(x_2))|_{x_2} > |g'(x_2, y_2)|_{x_2}$). Thus, the interreduction of $\mathcal{R}_3$ will result in a failure.
If the criterion of Kapur *et al.* (1988) is used, then the same problem will arise too. In that case the superposition o_1 will be eliminated and the same incomparable pair is generated by overlapping $l_2 \to r_2$ and $l_3 \to r_3$.

(ii) On the other hand, if the superposition o_1 is considered, that means if it is not eliminated, neither by the criterion of Kapur *et al.* (1988) nor by interreduction, then $\mathcal{R}_3$ will be completed with success by the Knuth-Bendix algorithm:

o_1:

$$
\begin{array}{l}
\nearrow \xrightarrow{l_1 \to r_1} f'(h(a), h(a)) \xrightarrow[2]{l_3 \to r_3} f'(h'(a), h'(a)) \xrightarrow{l_5 \to r_5} a \\
f(x_1, g(a, h(a), h(a))) \\
\searrow \xrightarrow[l_2 \to r_2]{} f(x_1, g'(a, h(a))) \xrightarrow[l_3 \to r_3]{} f(x_1, g'(a, h'(a))) \xrightarrow[l_4 \to r_4]{} h'(x_1)
\end{array}
$$

new rule: $h'(x_6) \to a$ $\quad (l_6 := h'(x_6);\ r_6 := a)$

This rule is essential: It can be used to reduce the incomparable pair mentioned before. Further reductions will lead to the following system $\mathcal{R}_3'$ which is complete and reduced:

$$\begin{aligned}\mathcal{R}_3' = \{ f(x_1, g(a, y_1, y_1)) &\to f'(y_1, y_1);\\ g'(x_2, y_2) &\to g(x_2, y_2, a);\\ h(x_3) &\to a;\\ f'(a, a) &\to a;\\ h'(x_6) &\to a\}\end{aligned}$$

We have constructed these counter-examples when comparing extensively the critical pair criterion of Kapur *et al.* (1988) with those of Winkler & Buchberger (1983) (see also Winkler, 1984) and Küchlin (1985). If the criterion of Winkler & Buchberger or that of Küchlin is used, these kinds of unnecessary stops with failure may not arise. Elimination of a superposition by one of those two criteria implies that the corresponding critical pair is connected below the term from which it is derived. Nevertheless, eliminating such a superposition may also result in an unnecessary failure. But, in all cases known to us these failures can be prevented by postponing critical pairs, by changing the reduction strategy used or by considering the superpositions in a different order (in a way that the same superpositions are still eliminated) (Sattler-Klein, 1987).

Furthermore, we have also transferred all these criteria on string rewriting systems and developed a new criterion that is in a certain sense stronger than the other ones (Sattler-Klein, 1987).

These criteria and some variants have been integrated in our completion systems CŌMTĒS (completion system for term rewriting systems) (Sonntag, 1988) and COSY (completion system for string rewriting systems) (Sattler-Klein, 1991). Extensive test series have been made. A report about our research on this topic will be available soon (Müller *et al.*, 1991).

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