Journal of Pure and Applied Algebra 216 (2012) 149–153

Contents lists available at SciVerse ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

# Locally tame plane polynomial automorphisms

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ARTICLE INFO

*Article history:*
Received 9 November 2010
Available online 14 June 2011
Communicated by C.A. Weibel

*MSC:* 14R10

ABSTRACT

For automorphisms of a polynomial ring in two variables over a domain $R$, we show that local tameness implies global tameness provided that every 2-generated locally free $R$-module of rank 1 is free. We give examples illustrating this property.



## 0. Introduction

A natural problem in commutative algebra and algebraic geometry is to understand the group $\mathrm{GA}_n(R)$ of algebraic automorphisms of the affine $n$-space $\mathbb{A}^n_R = \mathrm{Spec}(R[X_1, \ldots, X_n])$ over a ring $R$. This group is anti-isomorphic to the group of $R$-algebra automorphisms of the polynomial ring $R[X_1, \ldots, X_n]$ over $R$. Although much progress has been made in this direction during the past decades, one can state that only the case $n = 2$ and $R$ is a field is fully understood. A central and fruitful notion in the study of polynomial automorphisms is the notion of tameness: an automorphism is called *tame* if it can be written as a composition of affine and triangular ones, where by a triangular automorphism, we mean an automorphism $F = (F_1, \ldots, F_n) \in \mathrm{GA}_n(R)$ such that $F_i \in R[X_i, \ldots, X_n]$ for every $i = 1, \ldots, n$. Tame automorphisms form a subgroup $\mathrm{TA}_n(R)$ of $\mathrm{GA}_n(R)$ and a classical theorem due to Jung in characteristic zero [8] and van der Kulk in the general case [9] asserts that if $R$ is a field $k$ then $\mathrm{GA}_2(k) = \mathrm{TA}_2(k)$. The result is even more precise: $\mathrm{GA}_2(k)$ is the free product of the subgroups of affine and triangular automorphisms amalgamated over their intersection. In contrast, even the equality $\mathrm{TA}_2(R) = \mathrm{GA}_2(R)$ is no longer true for a general domain $R$, as illustrated by a famous example due to Nagata: for an element $z \in R \setminus \{0\}$ the endomorphism

$$F = (X - 2Y(zx + Y^2) - z(zX + Y^2)^2, Y + z(zX + Y^2))$$

of $R[X, Y]$ is in $\mathrm{GA}_2(R)$ and can be decomposed as

$$F = (X - z^{-1}Y^2, Y)(X, z^2X + Y)(X + z^{-1}Y^2, Y)$$

in $\mathrm{GA}_2(K(R)) = \mathrm{TA}_2(K(R))$. Such a decomposition being essentially unique, this implies in particular that if $z$ is not invertible in $R$, then $F$ cannot be tame over $R$. Note that more generally, given a prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$, $F \in \mathrm{TA}_2(R_\mathfrak{p})$ if and only if $z \notin \mathfrak{p}$.

Automorphisms $F \in \mathrm{GA}_n(R)$ such that $F \in \mathrm{TA}_n(R_\mathfrak{p})$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$ are said to be *locally tame*. Of course, every tame automorphism is locally tame. Furthermore, it has been recently proved in [1] that for plane polynomial automorphisms over an arbitrary base ring, local tameness implies stable tameness. In contrast, the Nagata automorphism is neither tame

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0022-4049/$ – see front matter 
doi:10.1016/j.jpaa.2011.05.013

nor locally tame, while stably tame [13]. This could suggest that, at least for plane polynomial automorphisms, tameness is a property that can be checked locally on the base ring. In particular, one could hope that the only reason why an automorphism $F \in \mathrm{GA}_2(R)$ is not tame is because there exists a prime $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $F$ is already nontame over $R_{\mathfrak{p}}$. It turns out that this hope is too optimistic, and that in general, some "global" properties of $R$ have to be taken into account to be able to infer tameness directly from local tameness. The main result of this article is the following characterization of rings for which global tameness can be checked locally:

**Theorem.** *For a domain R, the following assertions are equivalent:*

(1) $\mathrm{TA}_2(R) = \bigcap_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{TA}_2(R_{\mathfrak{p}})$,
(2) *Every 2-generated locally free R-module of rank 1 is free.*

In particular, it follows that over a unique factorization domain $R$, tameness is a local property of automorphisms.

The article is organized as follows. Section 1 is devoted to the proof of the above characterization, that we essentially derive from the fact that tame automorphisms of a polynomial ring in two variables can be recognized algorithmically. In Section 2, we consider many examples that illustrate condition (2) in the Theorem above.

## 1. From local tameness to global tameness

In this section, we characterize domains $R$ with the property that an automorphism $F = (F_1, F_2) \in \mathrm{GA}_2(R)$ is tame if and only if it is locally tame.

**Notations.** For an automorphism $F = (F_1, F_2) \in \mathrm{GA}_2(R)$, we let $\deg F = (\deg F_1, \deg F_2) \in (\mathbb{Z}_{>0})^2$ considered as equipped with the product order. We denote by $\overline{F_i}$ the homogeneous component of $F_i$ of degree $\deg F_i$, $i = 1, 2$. An automorphism with $\deg F = (1, 1)$ is affine, and we denote by $\mathrm{Aff}_2(R)$ the corresponding subgroup of $\mathrm{GA}_2(R)$.

### 1.1. *Properties of automorphisms*

Even if the equality $\mathrm{GA}_2(R) = \mathrm{TA}_2(R)$ is no longer true for a general domain $R$, tame automorphisms of a polynomial ring in two variables can be recognized algorithmically. Indeed, the following result quoted from [5, Prop. 1] (see also [4, Cor. 5.1.6]) says in essence that for every $F \in \mathrm{TA}_2(R)$ with $\deg F > (1, 1)$ there exists a linear or a triangular automorphism $G$ such that $\deg GF < \deg F$.

**Proposition 1.1.** *Let* $F = (F_1, F_2) \in \mathrm{TA}_2(R)$ *and let* $(d_1, d_2) = \deg F$. *Then the following holds:*

(a) $d_1 \mid d_2$ *or* $d_2 \mid d_1$.
(b) *If* $\max(d_1, d_2) > 1$ *then we have:*
  (i) *If* $d_1 < d_2$ *then* $\overline{F_2} = c\overline{F_1}^{d_2/d_1}$ *for some* $c \in R$,
  (ii) *If* $d_2 < d_1$ *then* $\overline{F_1} = c\overline{F_2}^{d_1/d_2}$ *for some* $c \in R$,
  (iii) *If* $d_1 = d_2$ *then there exists* $G \in \mathrm{Aff}_2(R)$ *such that* $GF = (F_1', F_2')$ *satisfies* $\deg F_1' = d_1$ *and* $\deg F_2' < d_1$.

**1.2.** In contrast to the tame case, for an arbitrary automorphism $F = (F_1, F_2) \in \mathrm{GA}_2(R)$ with $\deg F_1 = \deg F_2$ there is no guarantee in general that there exists $G \in \mathrm{Aff}_2(R)$ such that $\deg GF < \deg F$. Indeed, such a $G$ exists if and only if there exists a unimodular vector $(\alpha_1, \alpha_2) \in R^2$ such that $\alpha_1\overline{F_1} + \alpha_2\overline{F_2} = 0$, which is the case if and only if the $R$-module $R\overline{F_1} + R\overline{F_2}$ is free of rank 1. Combined with [4, Ex. 6 p. 94], this observation leads to a natural procedure to construct families of locally tame but not (globally) tame automorphisms, namely:

**Proposition 1.3.** *If* $z, w \in R$ *and* $q(T) \in R[T]$ *is a polynomial of degree at least 2, then*

$$F := (X + wq(zX + wY), Y - zq(zX + wY))$$

*is an element of* $\mathrm{GA}_2(R)$. *Furthermore, F is tame if and only if* $(z, w)$ *is a principal ideal of R.*

*In particular, if* $(z, w)$ *is a locally principal but not principal ideal, then F is a locally tame but not globally tame automorphism.*

**Proof.** A straightforward verification shows that

$$H = (X - wq(zX + wY), Y + zq(zX + wY))$$

is an inverse for $F$. Suppose that $(z, w) = aR$ for some $a \in R \setminus \{0\}$. Replacing $q(T)$, $z$ and $w$ by $aq(aT)$, $a^{-1}z$ and $a^{-1}w$ respectively, we may assume that $(z, w) = R$. But then if we take any $G \in \mathrm{SL}_2(R)$ having $zX + wY$ as its first component, one checks that $F = G^{-1}(X, Y - q(X))G \in TA_2(R)$. Conversely, if $F \in \mathrm{TA}_2(R)$, then, since $\deg F_1 = \deg F_2 = \deg Q > 1$, it follows from Proposition 1.1 and the above discussion that the $R$-module generated by $\overline{F_1} = w\overline{q(zX + wY)}$ and $\overline{F_2} = -z\overline{q(zX + wY)}$ is free of rank 1. Simplifying by $\overline{q(zX + wY)}$, we get that the $R$-module generated by $w$ and $-z$ is free of rank 1, i.e., $(w, -z)$ is a principal ideal. □

**1.4.** It follows that locally tame but not globally tame automorphisms abound: for instance, in the proposition above, one can take for $R$ the coordinate ring of a smooth nonrational affine curve $C$ and for $z, w$ a pair of generators of the defining ideal of a nonprincipal Weil divisor on $C$ (see also Section 2 below for more examples).

### 1.2. A criterion

It turns out that the examples discussed above illustrate the only global obstruction to infer global tameness from local tameness, namely, the existence of 2-generated locally free but not globally free modules of rank 1. Indeed, we have the following criterion.

**Theorem 1.5.** *For a domain R, the following assertions are equivalent:*

(1) $\mathrm{TA}_2(R) = \bigcap_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{TA}_2(R_{\mathfrak{p}})$,
(2) *Every* 2*-generated locally free R-module of rank* 1 *is free.*

***Proof.*** (1) ⇒ (2). Since $R$ is a domain, every locally free $R$-module of rank 1 is isomorphic to an $R$-submodule of the field of fractions $K(R)$ of $R$ (see e.g. [7, Prop. 6.15]). In turn, every such submodule is isomorphic to an ideal of $R$. In particular, if there exists a locally free but nonfree 2-generated $R$-module of rank 1, then there exists locally principal but not principal ideal $(z, w)$ of $R$. But then any $F \in \mathrm{GA}_2(R)$ as in Proposition 1.3 above is locally tame but not tame.

(2) ⇒ (1). Conversely, for any domain $R$, it is clear that

$$\mathrm{TA}_2(R) \subseteq \bigcap_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{TA}_2(R_{\mathfrak{p}}) \subseteq \bigcap_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{GA}_2(R_{\mathfrak{p}}) = \mathrm{GA}_2(R).$$

Let $F = (F_1, F_2) \in \mathrm{GA}_2(R)$ be a locally tame automorphism and let $d_i = \deg F_i$, $i = 1, 2$. We may assume that $d_1 \le d_2$. If $d_1 = d_2 = 1$ then $F$ is affine, whence tame. We now proceed by induction on $(d_1, d_2)$, assuming that every locally tame automorphism of degree $(d_1', d_2') < (d_1, d_2)$ is globally tame.

• Case 1: $d_1 < d_2$. Since $F \in \mathrm{TA}_2(R_{(0)}) = \mathrm{TA}_2(K(R))$, it follows from Proposition 1.1 that $e = d_2/d_1 \in \mathbb{Z}_{>0}$ and that there exists $\alpha \in K(R)$ such that $\overline{F_2} = \alpha\overline{F_1}^e$. But since $F \in \mathrm{TA}_2(R_{\mathfrak{p}})$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$, it follows that

$$\alpha \in \bigcap_{\mathfrak{p}\in\mathrm{Spec}(R)} R_{\mathfrak{p}} = R.$$

Now, the automorphism $(X, Y - \alpha X^e)F$ satisfies the induction hypothesis and we are done with case.

• Case 2: $d_1 = d_2$. Since for any $\mathfrak{p} \in \mathrm{Spec}(R)$, we have $F \in \mathrm{TA}_2(R_{\mathfrak{p}})$, it follows from Proposition 1.1 and the discussion 1.2 that for every $\mathfrak{p} \in \mathrm{Spec}(R)$, the $R_{\mathfrak{p}}$ module generated by $\overline{F_1}$ and $\overline{F_2}$ is free of rank 1. This means exactly that the $R$-module generated by $\overline{F_1}$ and $\overline{F_2}$ is locally free of rank 1. Our assumption implies that it is globally free, and so, we deduce from 1.2 that there exist $G \in \mathrm{Aff}_2(R)$ such that $\deg GF < \deg F$. □

**1.6.** Recall that the *Picard group* of a ring $R$ is the group $\mathrm{Pic}(R)$ of isomorphism classes of locally free $R$-modules of rank 1. In view of the above criterion, it is natural to introduce the subgroup $\mathrm{Pic}_2(R)$ of $\mathrm{Pic}(R)$ generated by isomorphism classes of locally free $R$-modules of rank 1 that can be generated by 2 elements. With this definition, property (2) in Theorem 1.5 is equivalent to the triviality of $\mathrm{Pic}_2(R)$. In particular, we obtain:

**Corollary 1.7.** *If* $\mathrm{Pic}_2(R) = \{1\}$ *and F belongs to* $\mathrm{GA}_2(R)$*, then F is tame if and only if it is locally tame.*

**Example 1.8.** The class of rings with $\mathrm{Pic}_2(R) = \{1\}$ contains in particular unique factorization domains since for these domains the Picard group itself is trivial. This also holds for *Bézout rings*, that is, domains in which every finitely generated ideal is principal (see e.g. [2]).

### 1.3. Minimal overring for tameness

Recall that $\mathrm{GA}_2(R) = \mathrm{TA}_2(R)$ if and only if $R$ is a field [4, Proposition 5.1.9]. If $F \in \mathrm{GA}_2(R)$, then $F$ is tame over the field of fractions $K$ of $R$, but, in general, there does not exist a smallest ring $S$ between $R$ and $K$ such that $F$ is tame over $S$. Indeed, letting $R = \mathbb{C}[z, w]$ every automorphism $F$ as in Proposition 1.3 with $zw \neq 0$ is tame over $R[z^{-1}]$ and $R[w^{-1}]$ but not over $R = R[z^{-1}] \cap R[w^{-1}]$. However, if we further assume that $R$ is a Bézout domain, we have the following result.

**Proposition 1.9.** *Let R be a Bézout domain and let* $(R_j)_{j\in J}$ *be a family of rings between R and K such that* $R = \bigcap_{j\in J} R_j$*. Then* $\mathrm{TA}_2(R) = \bigcap_{j\in J} \mathrm{TA}_2(R_j)$*.*

**Proof.** Similarly as in the proof of Theorem 1.5, we proceed by induction on the degree of $F = (F_1, F_2) \in \mathrm{GA}_2(R) \cap \bigcap_{j\in J} \mathrm{TA}_2(R_j)$, the case $\deg F = (1, 1)$ being obvious. Letting $d_i = \deg F_i$, we may assume that $d_1 \le d_2$.

• Case 1: $d_1 < d_2$. Then $e = d_2/d_1 \in \mathbb{Z}_{>0}$ and there exists $\alpha \in K$ such that $\overline{F_2} = \alpha\overline{F_1}^e$. Since $F \in TA_2(R_j)$, we have $\alpha \in R_j$ for every $j \in J$, and so $\alpha \in R = \bigcap_{j\in J} R_j$. Now the automorphism $(X, Y - \alpha X^e)F$ satisfies the induction hypothesis.

• Case 2: $d_1 = d_2$. Since $\overline{F_1}$ and $\overline{F_2}$ are $K$-linearly dependent, the $R$-module $R\overline{F_1} + R\overline{F_2}$ is isomorphic to a proper ideal of $R$. As $R$ is a Bézout domain, the latter is free of rank 1, and so, we conclude from 1.2 above that there exists $G \in \mathrm{Aff}_2(R)$ such that $\deg GF < \deg F$. □

**Proposition 1.10.** *If R is a Bézout domain and* $F \in \mathrm{GA}_2(R)$ *then there exists a smallest ring S between R and* $K(R)$ *such that* $F \in \mathrm{TA}_2(S)$*. Furthermore, S is a finitely generated R-algebra.*

*If we assume further that R is a principal ideal domain, then there exists* $r \in R \setminus \{0\}$ *such that* $S = R[r^{-1}]$*.*

**Proof.** Any ring between $R$ and $K(R)$ is again a Bézout domain [2, Theorem 1.3]. Therefore, the existence of $S$ is a consequence of the previous proposition. The fact that $S$ is finitely generated follows from Proposition 1.1 by easy induction. For the last assertion, since $S$ is finitely generated over $R$, there exists a finitely generated ideal $I \subset R$ and an element $r \in R \setminus \{0\}$ such that $S = R[I/r] = \{a/r^k \in K(R), a \in I^k, k = 0, 1 \dots\}$. Since $R$ is a p.i.d, $I$ is a principal ideal, say generated by an element $g \in R$. After eliminating common factors if any, we may assume that $r$ and $g$ are relatively prime and that $S = R[g/r] \subset R[r^{-1}]$. But by Bézout identity, there exists $u, v \in R$ such that $ur + vg = 1$ and so, $S = R[r^{-1}]$. □

**Example 1.11.** If $F \in \mathrm{GA}_2(\mathbb{C}[z])$, then there exists a smallest ring $S$ between $\mathbb{C}[z]$ and $\mathbb{C}(z)$ of the form $\mathbb{C}[z][r^{-1}]$ such that $F \in \mathrm{TA}_2(S)$.

## 2. Examples and complements

Here we give examples of domains $R$ that illustrate the property $\mathrm{Pic}_2(R) = \{1\}$.

### 2.1. *The condition* $\mathrm{Pic}_2(R) = \{1\}$ *for* 1*-dimensional noetherian domains*

If $R$ is a noetherian domain of Krull dimension 1, every locally free $R$-module of rank $j$ is generated by at most $j + 1$ elements (see e.g. [10, Th. 5.7]). In particular, we have $\mathrm{Pic}(R) = \mathrm{Pic}_2(R)$ for every noetherian domain of dimension 1. As a consequence, we get:

**Example 2.1.** If $R$ is a Dedekind domain, the following are equivalent:
(1) $\mathrm{Pic}_2(R) = \{1\}$; (2) $\mathrm{Pic}(R) = \{1\}$; (3) $R$ is a UFD; (4) $R$ is a p.i.d.

For the coordinate ring $R$ of an affine curve $C$ defined over an algebraically closed field, we have the following classical result:

**Proposition 2.2.** *The Picard group of $R$ is trivial if and only if $C$ is a nonsingular rational curve.*

**Proof.** Let $\tilde{C} = \mathrm{Spec}(\tilde{R})$ be the normalization of $C$. By virtue of [14, Theorem 3.2], the natural surjection $\mathrm{Pic}(C) \to \mathrm{Pic}(\tilde{C})$ is an isomorphism if and only if $R = \tilde{R}$. Therefore, if Pic(C) is trivial, then $C$ is necessarily a nonsingular curve. Now it is well known that a nonsingular curve has trivial Picard group if and only if it is rational (see e.g. [3, 11.4 p. 261]). □

**Corollary 2.3.** *Let $R$ be the coordinate ring of a rational affine curve and let $\tilde{R}$ be its integral closure in $K(R)$. If $F \in \mathrm{GA}_2(R)$ is locally tame, then $F \in \mathrm{TA}_2(\tilde{R})$.*

**Proof.** Indeed, with the notation of the previous proof, one has $F \in \mathrm{TA}_2(\mathcal{O}_p)$ for every $p \in C = \mathrm{Spec}(R)$ and so $F \in \mathrm{TA}_2(\tilde{\mathcal{O}}_p)$ for every $p \in C$. Since $\tilde{R} = \bigcap_{p \in C} \tilde{\mathcal{O}}_p$, it follows that $F$ is locally tame over $\tilde{R}$, whence tame by virtue of Proposition 2.2. □

**Example 2.4.** Let $R = \mathbb{C}[u, v]/(v^2 - u^3)$ be the coordinate ring of a cuspidal rational curve $C$. Via the homomorphism $\mathbb{C}[u, v] \to \mathbb{C}[t]$, $(u, v) \mapsto (t^2, t^3)$ we may identify $R$ with the subring $\mathbb{C}[t^2, t^3]$ of $\mathbb{C}[t]$ and the integral closure $\tilde{R}$ of $R$ with $\mathbb{C}[t]$. For every $a \in \mathbb{C}^*$, we let $I_a = (t^2 - a^2, t^3 - a^3)$ be the maximal ideal of the smooth point $(a^2, a^3)$ of $C$. In particular, $I_a$ is locally principal but one checks easily that it is not principal. So for $(z, w) = (t^2 - a^2, t^3 - a^3)$, any automorphism $F$ as in Proposition 1.3 is locally tame but not tame. On the other hand, $I_a\tilde{R}$ is principal, generated by $t - a$, and so, $F \in \mathrm{TA}_2(\mathbb{C}[t])$.

### 2.2. *Examples of rings with* $\mathrm{Pic}_2(R) = \{1\}$ *but* $\mathrm{Pic}(R) \neq \{1\}$

As observed above, for noetherian 1-dimensional domains $R$, the triviality of $\mathrm{Pic}_2(R)$ is equivalent to the one of $\mathrm{Pic}(R)$. Here we give examples of domains with $\mathrm{Pic}_2(R) = \{1\}$ and $\mathrm{Pic}(R) \neq \{1\}$ which are coordinate rings of smooth affine algebraic varieties.

**2.5.** Let $Q$ be a smooth quadric in the complex projective space $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$, $n \geq 2$, and let $U = \mathbb{P}^n \setminus Q$. As is well known, $U$ is smooth affine variety with Picard group isomorphic to $\mathbb{Z}_2$, generated by the restriction to $U$ of the invertible sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n$. Letting $R_n = \Gamma(U, \mathcal{O}_U)$ and $M_n = \Gamma(U, \mathcal{O}_{\mathbb{P}^1}(1))$, which is a locally free $R_n$-module of rank 1, we have the following result.

**Proposition 2.6.** *The minimal number of generators of $M_n$ as an $R_n$-module is $[n/2] + 1$. In particular, if $n \geq 4$ then $\mathrm{Pic}_2(R_n) = \{1\}$ whereas $\mathrm{Pic}(R_n) \simeq \mathbb{Z}_2$.*

**Proof.** Up to the action of $\mathrm{PGL}_{n+1}(\mathbb{C})$, we may assume that $Q \subset \mathbb{P}^n = \mathrm{Proj}(\mathbb{C}[x_0, \dots, x_n])$ is the hypersurface $q = 0$, where $q = x_0^2 + \dots + x_n^2 \in \mathbb{C}[x_0, \dots, x_n]$. Letting $\mathcal{Q} \subset \mathbb{A}^{n+1} = \mathrm{Spec}(\mathbb{C}[x_0, \dots, x_n])$ be the quadric defined by the equation $q = 1$, the natural map $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ restricts to an étale double cover $\mathcal{Q} \to \mathbb{P}^n \setminus Q$ expressing the coordinate ring $R_n$ of $\mathbb{P}^n \setminus Q$ as the ring of invariant functions of $A = \mathbb{C}[x_0, \dots, x_n]/(q - 1)$ for the $\mathbb{Z}_2$-action induced by $-\mathrm{id}$ on $\mathbb{A}^{n+1}$. With this description, $\mathcal{O}_{\mathbb{P}^n}(1) \mid_U$ coincides with the trivial line bundle $\mathcal{Q} \times \mathbb{A}^1$ equipped with the nontrivial $\mathbb{Z}_2$-linearization $\mathcal{Q} \times \mathbb{A}^1 \ni (x, u) \mapsto (-x, -u) \in \mathcal{Q} \times \mathbb{A}^1$ (see e.g. [11, 1.3]). It follows that we may identify regular functions on $U$ and global sections of $\mathcal{O}_{\mathbb{P}^n}(1) \mid_U$ with cosets in $A$ of even and odd polynomial functions on $\mathbb{A}^{n+1}$ respectively.

• Case 1: $n = 2m$ is even. Clearly, the $m + 1$ odd polynomials $p_j = x_{2j} + ix_{2j+1}$ for $0 \leq j \leq m - 1$ and $p_m = x_{2m}$ have no common zero on $\mathcal{Q}$. Therefore, the corresponding sections of $\mathcal{O}_{\mathbb{P}^n}(1) \mid_U$ generate $M_n$ as an $R_n$-module. Let us show that $M_n$ cannot be generated by less than $m + 1$ elements. Otherwise, we could find in particular $m$ odd polynomial functions

$s_1, \ldots, s_m$ on $\mathbb{A}^{n+1}$ with no common zero on $\mathcal{Q}$. Writing $s_j = a_j + ib_j$ for suitable odd polynomials $a_j, b_j \in \mathbb{R}[x_0, \ldots, x_n]$, this would imply in particular that the $n$ odd real polynomials $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ have no common zero on $\mathcal{Q} \cap \mathbb{R}^{n+1}$. This is impossible. Indeed, since $\mathcal{Q} \cap \mathbb{R}^{n+1}$ is the real $n$-sphere $\mathbb{S}^n$, it follows from Borsuk–Ulam theorem that the map $\phi = (a_1, \ldots, a_m, b_1, \ldots, b_m) : \mathbb{S}^n \to \mathbb{R}^n$ takes the same value on a pair of antipodal points, hence, being odd, vanishes on a pair of antipodal points.

• Case 2: $n = 2m+1$ is odd. One checks in a similar way as above that the $m+1$ global sections of $\mathcal{O}_{\mathbb{P}^1}(1) \mid_U$ corresponding the odd polynomials $p_j = x_{2j} + ix_{2j+1}, 0 \leq j \leq m$ generate $M_n$ as an $R_n$-module. Now if $M_n$ was generated by $m$ elements, then there would exists $m$ odd polynomials $s_j = a_j + ib_j$ as above for which the polynomials $a_j, b_j \in \mathbb{R}[x_0, \ldots, x_n], j = 1, \ldots, m$ have no common zero on the real $n$-sphere $\mathbb{S}_n$. But then the continuous map $\phi = (a_1, \ldots, a_m, b_1, \ldots, b_m, 0) : \mathbb{S}^n = \mathcal{Q} \cap \mathbb{R}^{n+1} \to \mathbb{R}^n$ would contradict the Borsuk–Ulam theorem. □

**Remark 2.7.** An argument very similar to the one used in the proof above shows that over the subring $\tilde{R}_n$ of $\mathbb{R}[x_0, \ldots, x_n]/(x_0^2 + \cdots + x_n^2 - 1)$ consisting of cosets of even polynomials, the module $\tilde{M}_n$ consisting of cosets of odd polynomials cannot be generated by less than $n + 1$ elements. This property seems to have been first observed by Chase (unpublished). Our proof is deeply inspired by an argument due to Gilmer [6] on a slightly different example.

### *2.3. Further research*

One may wonder if there exists a complete characterization of obstructions to infer global tameness from local tameness for higher dimensional polynomial automorphisms similar to Theorem 1.5. A good starting point would be to have in general an effective algorithmic way to recognize tame automorphisms. Unfortunately, at the present time, such an higher dimensional algorithm only exists for automorphisms of a polynomial ring in 3 variables over a field [12].

## Acknowledgements

The first author is funded by a free competition grant of the Netherlands Organisation for Scientific Research (NWO). The second author is partially supported by FABER grant 07-512-AA-010-S-179 and PHC Grant Van Gogh 18153NA. The third author is partially supported by PHC Grant Van Gogh 18153NA. The last author is funded by Veni-grant of council for the physical sciences, Netherlands Organisation for Scientific Research (NWO).

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