9. The N-Body Problem

9.1 Clockwork is rare

Newton knew that a force law for gravity falling as $1/r^2$ would give rise to closed elliptical orbits. In the last lecture we made the general proof only this law and that of the harmonic oscillator give rise to systems that behave like clockwork. Such stable mechanical systems are due to their integrability i.e. systems with n degrees of freedom are integrable whenever the solution can be reduced to n independent integrations by finding n independent (global) conservation laws.

It's worth noting the methodology used to solve the two body problem. We started with 6 degrees of freedom describing the positions of two particles. Eliminating the centre of mass reduced this to 3 degrees of freedom describing the separation. We then used conservation of the direction of $\mathbf{L}$ to reduce to 2 degrees of freedom (r and ϕ), and conservation of the magnitude of $\mathbf{L}$ to reduce to a single variable r. Finally conservation of E allowed us to solve the problem. You might now be getting an idea about how important conservation laws are to help us solve problems!

There are five independent constants of motion for the Kepler problem (not seven, since the Runge-Lenze vector is not independent of energy and angular momentum). By knowing that energy and angular momentum are conserved, one can solve the Kepler problem by two independent integrations. By knowing the Runge-Lenz vector and angular momentum are conserved one can solve the Kepler problem algebraically (without even considering energy of evaluating an integral)! That the Runge-Lenz vector is a conserved quantity is due to a hidden symmetry - the factor that the gravitational force law gives closed orbits, which when projected onto four dimensional space yields the hidden symmetry.

In mathematics, there is often unstable periodicity. Correspondingly there are are mechanical systems that cannot be solved by symmetry and are therefore non-integrable - these are systems where chaos is possible in portions of phase space. i.e. the three body problem which has far fewer conserved quantities than degrees of freedom.

9.1.1 Stable and unstable periodicity in math

The decimal expansion of any rational number is periodic in any number base. Irrational numbers have non-periodic expansions in any number base.

For example compare

$$\frac{1}{7} = \frac{10}{70} = 0.142857...142857... \tag{9.1}$$

which repeats a block of six digits (14285) infinite times whereas

$$\frac{10}{69} = 0.1449275362318840579710... \tag{9.2}$$

which repeats a 22 digit block infinite times yet the starting number is only different by one part in 463! The peroidic nature of rational numbers is part of the proof of Betrands theorem that we carried out in class.

Now compare

$$\sqrt{3} = 1.73205080........ \tag{9.3}$$

with

$$\sqrt{2.999} = 1.7317621....... \tag{9.4}$$

which give rise to non periodic unstable expansions.

It is interesting to ask which (conservative) mechanical systems have this kind of unstable periodicity and non-periodicity and how mechanics would have developed for example, if gravity did not give rise to closed orbits that repeat like clockwork.

Newton stated that departures from periodicity (precession of the apsidal angles) in the planetary orbits could be used to see if gravity deviated from $1/r^2$. Indeed - the perihelion of Mercury changes by 574" per century. Perturbations from the other planets were calculated to be responsible for 541" per century. The difference was finally accounted for by general relativity.

9.1.2 Solving the motion by numerical integration

We can learn a great deal about complex non-linear systems via numerical solutions. Solving the N-body problem using computers has been an active area of research since the 1960's, and is now used in fields such as astrophysics, plasma physics, molecular dynamics, protein folding etc.

The numerical techniques are similar and independent of the form of the potential. Here we look at the case of gravity. The force on the i'th particle can be written

$$F_i = \sum_{j \neq i} \frac{G m_i m_j (r_i - r_j)}{|r_i - r_j|^3} = \sum_{j \neq i} F_{ij} = m_i \frac{d^2 r_i}{dt^2} \tag{9.5}$$

where the forces, velocities and positions are vectors. This second order differential equation can be written as two first order equations which can be solved by differencing

$$v_i = \frac{dr_i}{dt} \qquad F_i = m_i \frac{dv_i}{dt}. \tag{9.6}$$

These equations are solved numerically by discretising time into $n = 1, 2, 3....$ steps $t_n = t_o + \Delta t$ and positions and velocities are calculated at those times. These two equations can then be solved numerically using a choice of integration schemes depending on what accuracy is needed. i.e. Taylor expand them to first order:

$$\frac{dx}{dt} = v \Rightarrow x_{n+1} - x_n = v_n \Delta t + 0(\Delta t)^2 \tag{9.7}$$

$$\frac{dv}{dt} = \frac{1}{m}F \Rightarrow v_{n+1} - v_n = a_n\Delta t + 0(\Delta t)^2 \tag{9.8}$$

so that

$$v_{n+1} = v_n + a_n\Delta t \qquad\qquad x_{n+1} = x_n + v_n\Delta t \tag{9.9}$$

This scheme is the *forward Euler method* which is unstable. The midpoint scheme is stable:

$$v_{n+1} = v_n + a_n\Delta t \qquad\qquad x_{n+1} = x_n + 1/2(v_n + v_{n+1})\Delta t \tag{9.10}$$

but the most powerful of the simplest schemes is the leapfrog:

$$v_{n+1/2} = v_{n-1/2} + a_n\Delta t \qquad\qquad x_{n+1} = x_n + v_{n+1/2}\Delta t \tag{9.11}$$

i.e. offset the postitions and velocities by half a step using the midpoint scheme, then "leapfrog" the calculations for the new positions and velocities.

There are many higher order accurate schemes, such as Runga-Kutta, but even though the leapfrog scheme is only 2nd order accurate, it is symplectic! We will find out exactly what this means later, but for now its enough to know that it conserves the important quantities and is time reversible! i.e. put $\Delta t = -\Delta t$ at the end of the calculation and it will exactly reverse its steps (asside from roundoff errors). The other techniques often have spurious systematic energy drifts, so that the systems appear non-conservative.

How long do N-body calculations take using the simplest direct method? The total length of a calculations is:

$$\mathrm{T} = \mathrm{n_{flops}N_{steps}N_{particles}(N_{particles} - 1)}\tau_{\mathrm{flop}} \tag{9.12}$$

where $n_{flops} \approx 10$ is the number of floating point operations to carry out the numerical integration, $\tau_{flop} \approx 10^{-13}$ seconds is the time for a fast cpu to do one floating point operation, $N_{steps} \approx 10^5$ is the total number of timesteps for the problem and $N_{particle} \approx 10^9$ for the largest calculations today. Thus $T \approx 3,000$ years!

As we will discuss in class the calculation can be done much more quickly by (i) using variable timesteps per particle (why move Pluto on the same timestep as Mercury?) (ii) writing parallel code and running on hundreds of processors of a supercomputer and (iii) not calculating the forces so accurately by summing over all the particles (you don't need to calculate the forces from all the atoms in the sun - i.e. replace distant regions by fewer points using a "data-tree"). These "tricks" can speed up the above calculation by a factor of order 10^5.

Sometimes it is easiest to calculate the force using the potential, specifically by using Poisson's equation which relates the density to the potential. We will derive this in class if there is time.

In the lecture we will look at computer simulations of the two and three body problem in different force laws.

- We will verify the form of the conic section orbits for the Kepler problem. We will show that circular orbits are stable for force laws shallower than $1/r^3$. We will show that closed orbits only occur for force laws $F \sim 1/r^2$ and $F \sim r$. We showed that other power laws gave orbits that filled the space between the turning points of the orbits, like the rosette in figure 13. We will look at how innacurate integration lead to energy loss and the orbits spiralled inwards. We will demonstrate that large a timestep caused the two particles to scatter unphysically. Finally we will look at the three body problem and sensitivity to initial conditions which leads to chaos.

In non-spherical potentials, i.e. the galaxy, then the potential comes from a nearly smooth and flattended distribution of ten billion stars. Rotational symmetry is not preserved so angular momentum is not conserved. Energy is the only conserved quantity. Understanding orbits in flattened potentials has to be studied using computers but solving newtons laws of motion.

9.2 The three body problem

"If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation approximately. If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon." - in a 1903 essay "Science and Method" by Henri Poincare.

9.2.1 History

In 1885, Poincare entered a contest formulated by the King Oscar II of Sweden in honour of his 60th birthday. One of the questions was to show the solar system, as modelled by Newton's equations, is dynamically stable. The question was nothing more than a generalization of the famous three body problem, which was considered one of the most difficult problems in mathematical physics. In essence, the three

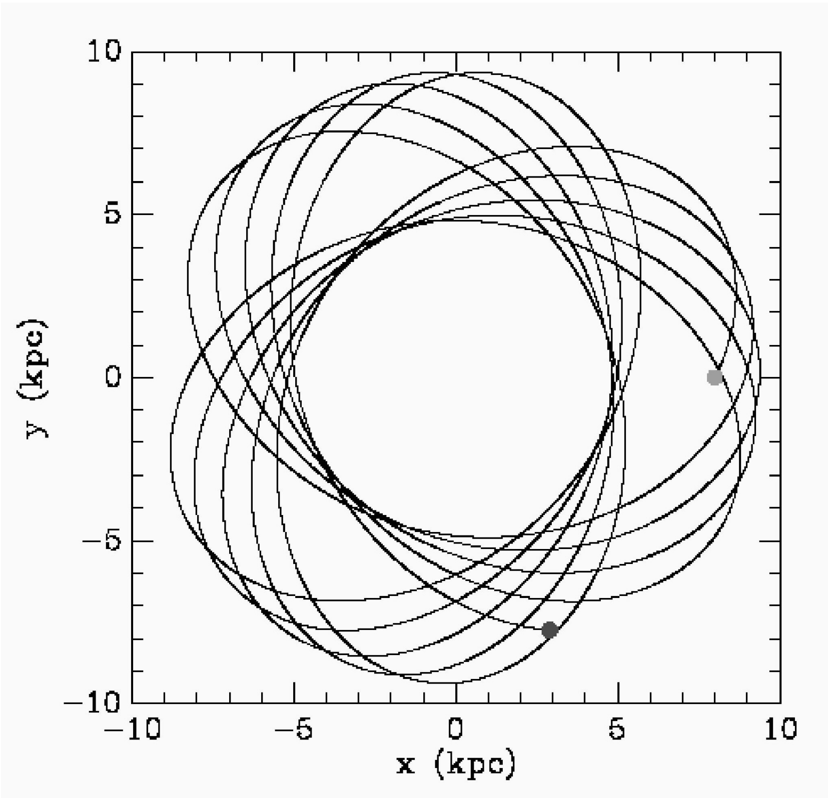

Figure 13: The typical rosette orbit that occurs in a power law force laws.

body problem consists of nine simultaneous differential equations. The difficulty was in showing that a solution in terms of invariants converges. (This isn't likely to happen today - that the birthday of any contemporary world leader is celebrated by a mathematical competition!) Henri Poincare was a favorite to win the prize, and he submitted an essay that demonstrated the stability of planetary motions in the three-body problem (actually the restricted problem, in which one test body moves in the gravitational field generated by two others). In other words, without knowing the exact solutions, we could at least be confident that the orbits wouldnt go crazy; more technically, solutions starting with very similar initial conditions would give very similar orbits. Poincares work was hailed as brilliant and he was awarded the prize.

But as his essay was being prepared for publication in Acta Mathematica, a couple of tiny problems were pointed out by Edvard Phragmn, a Swedish mathematician who was an assistant editor at the journal. Gsta Mittag-Leffler, chief editor, forwarded Phragmns questions to Poincare, asking him to fix up these nagging issues before the prize essay appeared in print. Poincare went to work, but discovered to his consternation that one of the tiny problems was in fact a profoundly devastating possibility that he hadnt really taken seriously. What he ended up proving was the opposite of his original claim three-body orbits were not stable at all. Not only were the orbits none periodic, they didnt even approach some sort of asymptotic fixed points. Now that we have computers to run simulations, this kind of behavior is less surprising, but at the time it came as an utter shock. In his attempt to prove the stability of planetary orbits, Poincar ended up inventing chaos theory.

But the story doesnt quite end there. Mittag-Leffler, convinced that Poincare would be able to tie up the loose threads in his prize essay, went ahead and printed it. By the time he heard from Poincare that no such tying-up would be forthcoming, the journal had already been mailed to mathematicians throughout Europe. Mittag-Leffler swung into action, telegraphing Berlin and Paris in an attempt to have all copies of the journal destroyed. He basically succeeded, but not without creating a minor scandal in elite mathematical circles across the Continent.