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A Proof of the Bundle Theorem for Certain Semimodular Locally Projective Lattices of Rank 4*

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The Bundle Theorem is proved for geometric locally projective lattices of rank 4 which for every given line (rank 2 element) do not contain too many lines that are on a common plane (rank 3 element) with this line, but on no common point (rank 1 element). By a result of J. Kahn (*Math. Z.* **175** (1980), 219–247), this implies that these lattices are projectively embeddable.

1. INTRODUCTION AND THEOREM

Let L be a projective geometry of rank n (considered as the lattice of its subspaces), and let Q be a nonempty set of points of L. Define $L(Q) := \{x \in L : \exists\, P \in Q \text{ with } P \leqslant x\} \cup \{0\}$ and consider $L(Q)$ as the lattice induced by the partial order of L. It is not hard to see that $L(Q)$ is a semimodular locally projective lattice of rank n. *Semimodular* means that if $x, y \in L$ cover $x \wedge y$, then they are covered by $x \vee y$. For lattice definitions see [1]. The rank 0, 1, 2, 3, and n elements are called "0," "points," "lines," "planes," and "1," respectively. A lattice is called *locally projective*, if for each point P the interval $[P, 1]$ is a projective geometry. A *geometric lattice* is a semimodular lattice of finite rank for which "Every element is the join of its points," holds. If a lattice M is isomorphic to some $L(Q)$, we say that M is *projectively embeddable.*

By a theorem due to Wille [9], every semimodular locally projective lattice L of rank $n > 4$, such that the whole space is the join of the set of its points, is projectively embeddable. (This is a generalization of a theorem of Mäurer [7] which states that Möbius geometries of dimension at least 3 are projectively embeddable; a similar result was obtained by Kantor [5].)

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Wille's result is false for L of rank 4: There are examples (finite as well as infinite) of semimodular locally projective lattices of rank 4 which are even geometric, but not projectively embeddable (see [3]). Kahn [4] has characterized the projectively embeddable lattices (under an additional assumption that on every given point there are not too many lines which contain no further point) by the following property, called "Bundle Theorem":

If 4 lines are such that no 2 are on a common point, no 3 are coplanar, and 5 of the 6 pairs of lines are coplanar, then so is the sixth pair.

On the other hand, it is well known that every affine geometry is projectively embeddable, and there are some generalizations of this fact (e.g., [6, 8]). This suggests that a geometric locally projective lattice of rank 4 should be projectively embeddable, if for each line l there are not too many lines which are on a common plane with l, but on no common point. For any nonincident point-line-pair (P, l) of a geometric lattice let $|l|$ and $z(P, l)$ denote the (possibly infinite) numbers of all points on l and of those lines on P which are on a common plane with l, but on no common point. We obtain the following

THEOREM. *Let L be a geometric locally projective lattice of rank 4 which satisfies for every nonincident point-line-pair (P, l)*

(1) $z(P, l) < |l|$,

(2) $(z(P, l) - 1)^2 < |l|$.

Then L is projectively embeddable.

In [2] this theorem serves to embed the "group space" of an orthogonal group into a projective space. The inequalities (1) and (2) are equivalent if $z(P, l)$ is infinite; if $|l| > 2$, then (2) implies (1).

2. Proof of the Theorem

In view of Kahn's result we only have to show the validity of the Bundle Theorem. A proof not using Kahn's result is given in [2]. Throughout Section 2, L is a geometric locally projective lattice of rank 4 satisfying conditions (1) and (2) of the theorem. The sets of points, lines, and planes of L are denoted L_1, L_2, and L_3, respectively. The minimum element of L is 0, and the maximum element is 1.

If P and Q are distinct points of L, then $P \vee Q$ is a line, hence a point of both $[P, 1]$ and $[Q, 1]$. The order of $[P, 1]$ ($[Q, 1]$) being one less than (so equal to if infinite) the number of planes on $P \vee Q$, it follows that all of the projective planes $[P, 1]$ have the same order. We denote this common order $O(L)$. Let P be a point and l a line not on P. Then $z(P, l)+|l|=O(L)+1$. If $O(L)$ is infinite, then (1) implies $|l|=O(L)$ and $z(P, l)<O(L)$. If $O(L)$ is finite, we get $2\cdot z(P, l)<O(L)+1$ and $2\cdot|l|>O(L)+1$. We conclude

LEMMA 1. *Let $P\in L_1$ and $l, m\in L_2$. Then $z(P, l)<|m|$.*

DEFINITION. Let E be a plane and S a collection of lines on E such that every point on E is on exactly one member of S. Then S is called a *spread* of E.

LEMMA 2. *Let $E\in L_3$ be a plane and $a, b\in L_2$ lines on E with $a\wedge b=0$. Then there is a unique spread of E which contains a and b.*

Proof. Choose a point $P\in L_1$ outside E. $P\vee a$ and $P\vee b$ are lines of the projective plane $[P, 1]$. This implies $(P\vee a)\wedge(P\vee b)=:f\in L_2$. We have $E\wedge f=a\wedge b=0$ and $E\wedge(X\vee f)\in L_2$ for every point X on E. Therefore $S:=\{E\wedge(X\vee f)\colon X\in L_1, X\leqslant E\}$ is a spread of E. Obviously S contains a and b. Let S' be another spread of E containing a and b, and assume $S'\neq S$. Then there is a line $c\in S-S'$. On every point on c there must be a line of S'. None of these lines is in S and no two of them coincide, for otherwise $c\in S'$. We will provide a contradiction by showing that the number of lines of S' which are not in S is smaller than the number of points on c.

For every $d\in S'-\{a, b\}$ define the lines $g(d):=(P\vee d)\wedge(P\vee a)$ and $h(d):=(P\vee d)\wedge(P\vee b)$. We have $g(d)\wedge a=d\wedge a=0$ and $h(d)\wedge b=d\wedge b=0$. For every line $d\in S'-S$ the lines f, $g(d)$, and $h(d)$, are pairwise distinct, and d is uniquely determined by $g(d)$ and $h(d)$.

Therefore $|S'-S|\leqslant(z(P, a)-1)\cdot(z(P, b)-1)$. If $|c|$ is infinite, then Lemma 1 implies $(z(P, a)-1)\cdot(z(P, b)-1)<|c|$. Now let $|c|$ be finite. Then the projective plane $[P, 1]$ is of finite order n, and $z(P, l)+|l|=n+1$ for every line $l\in L_2$ not on P. Without loss of generality we may assume $z(P, a)\geqslant z(P, b)$. Together with (2) this implies $|S'-S|\leqslant(z(P, a)-1)^2<|a|$. We are ready if $|a|\leqslant|c|$. Otherwise, $z(P, a)<z(P, c)$ and therefore (since $z(P, a)\geqslant 1$) $|S'-S|<(z(P, c)-1)^2<|c|$.

LEMMA 3 (Bundle Theorem). *Let $a, b, c, d\in L_2$ be four lines, no two on a common point and no three on a common plane. If $a\vee b$, $a\vee c$, $a\vee d$, $b\vee c$, $b\vee d\in L_3$, then also $c\vee d\in L_3$.*

Proof. $S_c:=\{(X\vee c)\wedge(a\vee b)\colon X\in L_1, X\leqslant a\vee b\}$ and $S_d:=\{(X\vee d)\wedge(a\vee b)\colon X\in L_1, X\leqslant a\vee b\}$ are spreads of $a\vee b$ because a common point of two lines of S_c (or of S_d) would be a common point on c (or on d) and $a\vee b$, and consequently a common point on a and b. We have

$a, b \in S_c$, S_d, and therefore $S_c = S_d$ by Lemma 2. If there is a point D on d with $(c \vee D) \wedge (a \vee b) \neq 0$, we conclude as follows: $g := (c \vee D) \wedge (a \vee b)$ is an element of S_c. Consequently $g \in S_d$, and therefore $g \vee d \in L_3$. We have $c \vee D = g \vee D = g \vee d$. This implies $c \vee d \in L_3$.

To complete the proof, we must show that such a point D does exist. We may assume that $c \vee d$ is not a plane, for otherwise we are ready. Choose points $A, B, C \in L_1$ with $A \leqslant a$, $B \leqslant b$, and $C \leqslant c$. For every point D on d, $l(D) := (A \vee B \vee C) \wedge (c \vee D)$ is a line on C which is on a common plane (namely, $A \vee B \vee C$) with $A \vee B$. Assume $l(D) \wedge (A \vee B) = 0$ for every point D on d. Since $l(D_1) \neq l(D_2)$ for $D_1 \neq D_2$ (otherwise $c \vee d \in L_3$), this implies $z(C, A \vee B) \geqslant |d|$. But $z(C, A \vee B) < |d|$ by Lemma 1. Consequently there is a point D_0 on d with $l(D_0) \wedge (A \vee B) \neq 0$. We have $l(D_0) \wedge (A \vee B) = (c \vee D_0) \wedge (A \vee B)$, and therefore $(c \vee D_0) \wedge (a \vee b) \neq 0$. Thus D_0 is the point we were looking for.

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