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Note

On the adjacent vertex distinguishing total coloring numbers of graphs with $\Delta = 3$☆

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Abstract

An adjacent vertex distinguishing total-coloring of a simple graph G is a proper total-coloring of G such that no pair of adjacent vertices meets the same set of colors. The minimum number of colors $\chi''_a(G)$ required to give G an adjacent vertex distinguishing total-coloring is studied. We proved $\chi''_a(G) \leqslant 6$ for graphs with maximum degree $\Delta(G) = 3$ in this paper.

MSC: 05C15

Keywords: Simple graph; Adjacent vertex distinguishing total-coloring

1. Introduction

Let G be a finite simple graph with no component K_2. Let C be a finite set of colors and let $\varphi : E(G) \to C$ be a proper edge coloring of G. The *color set* of a vertex $v \in V(G)$ with respect to φ, is the set of colors of edges incident with v. The coloring φ is *adjacent vertex distinguishing* (or *neighbor distinguishing*) if it distinguishes any two adjacent vertices by their color sets. The minimum number of colors $\chi'_a(G)$ (or $\mathrm{ndi}(G)$) required to give G an adjacent vertex distinguishing coloring has been studied in many papers, see for example [1,2,5,9].

The main conjecture related to adjacent vertex distinguishing coloring (formulated in [9]) is listed as follows.

Conjecture 1 (*Zhang et al. [9]*). For every connected graph G with order at least 6, we have $\chi'_a(G) \leqslant \Delta(G) + 2$.

This conjecture has been proved in [2] for bipartite graphs as well as for graphs with maximum degree at most three.

Let G be a finite simple graph. We say a proper total-coloring of G is *adjacent vertex distinguishing*-total coloring (or an avd-total coloring, total neighbors distinguishing coloring) if for any pair of adjacent vertices x and y, the set of colors meet to x (i.e., the set of colors of edges incident with x together with the color assigned to x. This set, denoted by $C(x)$, is called the color set of x with respect to the given total-coloring) is not equal to the set of colors meet to y. It is clear that an avd-total coloring exists for any graph G. A k-avd-total-coloring is an avd-total-coloring using at most

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k colors. Let $\chi''_a(G)$ be the minimum number of colors in an avd-total-coloring of G. In [8] the following conjecture was made.

Conjecture 2 (*Zhang et al.* [8]). For every connected graph G with order at least 2, we have $\chi''_a(G) \leqslant \Delta(G) + 3$.

The relationship between Conjectures 1 and 2 is similar to the relationship between the Vizing Theorem (for proper edge coloring, see [4]) and the Total Coloring Conjecture (see [7]).

Obviously $\chi''_a(G)$ is at least $\Delta(G)+1$; if G does have two distinct maximum degree vertices which are adjacent, then $\chi''_a(G)$ is at least $\Delta(G)+2$. For bipartite graph G the edge chromatic number is Δ (see [4]). We use two new colors to be properly assigned to vertices of G. Then we obtain $(\Delta(G)+2)$-avd-total-coloring of G. Thus we have the following proposition.

Proposition 1.1. *If G is a bipartite graph, then $\chi''_a(G) \leqslant \Delta(G) + 2$.*

So for bipartite graph Conjecture 2 is valid. For graphs with maximum degree $\Delta(G) = 3$ we have

Theorem 1.1. *If G is a graph with maximum degree $\Delta(G) = 3$, then $\chi''_a(G) \leqslant 6$.*

We know that $\chi''_a(G) = \Delta(G) + 3$ if $G = K_{2n+1}$ (Complete graph with odd order $2n+1$, $n \geqslant 1$). We will give another example with $\chi''_a(G) = \Delta(G) + 3$ in Section 2 and then prove Theorem 1.1 in Section 3.

2. An example with $\chi''_a(G) = \Delta(G) + 3$

Consider the joint $sP_3 \vee K_t$ of sP_3 and K_t, where sP_3 is the disjoint union of s paths $u_i v_i w_i$ $(i = 1, 2, \ldots, s)$ with length 2 and K_t is the complete graph with t vertices $x_1, x_2, \ldots, x_t$. Suppose s is an even positive integer and t is an odd positive integer and $t \geqslant 9s^2 + 2s - 1$. Obviously $\chi''_a(sP_3 \vee K_t) \geqslant \Delta + 2 = 3s + t + 1$. In the following we firstly prove that $sP_3 \vee K_t$ does not have $(3s + t + 1)$-avd-total-coloring and then prove that $sP_3 \vee K_t$ does have $(3s + t + 2)$-avd-total-coloring and therefore $\chi''_a(sP_3 \vee K_t) = 3s + t + 2 = \Delta + 3$.

Assume that we have an avd-total-coloring of $sP_3 \vee K_t$ using $3s + t + 1$ colors. Then for every x_i there is only one color which did not meet x_i. Obviously each color is assigned to at most $(3s + t - 1)/2$ edges. Meanwhile each color is assigned to at least $(t - 1)/2$ edges (Otherwise if some color is assigned to at most $(t - 3)/2$ edges then this color will meet at most $t - 2$ vertices x_i. So there are at least two vertices which have the same color set. This is a contradiction). Suppose that there are r_i colors such that each of these colors is assigned exactly to $(t-3)/2 + i$ edges, where $i = 1, 2, \ldots, (3s + 2)/2$. Thus we have

$$r_1 + r_2 + r_3 + \cdots + r_{(3s-2)/2} + r_{3s/2} + r_{(3s+2)/2} = 3s + t + 1;$$

$$\frac{t-1}{2} r_1 + \frac{t+1}{2} r_2 + \frac{t+3}{2} r_3 + \cdots + \frac{3s+t-5}{2} r_{(3s-2)/2} + \frac{3s+t-3}{2} r_{3s/2} + \frac{3s+t-1}{2} r_{(3s+2)/2}$$
$$= 2s + \frac{1}{2} t(t-1) + 3st.$$

From the above two equations we can deduce that

$$\begin{aligned} r_{(3s+2)/2} &= 2s + \frac{1}{2} t(t-1) + 3st - \frac{3s+t-3}{2}(3s+t+1) + \frac{3s-2}{2} r_1 \\ &\quad + \frac{3s-4}{2} r_2 + \cdots + 2r_{(3s-4)/2} + r_{(3s-2)/2} \\ &= -\frac{9}{2}s^2 + 5s + \frac{1}{2}t + \frac{3}{2} + \frac{3s-2}{2} r_1 + \frac{3s-4}{2} r_2 + \cdots + 2r_{(3s-4)/2} + r_{(3s-2)/2}. \end{aligned} \tag{1}$$

As $\{u_1, v_1, x_1, x_2, \ldots, x_t\}$ is a clique of $sP_3 \vee K_t$, we need at least $t+2$ colors to be assigned to vertices. Therefore, there are at most $3s-1$ colors which are not assigned to any vertices. Note that $r_{(3s+2)/2} \geqslant 3s-1$ (Using the condition $t \geqslant 9s^2+2s-1$ and Eq. (1)). In $r_{(3s+2)/2}$ colors, each of which is assigned exactly to $(3s+t-1)/2$ edges, there are at least $r_{(3s+2)/2}-3s+1$ colors such that each of which is assigned to some vertex and then meets all vertices. The other (at most) $6s+t-r_{(3s+2)/2}$ colors contain the t colors which are missing at vertices $x_1, x_2, \ldots, x_t$ respectively. Thus

$$t \leqslant 6s+t-r_{(3s+2)/2}$$

and using (1), we have

$$\frac{9}{2}s^2+s-\frac{1}{2}t-\frac{3}{2} \geqslant \frac{3s-2}{2}r_1+\frac{3s-4}{2}r_2+\cdots+2r_{(3s-4)/2}+r_{(3s-2)/2} \geqslant 0.$$

So $t \leqslant 9s^2+2s-3$. A contradiction. Thus $\chi''_a(sP_3 \vee K_t) > 3s+t+1$.

Construct a new graph G with $3s+t+1$ vertices by adding a new vertex y to $sP_3 \vee K_t$ such that y is connected to every vertex of $sP_3 \vee K_t$. From [3] we know that the vertex distinguishing proper edge coloring number of G is at most $3s+t+2$. Assigning the color of each edge zy to the vertex z for any vertex $z \in V(sP_3 \vee K_t)$, we will obtain the vertex distinguishing total-coloring of $sP_3 \vee K_t$. This is also the avd-total-coloring of $sP_3 \vee K_t$ using $3s+t+2$ colors. Thus $\chi''_a(sP_3 \vee K_t) = 3s+t+2 = \Delta(sP_3 \vee K_t)+3$.

3. Graphs with $\Delta = 3$

We start with the special case of regular graphs having a hamiltonian cycle.

Lemma 2.1. *If G is a 3-regular hamiltonian graph then G has a 6-avd-total-coloring.*

Proof. For K_4 (Complete graph with order 4), we may find its 6-avd-total-coloring easily. So we suppose that the order of G is at least 6 in the following. Let the six colors be $1, 2, 3, a, b, c$. Let $C = x_1x_2\cdots x_nx_1$ be a hamiltonian cycle of G and I be the remaining 1-factor of G. By Brooks' theorem G has vertex 3-coloring $f : V(G) \to \{a, b, c\}$. The edges of I are colored with 3. As the cycle C is even, the edges of C can be colored alternately by 1 and 2. For any pair of adjacent vertices x and y, the set of colors incident to x is equal to the set of colors incident to y whereas the colors of vertices x and y are distinct. So x is distinguished from y. $\square$

Theorem 2.2. *If G is a 3-regular graph containing 1-factor, then there exists a 6-avd-total-coloring of G.*

Proof. We may suppose that the order of G is at least 6. Without loss of generality we may assume G is connected. Decompose G as a 1-factor I and a union of cycles C_i. If there is only one cycle then G is hamiltonian and we are done by Lemma 2.1. Otherwise color G as follows.

By Brooks' theorem we properly color the vertices of G with a, b, c. The edges of I are colored by 3.

If C_i is an even cycle then the edges of C can be colored alternately by 1 and 2.

If $C_i = x_0x_1x_2\cdots x_nx_0$ is an odd cycle and each vertex of C_i is not adjacent to any vertex of the other odd cycle then the edges $x_0x_1, x_1x_2, x_2x_3, \ldots, x_{n-1}x_n$ can be colored alternately by 1 and 2 and x_nx_0 is colored by a color in $\{a, b, c\} \backslash \{f(x_0), f(x_n)\}$.

Suppose C_i is an odd cycle and some vertex of C_i is adjacent to some vertex of the other odd cycle. Construct a new graph M with vertex set $V(M)$ equal to the set of all odd cycles C_j and edges joining C_j and C_k when there is an edge of I joining some vertex of C_j to some vertex of C_k. Consider the nontrivial component S of M such that S contains the vertex corresponding to C_i. Suppose T is a spanning tree of S, We will color the edges of odd cycles corresponding to the vertices of T. Given any vertex $v \in V(T)$, the corresponding odd cycle is denoted by C_v. Starting with a vertex (of C_v) which is connected with one vertex of some other odd cycle by an edge of T, we color the edges of C_v using the method mentioned in the previous paragraph. For $u \in V(T)$, where $\mathrm{dist}_T(v, u) = 1$, the corresponding odd cycle is C_u. There is only one edge being in $E(T)$ and connecting one vertex x of C_v and one vertex y of C_u. Of course the edge is denoted by xy. If x meets 1 and 2 then starting with y we can color the edges of C_u using the method described in the previous paragraph. If x meets only one of 1 and 2, say 1, then starting with y we color the edges (except the last edge) of C_u alternately by 2 and 1 (not 1 and 2) and the last edge zy of C_u is colored by one color in $\{a, b, c\} \backslash \{f(z), f(y)\}$.

In the same way we can color the edges of C_w corresponding to the vertex $w \in V(T)$, where $\text{dist}_T(v, w) = i$, $i = 2, 3, \ldots$.

So far we obtain a proper total coloring.

Obviously in each odd cycle C there is at most one vertex which is not distinguished from some other vertex (this vertex do not belong to C). If there are two adjacent vertices z_1 and z_2 having the same color sets then z_1 and z_2 belong to different odd cycles and the edge z_1z_2 has color 3. Without loss of generality we assume the colors of z_1 and z_2 are c and a, respectively. There is one edge z_1x_1 which is incident to z_1 and has color a. Similarly there is one edge z_2x_2 which is incident to z_2 and has color c. There are two edges z_1y_1 and z_2y_2 which have the same colors 1 or 2, say 1. If y_1 has color b then recolor the vertex z_1 with 2; If y_1 has color a then recolor the vertex z_1 with 2 and the edge z_1x_1 with c.

After a series of modifications described above we obtain a 6-avd-total-coloring. □

Proof of Theorem 1.1. We shall prove Theorem 1.1 by induction on $|E(G)|$. Suppose the colors we will use are 1, 2, 3, 4, 5, 6.

From Proposition 1.1 we know that Paths on at least 2 vertices has a 4-avd-total-coloring and then cycles on at least 3 vertices has a 6-avd-total-coloring. So we may assume G is connected with maximum degree 3.

Assume x is a vertex of degree 1 in G. Let y be the neighbor of x. Then y is of degree 2 or 3. We can find a 6-avd-total-coloring of $G' = G - x$ by induction. In G', y has degree at most 2, so there are at least three colors that do not meet y. At most two of these colors cannot be used to color xy as they may result in y meeting the same set of colors as some neighbor in G'. Therefore, there is still at least one color that can be given to xy and then we color the vertex x properly so that the coloring is a 6-avd-total-coloring. Hence we may assume G contains no degree 1 vertex.

Assume two vertices of degree 2 are adjacent in G. Let $x_0x_1x_2 \cdots x_n$, $n > 2$, be a *suspended trail* in G, i.e., a trail with $d_G(x_0) = d_G(x_n) = 3$ and $d_G(x_i) = 2$ for $0 < i < n$. If $x_0 \neq x_n$ let G' be the graph obtained by contracting this path to x_0yx_n. If $x_0 = x_n$ let G' be the graph obtained by deleting the vertices $x_1, \ldots, x_{n-1}$ and connecting two new degree one vertices y, z to $x_0 = x_n$. By induction G' has a 6-avd-total-coloring. We may assume without loss of generality that the edge x_0y has color 1 and x_ny(or x_nz) has color 2. If the color of y is not 2 then without loss of generality we assume that y has color 6. Assign the color of y to the vertex x_1. The edges x_0x_1, $x_{n-1}x_n$ can be colored with 1 and 2, respectively. The sequence $x_1x_2, x_2, x_2x_3, x_3, \ldots, x_{n-2}, x_{n-2}x_{n-1}$ can be colored by 3–6 cyclically. So far x_{n-1} has not been colored. At least two colors can be used to color vertex x_{n-1} properly and at least one color can be used to color properly such that vertex x_{n-1} is distinguished from x_{n-2}.

Hence, we can assume that any vertex of degree two is adjacent only to vertices of degree 3. If G contains a bridge xy, let G_1 and G_2 be components of $G - xy$ with $x \in V(G_1)$ and $y \in V(G_2)$. Give $G_1 \cup xy$ and $G_2 \cup xy$ 6-avd-total-coloring by induction. By permuting the colors on $G_2 \cup xy$, we can assume that the vertices x, y and the edge xy receive the same colors in each coloring respectively and the color set of x in $G_1 \cup xy$ is not the same as the color set of y in $G_2 \cup xy$. This now gives a 6-avd-total-coloring.

Hence we can assume that G is a graph with maximum degree 3, no vertices of degree 1, no pair of adjacent degree 2 vertices, and bridgeless. If G does not have degree 2 vertices then G is 3-regular. G must have 1-factor for a cubic graph without a 1-factor must have at least three bridges. So G has a 6-avd-total-coloring by Theorem 2.2. If G does contain degree 2 vertices then let G' be the graph obtained by taking two copies of G and joining their corresponding degree 2 vertices by an edge. Then G' is 3-regular and contains at most one bridge. Hence G' has a 1-factor and so by Theorem 2.2 G' has a 6-avd-total-coloring. This coloring of G' induces a 6-avd-total-coloring of G since no two vertices of degree 2 are adjacent in G. □

Note that Theorem 1.1 had been also proved by Haiying Wang in [6]. But we have given a more short and more interesting proof in present paper.

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