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Note

Some results in square-free and strong square-free edge-colorings of graphs

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Abstract

The set of problems we consider here are generalizations of *square-free* sequences [A. Thue, Über unendliche Zeichenreichen, Norske Vid Selsk. Skr. I. Mat. Nat. Kl. Christiana 7 (1906) 1–22]. A finite sequence $a_1a_2 \ldots a_n$ of symbols from a set S is called *square-free* if it does not contain a sequence of the form $ww = x_1x_2 \ldots x_m x_1x_2 \ldots x_m$, $x_i \in S$, as a subsequence of consecutive terms. Extending the above concept to graphs, a coloring of the edge set E in a graph $G(V, E)$ is called *square-free* if the sequence of colors on any path in G is square-free. This was introduced by Alon et al. [N. Alon, J. Grytczuk, M. Hałuszczak, O. Riordan, Nonrepetitive colorings of graphs, Random Struct. Algor. 21 (2002) 336–346] who proved bounds on the minimum number of colors needed for a square-free edge-coloring of G on the class of graphs with bounded maximum degree and trees. We discuss several variations of this problem and give a few new bounds.

Keywords: Edge colorings; Square-free colorings

1. Introduction

The study of *square-free* sequences dates back over a century [14]. Graph coloring variants of such sequences were introduced by Alon et al. [1].

A finite sequence $a_1a_2 \ldots a_n$ of symbols from a set S is called *square-free* if it does not contain a sequence of the form $ww = x_1x_2 \ldots x_m x_1x_2 \ldots x_m$, $x_i \in S$, as a subsequence of consecutive terms. 010 and 101 are square-free sequences on two symbols and we can see that they are the longest possible on two symbols. 01021020 is not square-free since it contains 102102 as a subsequence. A theorem of Thue [14] states that there exist arbitrarily long square-free sequences on three symbols. The proof is constructive, and uses simultaneous substitutions over a given set of symbols. For instance, the substitution $1 \to 12312$, $2 \to 131232$ and $3 \to 1323132$ preserves the square-free property on the set of finite sequences over the set $\{1, 2, 3\}$. That is, if we have a square-free sequence on these three symbols, we can get a longer square-free sequence by replacing each of the symbols in the original sequence by the substitution block assigned to it.

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Square-free sequences have found applications in various fields such as group theory, universal algebra, number theory, dynamical systems, formal language theory, etc. (see [3–5,8,11–13]).

There has also been many generalizations of such sequences. One such generalization is the *strong square-free* sequences, in which a consecutive subsequence ww' is forbidden if w' is a permutation of w. It is now known [10] that arbitrarily long such sequences exist on a set of four symbols and it is the best possible.

The concept of *square-free colorings* of graphs, introduced by Alon et al. [1], is a graph-theoretical variant of the *square-free sequences* of Thue [14]. Extending the above concepts to graphs, we call a coloring of the edge set E in a graph $G(V, E)$ *square-free* if the sequence of colors on any path in G is square-free. Similarly, a *strong square-free* coloring is where the sequence of colors on any path is strongly square-free. We also introduce the concept of *cyclic square-free* edge colorings. A coloring of the edge set E of a graph G is called *cyclic square-free* if no path in G has a consecutive subsequence of the form ww or ww' where w' is a reverse cyclic permutation of w. 01232103 is an example of a cyclic square, as 2103 is a cyclic permutation of 3210, the reflection of the beginning block 0123.

Alon et al. [1] defined *Thue number* of a graph G, denoted by $\pi(G)$, as the minimum number of colors needed for a square-free edge-coloring of G and showed that the Thue number is bounded on the class of graphs with bounded maximum degree. In particular, there exists an absolute constant c such that $\pi(G) \leqslant c\Delta^2$ for all graphs G with maximum degree at most Δ. We denote *strong Thue number* of a graph G by $\pi_s(G)$ and define it as the minimum number of colors needed for a strong square-free edge-coloring of G. Likewise, *cyclic Thue number* $\pi_c(G)$ is defined as the minimum number of colors needed for a cyclic square-free coloring of the edges of G.

There are some classes of graphs for which better bounds are known. Currie [6] recently showed that $\pi(C_n) = 3$ except for values $n = 7, 9, 10, 14$ and 17, proving an old conjecture apparently due to R. J. Simpson. Note that from Thue's theorem it is trivial that $\pi(C_n) \leqslant 4$. Alon et al. [1] give an explicit coloring that uses $2^k - 1$ colors for complete graphs on 2^k vertices, thus giving an upper bound of $2n - 3$ for K_n. They then give an upper bound of $4(\Delta(T) - 1)$ colors for trees T with $\Delta(T) \geqslant 2$. The base of this coloring is the Thue strings of arbitrary lengths on three symbols, which can be made *palindrome-free* with one additional color. The paper also shows that the Thue number of the Cartesian product of a collection of trees is at most the sum of their individual Thue numbers.

2. Summary of results

We give a lower bound on $\pi(T)$ as a function of the maximum degree of the tree, improving the previously known bound.

We then look at explicit colorings on certain classes of graphs. Each vertex v in a d-dimensional hypercube $H(d)$ has degree d, and we observe that there is a square-free coloring of the cube that uses exactly $\Delta = d$ colors. A similar technique is used to color the complete graph on 2^k vertices with $2^k - 1$ colors, leading to an upper bound of $2n - 3$ on π_s on graphs that have at most n vertices. It turns out that with $2d$ colors, squares can be avoided on cycles as well as paths in $H(d)$. We prove that this is not too far from the best possible, as it takes at least $\frac{3}{2}d$ colors to avoid squares on paths and cycles in $H(d)$.

A trivial strong square-free coloring of k-ary trees of height h uses kh colors, and we show that this coloring is nearly tight for trees, as a complete k-ary tree of height h cannot have a strong square-free coloring of edges that uses $h(k/e - 1)$ colors or less.

On cyclic square-free colorings, we show that $\pi_c(T) \leqslant 5(\Delta(T) - 1)$ for trees T with $\Delta(T) \geqslant 2$. We also show that there are trees for which a cyclic square-free coloring needs at least $1.8844(\Delta(T) - 1)$ colors.

3. Square-free colorings: a lower bound on trees

For the class of trees, Grytczuk [9] says the best known general lower bound on $\pi(G)$ is $\frac{4}{3}\Delta$. We first prove the following simple proposition which we improve afterwards.

Proposition 1. *For a complete k-ary tree T of height* 2, $\pi(T) > \frac{3}{2}k$.

Proof. Consider a complete k-ary tree of height 2. Note that $\Delta = k + 1$ for this graph.

Assume, for a contradiction, that a coloring with at most $\frac{3}{2}k$ colors exists. Assume without loss of generality that the edges in the first level of the tree use colors 1 through k. We define a *child* of an edge as an adjacent edge that belongs to the next level. If color j ($1 \leqslant j \leqslant k$) is used on any of the k children of the edge colored i, the square-free property demands that

(i) $i \neq j$, and
(ii) color i is ruled out on children of the first level edge colored j.

There are at most $k/2$ colors that are unused in the first level, because the total number of colors is at most $(3/2)k$. So at least $k/2$ of the k children of a first level edge colored i use colors from the set $\{1, 2, \ldots, k\}$. But (ii) rules out using color i on children of edges corresponding to all of these (at least $k/2$) colors. Also by (i), i is forbidden below itself as well. This means that i is forbidden below at least $k/2 + 1$ edges. That is, in level 2, color i can be used below at most $k/2 - 1$ edges of level 1. This is true for every i, and hence the total number of edges at level 2 which can use an *old* color (from 1 through k) is at most $k \cdot (k/2 - 1)$. But we have observed earlier that at least $k/2$ children each of every edge in the first level must use an old color, leading to a contradiction. $\square$

We can further show that this lower bound is tight for complete k-ary trees of height 2 by explicitly giving a coloring that uses exactly $\lceil (3k+1)/2 \rceil$ colors. We use colors 0 through $k-1$ at level one, and all of the $\lceil (k+1)/2 \rceil$ new colors below every edge of level 1. On children of an edge colored i in the first level, fill the remaining edges with colors $(i+1) \bmod k$ through $(i + \lfloor (k-1)/2 \rfloor) \bmod k$. It can be verified easily that if j appears below i, i does not appear below j. Thus there are no paths of the form *ijij*, and the coloring is square-free.

Improving the above result, we now prove the following proposition.

Proposition 2. *For a complete k-ary tree T of height* 3, $\pi(T) > ((\sqrt{5}+1)/2)k$.

Proof. Assume for a contradiction that the complete k-ary tree of height 3 admits a square-free coloring of edges using $((\sqrt{5}+1)/2)k$ colors or less. Note that $\Delta = k+1$ for this graph. A coloring of the edges of this tree can be looked upon as a collection of triplets of the form (a, b, c) where a is a color that appears on an edge on the first level, b on the next level and c on an edge on the third and final level of the tree. We denote the set of colors that are used at the first level by S. Note that $|S| = k$. Let S' be the set of colors that do not belong to S but are used in coloring some edge in the tree. Note that $|S'| \leqslant ((\sqrt{5}-1)/2)k$.

Consider a square-free edge-coloring C of the tree. Let $(a, b, c) \in C$. To make the analysis simple, we assume that $a \neq c$. Note that this restriction increases the minimum required number of colors by at most one. This is because if an optimum coloring has a number of edges at the third level that are colored the same as its pre-predecessor, then replacing the colors on all such edges by a new color r does not introduce any squares in the tree. A path of length less than 5 has at most one edge colored r in the path, and the paths of length 6 can have color r only at its end edges.

When (a, b, c) is present in C, (b, a, x) cannot be in C, $x \in S \cup S'$. Also, $(a, c, b), (c, b, a) \notin C$. Forbidding triplets of the form $(b, a, *)$ correspond to the square-free property on paths that lie in the first two levels. Similarly, the triplet (a, c, b) being forbidden corresponds to paths in levels two and three, and absence of (c, b, a) implies there are no squares on paths of length six which consist of edges from every level. Simpler constraints, like $b \neq a$ and $c \neq b$ in any triplet a, b, c in a square-free coloring, are implied by the more general constraints stated above.

To show that $((\sqrt{5}+1)/2)k$ colors do not suffice, we bound from above the number of triplets that could be picked from $S \times S' \times \{S \cup S'\}$, i.e., triplets of the form (a, b, c) where $a \in S$ and $b \in S'$. The number of triplets from $S \times S' \times S$ and $S \times S' \times S'$ are counted separately.

Since we have assumed that no two colors in any triplet is the same, there are at most $\binom{k}{2} \cdot |S'| < ((\sqrt{5}-1)/4)k^3$ triplets that belong to $S \times S' \times S$. Similarly, the number of triplets from $S \times S' \times S' \leqslant k \cdot \binom{|S'|}{2} < ((\sqrt{5}-1)^2/8)k^3$.

Adding up the above two yields that the number of triplets from $S \times S' \times \{S \cup S'\}$ is less than $((\sqrt{5}-1)/4 + (\sqrt{5}-1)^2/8)k^3 = k^3/2$. But the square-free property for the first two levels means that we can have either (a, b) or (b, a) for the first two levels for any two colors a and b in S. Thus we have at most $\binom{k}{2}$ colors from the set S at level 2, restricting the total number of triplets from $S \times S \times \{S \cup S'\}$ in a coloring to less than $k^3/2$. This leaves more than $k^3/2$ to be picked from the set $S \times S' \times \{S \cup S'\}$, and we fall short. $\square$

4. Coloring certain classes of graphs

There are some classes of graphs for which we can get better bounds. For instance, the d-dimensional hypercube $H(d)$. Coloring the edges of index i with color i, we get a coloring of the d-dimensional hypercube $H(d)$ that uses exactly d colors. It is easy to see that this is a strong square-free coloring. Indeed, consider any path in the hypercube that is colored ww', where w' is a permutation of w. The sequence has every index appearing an even number of times. Hence if we take a walk along the path, none of the co-ordinates are altered, and we are back at the same vertex as where we started. This contradicts our assumption and shows that the coloring is strong square-free.

This coloring could be extended to other graphs on the same vertex set by allowing the *index* of an edge to be any subset of the set of positions $\{1, 2, \ldots, d\}$. For any edge e that connects two vertices $u, v \in \{0, 1\}^d$, we define the *index* $I(e)$ of e as $\{i_1, i_2, \ldots, i_e\}$ if the sequences u and v differ at positions $i_1, i_2, \ldots, i_e$. By coloring an edge e that has index $\{i_1, i_2, \ldots i_e\}$ where $i_1 < i_2 < \cdots < i_e$ by color $i_1 i_2 \ldots i_e$, we can use the same argument as above to see that the coloring is strong square-free.

Thus a complete graph K_{2^d} can be colored with $2^d - 1$ colors, as the indices of edges here are all non-zero subsets of $\{1, 2, \ldots, d\}$. This immediately gives an upper bound of $2n - 3$ for a complete graph K_n on n vertices. Thus we have:

Proposition 3. $\pi_s(K_{2^d}) = 2^d - 1$ *and* $\pi_s(K_n) \leqslant 2n - 3$.

While the above coloring of the hypercube avoids squares on paths, it is seen that sequences of the form ww appear along cycles in this graph. We show below that we can avoid this by doubling the number of colors.

Proposition 4. *The edges of $H(d)$ can be $(2d - 1)$-colored so that the sequence of colors on every path as well as every cycle is square-free.*

Proof. Consider the coloring where an edge e is assigned color i_p where i is the index of e and p is the *parity* of the edge, defined as follows. If e connects $u = x_{k-1} \ldots x_{i+1} \mathbf{1} x_{i-1} \ldots x_0$ and $v = x_{k-1} \ldots x_{i+1} \mathbf{0} x_{i-1} \ldots x_0$, then $p = 1$ if $x_{i-1} \ldots x_0$ has an odd number of zeros, and 0 otherwise. The parity of the index 0 edges is always 0, and hence we use exactly $(2k - 1)$ colors.

We have already seen that any coloring that distinguishes edges by their indices avoids squares on paths. To show that there are no cycles colored ww, we claim that if there is any walk along a sequence ww, there is a closed circuit that is properly contained in the walk. To see this, note that if every index appears an even number of times in w, the sequence w itself corresponds to a circuit. So we may assume that i is the smallest index that appears an odd number of times. Now consider an occurrence of i in w and the two positions in the walk that correspond to this occurrence. If all indices other than i make even appearances in w, then we have every index appearing an even number of times between these two positions, which corresponds to a circuit in the walk. Otherwise, let j be smallest index such that $j > i$ and j appears an odd number of times in w. Between any two corresponding occurrences of j in the two halves of the walk, i appears an odd number of times, and all other indices less than j appear an even number of times. This means that if the edge that corresponds to this j in the first half of the walk has parity p, then the corresponding edge in the second half has parity $\bar{p}$. So we cannot have ww if there is such a j. This completes the proof of the claim, and hence the proposition. □

The coloring described above does not avoid permuted squares on cycles. On $H(3)$, we can have the cycle $2_0 0_0 1_0 2_0 1_0 0_0$. However, we see below that the $2d - 1$ colors we used to avoid squares on both paths and cycles is close to the minimum number of colors needed.

Proposition 5. *Any edge-coloring of the d-dimensional hypercube $H(d)$ with $d > 1$, that uses less than $\frac{3}{2}d$ colors has a square ww on a path, or a cycle of length at most* 4.

Proof. The origin $\{0\}^d$ and all vertices at distance at most two from the origin in $H(d)$ can be laid out as a rooted tree of height 2. The tree has the origin $\{0\}^d$ at the root and all vertices that are adjacent to it in the level below. Below each of them there are vertices that have two ones and $d - 2$ zeros. To avoid cycles, we keep two copies each of these vertices, one for each path from $\{0\}^d$ to the vertex. Now the root has d children and the first level edges have $d - 1$

children each. Note that any coloring of the edges of the hypercube has a corresponding coloring of the edges of this tree, though a coloring of the tree edges does not necessarily correspond to a coloring of the edges of the cube because of the duplication of edges in the tree. A square ww on any path in this tree corresponds to either a path or a cycle of length at most 4 in $H(d)$. For reasons identical to those used in the proof of Proposition 1, any square-free coloring of these edges uses at least $3d/2$ colors. □

5. Strong square-free coloring of trees

In this section, we first give an explicit strong square-free coloring for a complete k-ary tree. We then go on to prove that there are trees that need at least $c\Delta \log n$ colors, and it suggests that the trivial coloring we have is nearly tight for trees.

Proposition 6. *For a complete k-ary tree T_k of height h,* $\pi_s(T_k) \leqslant kh$.

Proof. The edges of T_k can be naturally partitioned into levels 1 through h. Without loss of generality, we may also assume that the k children of every edge other than those in the last level are numbered with indices 1 through k. Then coloring edge e with the ordered pair $(level, index)$ results in a strong square-free coloring of the tree. This is because a color L_a, i_b is repeated on a path only if the path has two consecutive edges at a level $L < L_a$, and these two edges differ in their indices. □

Theorem 7. *There exists a tree T on n vertices with maximum degree Δ for which* $\pi_s(T) > \log n(c(\Delta - 1) - 1)$, *where* $0 < c < 1$ *is a constant.*

Proof. We show that a complete k-ary tree of height h cannot have a strong square-free coloring of edges that uses $h(k/e - 1)$ colors or less. Note that $\Delta = k + 1$ for this graph.

Consider any strong square-free coloring of the edges of this tree that uses C colors. The colors of a path from the root of the tree to any leaf node can be looked at as an h-element multiset taken from a set of C elements. Since the coloring avoids permutations of the same sequence appearing consecutively on any path, we must have different such multisets on every such path. The number of h-element multisets taken from a set of C elements is given by $\binom{h+C-1}{h}$. We need this to be at least k^h, the number of leaves (and hence the number of root-leaf paths) in the tree. Using the inequality $\binom{n}{r} < (en/r)^r$, we get $(e(h + C - 1)/h)^h > k^h$, or $e(h + C)/h > k$, or $C > h(k/e - 1)$. □

6. Cyclic Thue number

A sequence U is called *cyclic square-free* if it contains no consecutive subsequence $u_1u_2 \ldots u_{2n}$ such that either (i) $u_{i+n} = u_i$ $(1 \leqslant i \leqslant n)$ or (ii) $u_{n+1}u_{n+2} \ldots u_{2n}$ is a cyclic permutation of $u_nu_{n-1} \ldots u_1$. The *cyclic Thue number* $\pi_c(G)$ of a graph G is the minimum number of colors needed for assigning colors to the edges of G so that the sequence of colors on every path is cyclic square-free.

To show that trees with degree at most Δ admit a cyclic square-free $5(\Delta(T) - 1)$ edge-coloring, we start with a strong square-free sequence that is *palindrome-free*. The coloring technique is adapted from Alon et al.'s [1] square-free edge-coloring of trees that uses $4(\Delta(T) - 1)$ colors, though the proof is more complicated here.

A sequence $a = a_1a_2 \ldots a_n$ is a *palindrome* if it is equal to its own reflection $\tilde{a} = a_na_{n-1} \ldots a_1$. An example is the word *pop*. Any strong square-free sequence can be made palindrome-free by inserting an additional symbol between consecutive blocks of length 2. The sequence 1234323 becomes 1253453253 under this transformation. The sequence that results is palindrome-free because the new symbol breaks the symmetry in the middle unless the sequence has odd number of symbols and the middle symbol is the new symbol 5. But we cannot have $a5a$ in the middle because the sequence is originally square-free. It may also be verified that the sequence remains strong square-free. Keränen's [10] strong square-free sequences on four symbols can thus be made palindrome-free with a fifth symbol. With this sequence on five symbols that can be made arbitrarily long, we are now prepared for a proof of the following proposition.

Proposition 8. *Let T be any tree with* $\Delta(T) \geqslant 2$. *Then* $\pi_c(T) \leqslant 5(\Delta(T) - 1)$.

Proof. Let T be a tree of maximum degree $\Delta \geqslant 2$. Choose a vertex of degree strictly less than Δ as the root of T. We may assume the tree to be complete k-ary where $k = \Delta - 1$. The edges of T are arranged in levels and except at the leaf level, every edge has k children. Let the children of each edge be indexed with numbers 1 through k. Let $b = b_1 b_2 \ldots$ be a strong square-free and palindrome-free sequence over the set $\{1, 2, 3, 4, 5\}$. We assign this sequence to the levels of T in sequential order. Now color each edge e in level j with color $\langle b_j, i\rangle$ where i is the index of the edge.

Now we show that the coloring is cyclic square-free. Assuming to the contrary, let there be a sequence $u_1 u_2 \ldots u_{2n}$ on some path P in T so that either $u_{i+n} = u_i$ $(1 \leqslant i \leqslant n)$ or $u_{n+1} u_{n+2} \ldots u_{2n}$ is a cyclic permutation of $u_n u_{n-1} \ldots u_1$. Since b is strongly square-free, P cannot be monotone. We say P has a *bend*, i.e., it has two consecutive edges that are siblings at some level j. Colors assigned to both these edges are of the form $\langle b_j, *\rangle$. There cannot be more than one bend in the path because T is a tree.

Case 1: $u_{i+n} = u_i$, $1 \leqslant i \leqslant n$ (P is of the form ww).

Establishing the non-existence of such a path P amounts to proving that the coloring is square-free. We reproduce the arguments from [1].

The bend has to be at the middle, because P cannot have more than one bend and hence we cannot have two colors of the form $\langle b_j, *\rangle$ on consecutive edges twice in P. Consider the sequence $c = c_1 c_2 \ldots c_{2n}$ defined by $c_k = j$ if u_k is of the form $\langle j, *\rangle$. Since $u_{i+n} = u_i$, $c_{i+n} = c_i$. And from the way P is laid out in the tree, the second half $c_{n+1} \ldots c_{2n}$ of c is the reflection of the first half $c_1 c_2 \ldots c_n$. Thus $c_1 c_2 \ldots c_n$ is a palindrome, which is a contradiction since b is palindrome-free.

Case 2: $u_{n+1} u_{n+2} \ldots u_{2n}$ is a cyclic permutation of $u_n u_{n-1} \ldots u_1$.

Let the bend $u_i u_{i+1}$ in P be at level j and let $b_j = l$. Suppose $u_i = \langle l, x\rangle$ and $u_{i+1} = \langle l, y\rangle$.

Case 2.1: $i < n$.

The block $\langle l, y\rangle\langle l, x\rangle$ must appear cyclically in $u_{n+1} u_{n+2} \ldots u_{2n}$. Since there is no bend in the second half of P, it must be the case that $u_{2n} = \langle l, y\rangle$ and $u_{n+1} = \langle l, x\rangle$.

Now consider the subsequence $c = c_1 c_2 \ldots c_{2(n-i)}$ of b, defined as $c_k = j$ if u_{i+k} is of the form $\langle j, *\rangle$. We know that $c_1 = c_{n-i+1} = l$.

Case 2.1.1: $i \geqslant n/2$.

Note that $u_{n+1} u_{n+2} \ldots u_{n+(n-i)} = u_i u_{i-1} \ldots u_{i-(n-i-1)}$. The left-hand side corresponds to the sequence $c_{n+1-i} c_{n+2-i} \ldots c_{n+(n-i)-i}$. Since the bend is at i, for $0 \leqslant r \leqslant i$, if u_{i-r} is of the form $\langle j, *\rangle$ then u_{i+r+1} is also of the form $\langle j, *\rangle$, and $c_{r+1} = j$. This implies that the sequence on the right corresponds to $c_1 c_2 \ldots c_{n-i}$. Now we have $c_{n-i+1} c_{n-1+2} \ldots c_{2(n-i)} = c_1 c_2 \ldots c_{n-i}$, i.e., c is of the form ww. This contradicts the square-free property of b.

Case 2.1.2: $i < n/2$.

(i) $u_{n+1} u_{n+2} \ldots u_{n+i} = u_i u_{i-1} \ldots u_1$ and (ii) $u_{n+i+1} u_{n+i+2} \ldots u_{n+i+(n-2i)} = u_n u_{n-1} \ldots u_{2i+1}$.

From (i), we have $c_{k+(n-i)} = c_k$, $1 \leqslant k \leqslant i$. From (ii), $c_{n+1} c_{n+2} \ldots c_{n+(n-2i)} = c_{n-i} c_{n-i-1} \ldots c_{i+1}$. Together they imply that $c_{n-i+1} \ldots c_{2(n-i)}$ is a permutation of $c_1 \ldots c_{n-i}$, contradicting our assumption that b is a strong square-free sequence.

Case 2.2: $i = n$.

Let $c = c_1 c_2 \ldots c_n$ be such that $c_k = j$ if u_{n+k} is of the form $\langle j, *\rangle$.

For some $2 \leqslant p \leqslant n$, $u_{n+1} u_{n+2} \ldots u_{2n} = u_{((n-p) \bmod n)+1} u_{((n-1-p) \bmod n)+1} \ldots u_{((1-p) \bmod n)+1}$. $p \neq 1$ because $u_n \neq u_{n+1}$.

Case 2.2.1: $p \leqslant n/2$.

$c_{p+d} = c_d$ for all $1 \leqslant d \leqslant p$, a contradiction because we have chosen a square-free b.

Case 2.2.1: $p > n/2$.

$c_{n-p+d} = c_d$ for all $1 \leqslant d \leqslant (n-p)$, a contradiction again.

Case 2.3: $i > n$.

This case is symmetrical to Case 2.1, and the same arguments apply. $\square$

On a path of length at most 6, we make the following observation. A sequence ww', where w' is a permutation of w, either contains a consecutive subsequence ww or w' is a reverse cyclic permutation of w. Thus the cyclic Thue number and the strong Thue number are the same for trees of height at most 3. The proposition below gives a lower bound on cyclic square-free colorings on trees.

Proposition 9. *Any cyclic square-free coloring of the edges of a complete k-ary tree of height 3 uses at least $((\sqrt{69}+3)/6)k$ colors.*

Proof. As in the proof of Proposition 2, we regard a strong square-free coloring as a collection of triplets of the form (a, b, c) where a is a color that appears on an edge on the first level, b on the next level and c on an edge on the third level of the tree. S denotes the set of colors used at the first level, $|S| = k$. We define S' as the set of remaining colors. We show that $|S'| > (\sqrt{69}-3)/6k$.

As before, we count the triplets that belong to $S \times S' \times S$, $S \times S' \times S'$ and $S \times S \times S$ separately. Note that here, the presence of a triplet (a, b, c) rules out triplets (b, c, a), (c, a, b), (a, c, b), (c, b, a) and triplets of the form $(b, a, *)$.

There are $\binom{k}{2} \cdot |S'|$ ways of picking three colors a, b and c where $a, b \in S$ and $c \in S'$, and for each choice we may pick either (a, c, b) or (b, c, a), and we have at most $\binom{k}{2} \cdot |S'|$ triplets that belong to $S \times S' \times S$. For every choice of three colors a, b and c such that $a \in S$ and $b, c \in S'$, we can pick either (a, b, c) or (a, c, b). These add up to $k \cdot \binom{|S'|}{2}$ triplets. From $S \times S \times S$ we can have at most $\binom{k}{3}$ triplets, one for every choice of three colors. Adding up the above three we must have a total of k^3 triplets. We get a quadratic in $|S'|$, solving which completes the proof. □

As mentioned earlier, Alon et al. [1] proved that the Thue number is bounded for graphs with bounded maximum degree. The same proof with slight modifications gives the same asymptotic bound for cyclic Thue number. We leave this to the reader as an exercise.

7. Open problems

There are gaps between the upper and lower bounds for almost all problems in this area. The most important being the Thue number for graphs with degree bounded by Δ. The correct constant for k-ary trees is to be determined. We believe that both the upper and lower bounds here can be improved.

For avoiding both cycles and paths on the cube, we believe that the upperbound is the correct one. We hope that this paper will lead to further results on square-free colorings on graphs.

References

[1] N. Alon, J. Grytczuk, M. Hałuszczak, O. Riordan, Nonrepetitive colorings of graphs, Random Struct. Algor. 21 (2002) 336–346.
[3] D.R. Bean, A. Ehrenfeucht, Avoidable patterns in strings of symbols, Pacific J. Math. 85 (1979) 261–294.
[4] Ch. Choffrut, J. Karhumäki, Combinatorics of words, in: G. Rozenberg, A. Salomaa (Eds.), Handbook of Formal Languages, Springer, Berlin, Heidelberg, 1997, pp. 329–438.
[5] J.D. Currie, Open problems in pattern avoidance, Amer. Math. Monthly 100 (1993) 790–793.
[6] J.D. Currie, There are ternary circular square-free words of length n for $n \geqslant 18$, Electron. J. Combin. 9 (2002).
[8] W.H. Gottschalk, G.A. Hedlund, A characterization of the Morse minimal set, Proc. Amer. Math. Soc. 15 (1964) 70–74.
[9] J. Grytczuk, Thue-type problems for graphs, points and numbers, manuscript, 2003.
[10] V. Keränen, Abelian squares are avoidable on 4 letters, Automata, Languages and Programming, vol. 623, 1992, pp. 298–301, Lecture Notes in Computer Science, vol. 4152, Springer, Berlin.
[11] J. Larson, R. Laver, G. McNulty, Square-free and cube-free colorings of the ordinals, Pacific J. Math. 89 (1980) 137–141.
[12] C. Mauduit, Multiplicative properties of the Thue–Morse sequence, Period. Math. Hung. 43 (2001) 137–153.
[13] M. Morse, A one-to-one representation of geodesics on a surface of negative curvature, Amer. J. Math. 43 (1921) 35–51.
[14] A. Thue, Über unendliche Zeichenreichen, Norske Vid Selsk. Skr. I. Mat. Nat. Kl. Christiana 7 (1906) 1–22.