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# On Parallel Sum and Difference of Matrices

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## 1. INTRODUCTION

Anderson and Duffin [1] have recently introduced the concept of parallel sum of a pair of matrices and established interesting and important properties of this operation when the matrices concerned are nonnegative definite (n.n.d.). Applications of this concept in Electrical Network Theory were discussed by the same authors and in some statistical problems connected with the estimation of parameters in a dynamic linear system by Anderson *et al.* [3]. The concept of parallel sum was extended and its elegance further demonstrated by Rao and Mitra [7], who showed that most of the properties proved by Anderson and Duffin [1] are indeed true for a much wider class of pairs of matrices designated by these authors as "parallel summable." The object of the present paper is to explore additional properties of the parallel sum to obtain the condition for consistency together with the complete class of solutions ($X$) of the parallel sum equation

$$A \mp X = C.$$

A special case of this problem (when $A$, $C$ and $X$ are n.n.d.) was recently solved in [2], [4] and [5].

We shall be primarily concerned with complex matrices, though most of the results will remain valid with trivial modifications for matrices over a more general field. For a matrix $A$, $\mathscr{M}(A)$, $\mathscr{N}(A)$, $R(A)$ and $A^*$ denote its column space, null space, rank, and conjugate transpose, respectively. A generalized inverse $A^-$ satisfies the equation $AA^-A = A$, while the Moore

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Penrose inverse $A^+$ is the unique solution to $AA^+A = A$, $A^+AA^+ = A^+$, $AA^+ = P_A$, $A^+A = P_{A^*}$ where $P_A$ and $P_{A^*}$ are the orthogonal projectors onto $\mathcal{M}(A)$ and $\mathcal{M}(A^*)$. Unless explicitly stated otherwise, the norms involved are assumed to be those induced by the inner product $(x, y) = y^*x$ where $x$ and $y$ are column vectors of appropriate dimension. Matrices $A$ and $B$ of like order are said to be disjoint if $\mathcal{M}(A)$ and $\mathcal{M}(B)$ are virtually disjoint, that is, have only the null vector in common and so are $\mathcal{M}(A^*)$ and $\mathcal{M}(B^*)$.

## 2. Parallel Sum of Matrices

Definition. Matrices $A$ and $B$ of order $m \times n$ each are said to be parallel summable (p.s.) [7] if $A(A + B)^- B$ is invariant under the choice of the generalized inverse $(A + B)^-$. If $A$ and $B$ are p.s. $A \mp B = A(A + B)^- B$ is called the parallel sum of $A$ and $B$.

Theorem 2.1, due to Rao and Mitra [7], is easy to establish.

Theorem 2.1. *$A$ and $B$ are p.s. iff*

$$\mathcal{M}(A) \subset \mathcal{M}(A + B), \qquad \mathcal{M}(A^*) \subset \mathcal{M}(A^* + B^*),$$

*or equivalently*

$$\mathcal{M}(B) \subset \mathcal{M}(A + B), \qquad \mathcal{M}(B^*) \subset \mathcal{M}(A^* + B^*).$$

Theorem 2.2 lists certain known properties of the parallel sum (see [1, 7]).

Theorem 2.2. *If $A$ and $B$ are p.s. matrices of order $m \times n$ each, then*

(a) $A \mp B = B \mp A$,

(b) *$A^*$ and $B^*$ are also p.s. and $A^* \mp B^* = (A \mp B)^*$,*

(c) *$A \mp B$ is n.n.d., when $m = n$ and $A$, $B$ are n.n.d.,*

(d) *for a matrix $C$ of order $p \times m$ and rank $m$, $CA$ and $CB$ are p.s. and*

$$CA \mp CB = C(A \mp B),$$

(e) *$A^- + B^-$ is one choice of $(A \mp B)^-$ and conversely every g-inverse of $A \mp B$ is expressible as $A^- + B^-$ for suitable choices of $A^-$ and $B^-$,*

(f) $\mathcal{M}(A \mp B) = \mathcal{M}(A) \cap \mathcal{M}(B)$,

(g) *If $P_*$ is the orthogonal projector onto $\mathcal{M}(A^*) \cap \mathcal{M}(B^*)$ and $P$ is the orthogonal projector onto $\mathcal{M}(A) \cap \mathcal{M}(B)$,*

$$(A \mp B)^+ = P_*(A^- + B^-)\, P,$$

(h) $(A \mp B) \mp C = A \mp (B \mp C)$, *if all the parallel sum operations are defined,*

(i) *if* $P_A$ *and* $P_B$ *are the orthogonal projectors onto* $\mathcal{M}(A)$ *and* $\mathcal{M}(B)$, *the orthogonal projector* $P$ *onto* $\mathcal{M}(A) \cap \mathcal{M}(B)$ *is given by*

$$P = 2(P_A \mp P_B).$$

We shall prove the following.

THEOREM 2.3. *Let* $A$ *and* $B$ *be p.s. matrices of order* $m \times n$ *each and* $A \mp B = C$. *Then*

(a) *any one of* (i) $\mathcal{M}(B) \subset \mathcal{M}(A)$ *or* (ii) $\mathcal{M}(B^*) \subset \mathcal{M}(A^*)$ *implies the other and*

$$R(A - C) = R(A), \tag{2.1}$$

(b) $R(A - C) = R(A)$ *implies* $A$ *and* $-C$ *are p.s. and*

$$B = -[A \mp (-C)] + W$$

*where* $W$ *and* $A$ *are disjoint matrices.*

*Proof of* (a). Since $A$ and $B$ are p.s., $\mathcal{M}(A) \subset \mathcal{M}(A + B)$. Hence, (i) $\Rightarrow R(A + B) = R(A) \Rightarrow$ (ii). Also observe that

$$A - C = A(A + B)^- A$$

and that (i) $\Rightarrow B = AU$ for some $U$

$$\Rightarrow A = A(A + B)^- (A + B) = (A - C)(I + U)$$
$$\Rightarrow R(A - C) = R(A).$$

The other part of (a) is similar.

*Proof of* (b). Since $\mathcal{M}(C) \subset \mathcal{M}(A)$ and $\mathcal{M}(C^*) \subset \mathcal{M}(A^*)$,

$$R(A - C) = R(A) \Rightarrow \mathcal{M}(A) \subset \mathcal{M}(A - C), \qquad \mathcal{M}(A^*) \subset \mathcal{M}(A^* - C^*)$$
$$\Rightarrow A \text{ and } -C \text{ are p.s.}$$

Write

$$B_0 = -[A \mp (-C)] = A(A - C)^- C,$$

check that $A + B_0 = A(A - C)^- A$ and that

$$R(A + B_0) = R[A(A - C)^- (A - C)] = R(A). \tag{2.2}$$

This shows $A$ and $B_0$ are p.s. By direct multiplication it is seen that $A^-(A - C) A^-$ is one choice of $(A + B_0)^-$ and that

$$\begin{aligned} A \mp B_0 &= A(A + B_0)^- B_0 \\ &= A[A^-(A - C) A^-] A(A - C)^- C = C. \end{aligned} \tag{2.3}$$

(2.2) and (2.3) $\Rightarrow A(A+B)^- A = A(A+B_0)^- A$

$\Rightarrow$ every $g$-inverse of $A+B$ is a $g$-inverse of $A+B_0$

$\Rightarrow A+B = A+B_0+W,$

where $W$ and $A+B_0$ are disjoint matrices [6, Lemma 2.7]. This completes the proof of Theorem 2.3.

*Note* 1. When $R(A-C) = R(A)$, $\mathcal{M}(C) \subset \mathcal{M}(A)$ and $\mathcal{M}(C^*) \subset \mathcal{M}(A^*)$, $A(A-C)^- C$ is the unique solution ($X$) of the equation $A \mp X = C$ with row and column spaces contained in those of $A$. In such cases, therefore, $A(A-C)^- C$ may be legitimately called the parallel difference of $A$ and $C$ and denoted by the symbol $A \equiv C$.

*Note* 2. When $A$ and $B$ are n.n.d. and $A \mp B = C$, $A - C$ is n.n.d. with rank equal to $R(A)$. This shows that when $A$ and $C$ are n.n.d. and so is $A - C$, the equation $A \mp X = C$ has a n.n.d. solution $X$ iff $R(A-C) = R(A)$. When this condition is satisfied, Theorem 2.3 provides an alternative method for solving $A \mp X = C$, and a general n.n.d. solution is seen to be

$$A(A-C)^- C + W,$$

where $A$ and $W$ are disjoint matrices and $W$ is n.n.d. (see [2], [4] and [5] in this connection).

THEOREM 2.4. (a) *Let $A$ and $B$ be p.s. matrices of order $m \times n$ and $C = A \mp B$. Then*

$$R(A-C) \geqslant 2R(A) - R(A+B). \tag{2.4}$$

*Conversely*, (b) *let $A$ and $C$ be matrices of order $m \times n$ each such that* (i) $\mathcal{M}(C) \subset \mathcal{M}(A)$, $\mathcal{M}(C^*) \subset \mathcal{M}(A^*)$ *and* (ii) $R(A-C) \geqslant 2R(A) - \min(m, n)$. *Then there exists a matrix $X$ of order $m \times n$ such that $A$ and $X$ are p.s. and*

$$A \mp X = C. \tag{2.5}$$

*Proof of* (a). Let $R(A+B) = p$ and $A + B = LR$ be a rank factorization of $A+B$. Since $A$ and $B$ are p.s. it follows that $A = LDR$ where $R(D) = R(A)$

$$\Rightarrow B = L(I-D)R$$
$$\Rightarrow C = LD(I-D)R$$
$$\Rightarrow A - C = LD^2R$$
$$\Rightarrow R(A-C) = R(D^2).$$

An application of Frobenius inequality now leads to (2.4).

*Proof of* (b). Let $R(A) = r$ and $A = L_1R_1$ be a rank factorization of $A$. Write

$$A - C = L_1A_1R_1 .$$

Clearly $R(A - C) = R(A_1) = q$ (say). Let the integer $p$ be such that $p \leqslant \min(m, n)$ and $R(A - C) \geqslant 2R(A) - p$. Let $F$ be a matrix of order $p \times r$ and rank $r$ and $E$ of order $r \times p$ and rank $r$ be such that ([6],

$$EF = A_1 .$$

For $F$ with $\mathscr{M}(A_1^*) \subset \mathscr{M}(F^*)$ one choice of $E$ is $A_1F^- + UV$ where $U$ is a matrix of order $r \times (r - q)$ and rank$(r - q)$, such that $R(A_1 \vdots U) = R(A_1) + R(U) = r$ and $V$ a matrix formed by $(r - q)$ independent rows of $(I - FF^-)$.

Let $L$ and $R$ be matrices of full rank similarly obtained such that $LF = L_1$ $ER = R_1$. Then

$$A = L_1R_1 = L(FE)\,R \qquad \text{and} \qquad A - C = L_1A_1R_1 = L(FE)^2\,R.$$

Check that $X = LR - A$ satisfies the equation

$$A \mp X = C.$$

*Note.* Arbitrary choices of $E, F, L$ and $R$ in the proof of (b) given above lead to a general solution of (2.5).

THEOREM 2.5. (a) *Let $A = A_1 + A_2$ and $B = B_1 + B_2$ be p.s. matrices of order $m \times n$ such that* (i) *$A_i$ and $B_i$ are p.s.,* $(i = 1, 2)$, (ii) $(A_1 + B_1)$ *and* $(A_2 + B_2)$ *are pairwise disjoint. Then*

$$A \mp B = (A_1 \mp B_1) + (A_2 \mp B_2).$$

*Conversely,* (b) *let $A = A_1 + A_2$ and $C = C_1 + C_2$ be matrices of order $m \times n$ each such that $A_1$ and $A_2$ are pairwise disjoint, $\mathscr{M}(C_i) \subset \mathscr{M}(A_i)$, $\mathscr{M}(C_i^*) \subset \mathscr{M}(A_i^*)$, $i = 1, 2$. Then if $A \mp B = C$, $B$ can be expressed as $B = B_1 + B_2$ such that $(A_1 + B_1)$ and $(A_2 + B_2)$ are pairwise disjoint and*

$$A_i \mp B_i = C_i .$$

*Proof of* (a). Write $E_i = (A_i + B_i)$ and check that

$$G_i = (E_1^*E_1 + E_2^*E_2)^- E_i^*E_iE_i^*(E_1E_1^* + E_2E_2^*)^-$$

is a $g$-inverse of $E_i$ such that

$$E_jG_i = 0, \qquad G_iE_j = 0$$

whenever $i \neq j$. This shows $G_1 + G_2$ is a $g$-inverse of $E_1 + E_2 = A + B$ and

$$\begin{aligned} A \mp B &= A(G_1 + G_2)\, B \\ &= A_1G_1B_1 + A_2G_2B_2 \\ &= (A_1 \mp B_1) + (A_2 \mp B_2). \end{aligned}$$

*Proof of* (b). Let $A + B = LR$ be a rank factorization of $A + B$ and $A_i = LD_iR$, $i = 1, 2$. We put $D = D_1 + D_2$.

$$A - C = (A_1 - C_1) + (A_2 - C_2)$$

and

$$\mathcal{M}(A_i - C_i) \subset \mathcal{M}(A_i), \qquad \mathcal{M}(A_i^* - C_i^*) \subset \mathcal{M}(A_i^*) \Rightarrow D_1D_2 = D_2D_1 = 0.$$

This implies the existence of a non singular $T$ such that

$$T^{-1}D_1T = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad T^{-1}D_2T = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_2 \end{pmatrix}.$$

Let

$$T = (T_1 \vdots T_2) \qquad \text{and} \qquad T^{-1} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

be the corresponding partitions of $T$ and $T^{-1}$. Put $B_i = LT_i(I - \Lambda_i)\, U_iR$ and check that $B_1$ and $B_2$ satisfy (b).

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